Decomposition of Steiner Triple Systems into Triangles

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ABSTRACT

A triangle in a Steiner triple system S is a triple of blocks from S which meet pairwise and whose intersection is empty. If S contains b blocks, and $b = 3q + s$, where $0 \le s \le 2$, then a triangulation of S is a collection of q triangles $\{T_1, T_2, \ldots, T_q\}$ in S such that no two distinct triangles share a common block. It is shown that, for $v \equiv 1$ or $3 (mod\, 6)$, there exists a Steiner triple system which admits a triangulation. Moreover, if $s = 2$, there is a triangulated triple system in which the pair of blocks not occurring in a triangle are disjoint, and a triangulated triple system in which they intersect.

1. Introduction.

A *Steiner triple system* (*STS*) of order v is a pair (V,B), where V is a v-set and B is a collection of triples from B called blocks, which has the property that every pair of distinct elements (points) from V occurs in exactly one block of B. It is well known that a necessary and sufficient condition for the existence of a Steiner triple system of order v ($STS(v)$) is that v be a positive integer congruent to 1 or 3 modulo 6. A $STS(v)$ contains $b = v(v-1)/6$ blocks and distinct blocks meet in at most one point.

Let S be an $STS(v)$ and suppose $b = 3q + s$, where $0 \le s \le 2$. A *triangle* in S is a set of three blocks of S which meet pairwise but whose intersection is empty. A *triangulation* T of S is a set $\{T_1, T_2, \ldots, T_q\}$ of q triangles of S such that $T_i \cap T_j = \emptyset$ for $i \neq j$. Let R denote the set of s blocks which occur in no triangle of a triangulation T. We say that T is of *type I* if any two distinct blocks from R are disjoint; otherwise T is of *type II*. Vacuously, if $s = 0$ or 1, then any triangulation is of both type I and type II. A *triangularized Steiner triple system* (*TSTS*) of order v ($TSTS(v)$) is a triple (V,B,T) where (V,B) is a Steiner triple system S of order v and T is a triangulation of S. A *TSTS* is said to be of type I or type II according as T is of type I or type II. It is our object to show that for every positive $v \equiv 1$ or $3 (mod\, 6)$ there exists a $TSTS(v)$ of type I and type II whose underlying $STS(v)$'s are isomorphic.

2. Notation and Preliminary Results.

2.1 Notation. Let N_i be the set of integers $\{v\}$ such that there exists an $STS(v)$ with its blocks decomposable into triangles such that no two triangles have common blocks and there are i disjoint blocks left over, where $i = 0,1,2$, or 3. Note that N_0, N_1, and N_2 are the set of integers $\{v\}$ such that there exists a $TSTS(v)$ of type I. Also, we let M_2 be the set of integers $\{v\}$ such that there exist a $TSTS(v)$ of type II. Note that in this paper we refer to blocks left over as *remaining* blocks.

Throughout this paper we will use the convention that, for $STS(v)$ a Steiner triple system S, when we make use of the expression $S \in N_i$, we are identifying a decomposition of S such that its blocks are decomposed into triangles such that no two triangles have common blocks and there are i disjoint remaining blocks.

2.2 Preliminary Results.

The following necessary conditions are easily established.

2.2.1 Necessary Conditions.

2.2.1.1. $v \in N_0$ implies that $v \equiv 1,9 (mod\ 18)$.

2.2.1.2. $v \in N_1$ implies that $v \equiv 3,7 (mod\ 18)$.

2.2.1.3. $v \in N_2$ implies that $v \equiv 13,15 (mod\ 18)$.

2.2.1.4. $v \in M_2$ implies that $v \equiv 13,15 (mod\ 18)$.

We will first concern ourselves with proving the sufficiency of the first three necessary conditions above. We will use recursive constructions in each of the six cases; then only four small cases need to be done directly, namely, $v = 7,9,13$, and 21.

2.2.2 Sufficiency for the cases v = 7,13, and 21.

Lemma 2.2.2.1. *7 is in* N_1.

Proof. Figure 1 illustrates the blocks in triangles by grouping them in collections of three. The one remaining block is (7,1,3).

Lemma 2.2.2.3. *21 is in* N_1.

Proof. The triangulation of an $STS(21)$ is illustrated in Figure 3.

1	2	4	4	5	7
2	3	5	5	6	1
3	4	6	6	7	2

7 1 3

Figure 1

Lemma 2.2.2.2. *13 is in* N_2.

Proof. Figure 2 illustrates this. The two remaining blocks are (5,6,9) and (10,11,1).

1	2	5	6	7	10	2	3	6
11	12	2	1	6	8	2	7	9
5	10	12	7	12	1	9	1	3
3	4	7	4	5	8	9	10	13
4	9	11	8	9	12	12	13	3
11	3	5	12	4	6	3	8	10
13	1	4	7	8	11			
8	13	2	6	11	13			
10	2	4	13	5	7			
5	6	9	10	11	1			

Figure 2

Remark. In the *STS*(21) shown in Figure 3, if we develop the block (0,7,14) mod 21, we get 7 blocks forming a parallel class of blocks and containing the remaining block (0,7,14).

Let L be a Latin square of order n based on these symbols: $\{a_1,a_2,\ldots,a_n\}$. Then a transversal in L is a set n of cells, one from each row and column, with the property that $a_1,a_2,\ldots,$ and a_n each appear in a cell.

Lemma 2.2.3. *For $n > 2$ there is an idempotent Latin square of order n with three disjoint transversals, one of which is the main diagonal.*

0	6	16	1	7	17	2	8	18	11	19	2
16	17	19	11	17	6	1	2	4	19	20	1
18	19	0	6	7	9	1	8	15	20	0	2
3	9	19	4	10	20	5	11	0	17	4	8
14	20	9	13	0	4	14	1	5	3	4	6
12	20	3	15	0	10	0	1	3	3	10	17
6	12	1	7	13	2	8	14	3	20	7	11
16	1	11	13	19	8	8	16	20	7	8	10
4	12	16	7	15	19	16	3	7	8	9	11
9	15	4	10	16	5	12	18	7	10	11	13
15	2	6	9	10	12	4	5	7	11	12	14
2	9	16	5	12	19	4	11	18	12	13	15
17	2	12	18	3	13	19	4	14	13	14	16
9	17	0	1	9	13	19	6	10	14	15	17
0	8	12	10	18	1	2	10	14	15	16	18
20	5	15	5	13	17	6	14	18			
3	11	15	17	18	20	18	5	9	0	7	14
2	3	5	6	13	20	5	6	8			

Figure 3

Proof. For $n > 2$ and $n \neq 6$, there is a pair of orthogonal Latin squares of order n. Take three elements in the second square, each of which determines a transversal in the first square. Some permutation will take one of them to the main diagonal so that the (i,i) entry is i, for $i = 1,\ldots,n$. A required square of order 6 is shown in Figure 4. Two transversals other than the main diagonal are marked with circles and squares.

1	(4)	[2]	5	6	3
5	2	4	(6)	[3]	1
[6]	1	3	2	4	(5)
(3)	[5]	6	4	1	2
2	6	(1)	3	5	[4]
4	3	5	[1]	(2)	6

Figure 4

A *group divisible design* $GD(n\ell;n;k)$ consists of a triple (V,G,F) where V is an $n\ell$ set, G is a partition of V into ℓ n-subsets (called groups), and F is a family of k-subsets (called blocks) of V with the property that every pair of points of V from distinct groups of G occurs in precisely one block of F, and no pair of points from the same group of G occurs in any block of F. Let A be a Latin square based on $\{1,2,\ldots,n\}$. If the (i,j) entry of A is k, and we associate a block (x_i,y_j,z_k) with it, we get a group divisible design $GD(3n;n;3)$, based on the set $\{(x_i,y_i,z_i)\,|1\leq i\leq n\}$. If A has a subsquare with the row indices, column indices, and entries in the same set, then the deletion of the subsquare will result in an incomplete GD, that is, $GD(3n;n;3)-GD(3k;k;3)$. Deletion of two disjoint such subsquares of orders k and h will result in an incomplete GD, that is, $GD(3n;n;3)-GD(3k;k;3)-GD(3h;h;3)$. We write $n \in C_i$, or $(n;k) \in C_i$, or $(n;k;h) \in C_i$ if there is a $GD(3n;n;3)$, or $GD(3n;n;3)-GD(3k;k;3)$, or $GD(3n;n;3)-GD(3k;k;3)-GD(3h;h;3)$, respectively, whose blocks can be decomposed into triangles such that no two triangles have common blocks and there are i remaining blocks, $i = 0$ or 1 (since $n^2 \equiv 0$ or $1 (mod\, 3)$). Similarly we can have an incomplete GD with several sub-GD deleted. For example, the notation $(n;k;h;\ell) \in C_i$ is also well defined.

Lemma 2.2.4. *If* $n \equiv 0(mod\,3)$, *then*

1. $n \in C_0$, $n \geq 3$
2. $(n;3) \in C_0$, $n \geq 6$
3. $(n;3;3) \in C_0$, $n \geq 9$
4. $(n;3;3;3) \in C_0$, $n \geq 9$

Proof. Suppose some Latin square A has the same element k as its (i,j) entry and its (r,s) entry. Then $i \neq r$ and $j \neq s$. If the (i,s) and (r,j) entries are t and h, then we have two triangles as shown in Figure 5.

$$\begin{array}{ccc} x_i\ y_j\ z_k & \qquad & x_i\ y_j\ z_k \\ x_r\ y_s\ z_k & & x_r\ y_s\ z_k \\ x_i\ y_s\ z_t & & x_r\ y_j\ z_h \end{array}$$

Figure 5.

With the above statement, we can decompose a $GD(9;3;3)$ into triangles as shown in Figure 6.

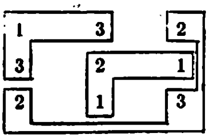

Figure 6.

For $n \equiv 0 \pmod 3$, write $n = 3t$, $t \geq 1$. From Lemma 2.2.3, for $t = 2$ and $k = 1$, and for $t \geq 3$ and $k = 1,2$, and 3, there is an idempotent Latin square of order t with the (i,i) entry i ($i = 1,\ldots,k$). We can delete the entries $(1,1),\ldots,(k,k)$. Thus, the direct product of the square of order 3 and these squares of order t (with and without deletions) results in $GD(3n;n;3)$, $GD(3n;n;3)-GD(9;3;3)$, $GD(3n;n;3)-GD(9;3;3)-GD(9;3;3)$, $GD(3n;n;3)-GD(9;3;3)-GD(9;3;3)-GD(9;3;3)$. In each case the decomposition for $GD(9;3;3)$ applies, and the four results follow.

Lemma 2.2.5. *If* $n \equiv 1,2 \pmod 3$, *then*

1. $n \in C_1$, $n \geq 1$
2. $(n;3) \in C_1$, $n \geq 7$
3. $(n;3;3) \in C_1$, $n \geq 10$
4. $(n;3;3;3) \in C_1$, $n \geq 10$

Proof. First, we know from the decomposition in Figure 7 that $2 \in C_1$.

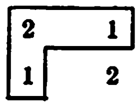

Figure 7.

If $t > 2$ and $n = 3t + j$, $j = 1$ or 2, then from Lemma 2.2.3 there is an idempotent Latin square of order t with $j + 1$ disjoint transversals, one of which is the main diagonal. Take the product of a square of order 3 with the square of order t. Add one one new row and new column by projecting one transversal and adjoining a new element α in its place. In each cell on the transversal, use the decomposition of Figure 8. Note that, from here on, in this paper, the term 'Latin square' is replaced by 'square'.

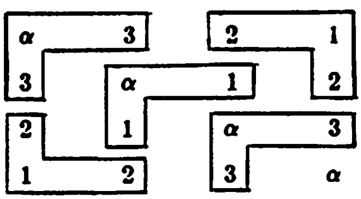

Figure 8.

We choose an off-diagonal transversal and the square of order $n = 3t+1$ will be as in Figure 9.

3×3	α 6 5 6 α 4 5 4 α	3×3	. . .	3×3	. . .	4 5 6
3×3	3×3	3×3	. . .	α 3 2 3 α 1 2 1 α	. . .	1 2 3
α 9 8 9 α 7 8 7 α	3×3	3×3	. . .	3×3	. . .	7 8 9
⋮	⋮	⋮	. . .	⋮	. . .	
7 8 9	4 5 6			1 2 3		α

Figure 9

Thus the conclusion is true for $n = 3t+1$, $t > 2$. When $n = 3t+2$ and $t > 2$, we use two transversals and two new elements α and β. We choose the two transversals such that neither is the main diagonal. The decomposition (Figure 11) will contain decompositions of order 3, decompositions of order 4, and the decomposition of order 2 in the lower right corner, shown in Figure 10.

β	α
α	β

Figure 10.

The cells of the two off-diagonal transversals combine with the two added rows and columns to give two sets of decompositions of order 4.

Finally, we consider the cases $t = 2$ and $t = 1$. A decomposition for a $GD(21;7;3)-GD(9;3;3)$ is shown below. Thus $(7;3) \in C_1$ and $7 \in C_1$ as shown in Figure 12.

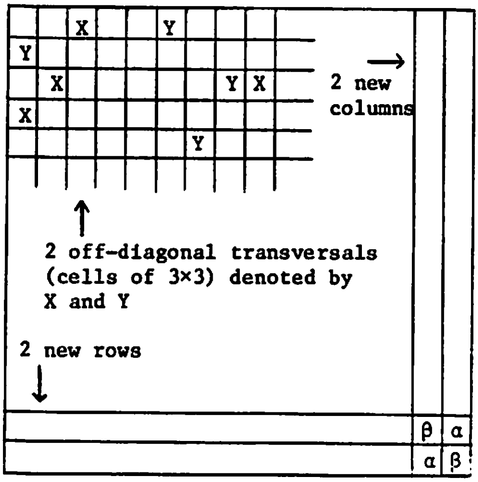

Figure 11.

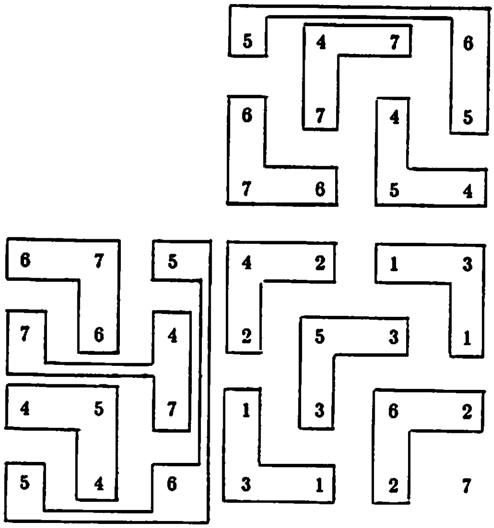

Figure 12.

Similarly, (8;3) $\in C_1$ and 8 $\in C_1$ come from the decomposition in Figure 13.

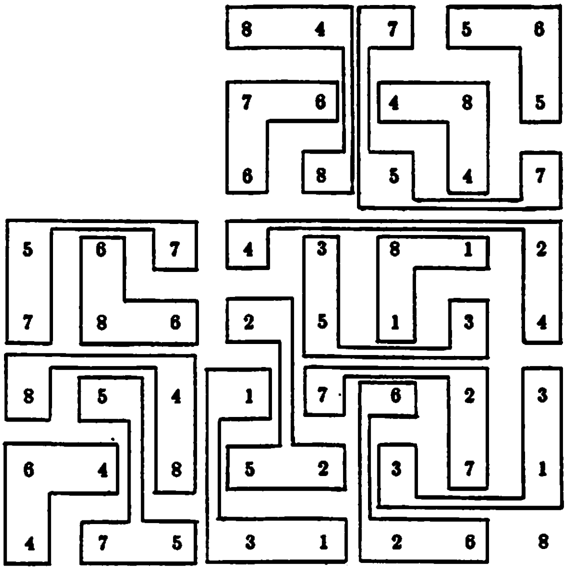

Figure 13.

For $t = 1$, $4 \in C_1$ and $5 \in C_1$, as shown in Figures 14 and 15 respectively.

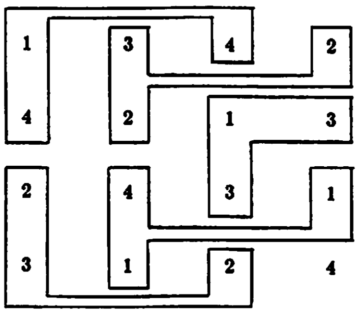

Figure 14.

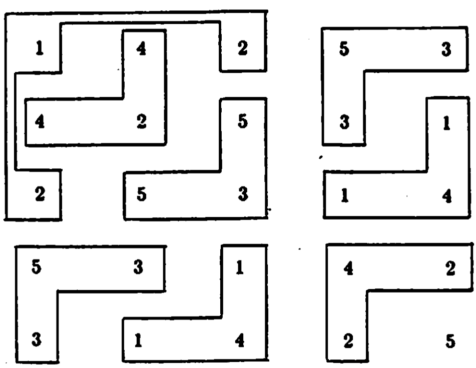

Figure 15.

Remark. In Lemma 2.2.5, the remaining block is always of the form (x_n, y_n, z_n). Also, we choose our square of order 3 to be that of in Figure 6, that is, a transversal is on the main diagonal. Take the product of this square with the idempotent square of order t. The result is that, when $n \equiv 0,1 (mod\, 3)$, the (i,i) entry, except in some hole, is always i unless for the case $(6;3) \in C_0$.

Lemma 2.2.6. *9 is in N_0 and N_3.*

Proof. Since $3 \in C_0$ from Lemma 2.2.4, $9 \in N_3$. The remaining blocks are (x_1,x_2,x_3), (y_1,y_2,y_3), and (z_1,z_2,z_3). A modification of the decomposition in Figure 6 is shown in Figure 16, from which we know $9 \in N_0$.

$x_1\ y_1\ z_1$	$x_2\ y_2\ z_2$	$x_3\ y_3\ z_3$
$x_1\ y_2\ z_3$	$y_1\ y_3\ z_2$	$x_2\ y_3\ z_1$
$y_1\ y_2\ y_3$	$x_1\ x_2\ x_3$	$x_3\ y_2\ z_1$
$x_3\ y_1\ z_2$		
$x_2\ y_1\ z_3$		
$z_1\ z_2\ z_3$		

Figure 16

Remark. $STS(9)$ contains a parallel class of blocks (x_1,x_2,x_3), (y_1,y_2,y_3), and (z_1,z_2,z_3). These blocks are the three remaining blocks when we use an $STS(9) \in N_3$.

3. Lemmas used recursively to get sufficient conditions for the first 3 necessary conditions of Lemma 2.2.1.

Lemma 3.1. *If $6t+1 \in N_3$ or $6t+1 \in N_i$, $i = 1$ or 2, then $18t+1 \in N_3$.*

Proof. If $6t+1 \in N_3$, we add one point to each of the three groups of a $GD(18t;6t;3)$ and use the decomposition of an $STS(6t+1)$ for each group. Now we use a $(6t;3;3;3) \in C_0$, $t > 1$. For the incomplete GD, fill in two sub-$GD(9;3;3)$s with an $STS(9) \in N_0$, and the third one with an $STS(9) \in N_3$. We then get $18t+1 \in N_3$.

If $6t+1 \in N_i$, $i = 1,2$, we use an incomplete $GD(18t;6t;3)$ (again for $t > 1$) with i holes and fill in one hole with an $STS(9) \in N_3$, and hole $(i-1)$ with an $STS(9) \in N_0$. We get $18t+1 \in N_3$.

For $t = 1$, we have $6t+1 = 7 \in N_1$ (Lemma 2.2.2.1), and we use an incomplete $GD(18;6;3)-GD(9;3;3)$. We fill in the *one* hole with an $STS(9) \in N_3$, and we get $18t+1 \in N_3$.

Lemma 3.2. *If* $6t+3 \in N_0 \cap N_3$, *or* $6t+3 \in N_i$, $i = 1$ *or* 2, *then* $18t+9 \in N_0 \cap N_3$.

Proof. The case $t = 0$ is done in Lemma 2.2.6. Suppose $t \geq 1$. We have a decomposition for an incomplete $GD(18t+9;6t+3;3)$ with i holes, $i = 1,2$, or 3, since $(6t+3;3;3;3) \in C_0$ from Lemma 2.2.4. Fill in each of the $i-1$ holes with an $STS(9) \in N_0$ and one hole with an $STS(9) \in N_j$, $j = 0$ or 3. We get $18t+9 \in N_0 \cap N_3$. Note that, if $i = 0$, we use the fact that $6t+3 \in C_0$ (no holes), and thus $18t+9 \in N_0$.

Remark. The $STS(18t+9)$ in Lemma 3.2 contains a parallel class of blocks, each of which is associated with some diagonal element of a square of order $6t+3$, or from some $STS(9)$ corresponding to i holes (see Remark, Lemma 2.2.5), that is, when we use an $STS(9) \in N_0$ choose (x_i,y_i,z_i), $(x_{i+1},y_{i+1},z_{i+1})$, and $(x_{i+2},y_{i+2},z_{i+2})$. When we use an $STS(9) \in N_0$ for the final hole, $18t+9 \in N_0$. When we use an $STS(9) \in N_3$, $18t+9 \in N_3$, and the parallel class contains the three remaining blocks, that is, $(x_{6t+1},x_{6t+2},x_{6t+3})$, $(y_{6t+1},y_{6t+2},y_{6t+3})$, and $(z_{6t+1},z_{6t+2},z_{6t+3})$.

Lemma 3.3. *If* $6t+3 \in N_i$, $i = 1,2$ *or* 3, *then* $18t+3 \in N_1$.

Proof. The case for $t = 0$ is trivial;the case for $t = 1$ is established in Lemma 2.2.2.3. For $t > 1$, we have an incomplete $GD(18t;6t;3)$ with $i-1$ holes. Add three points to each group and use $6t+3 \in N_i$; the three points form a block, one of the remaining blocks. As usual, the $i-1$ remaining blocks should match the $i-1$ holes. Fill in each hole with an $STS(9) \in N_0$. Thus $18t+3 \in N_1$.

Remark. The $STS(18t+3)$ in Lemma 3.3 contains a parallel class of blocks, $6t$ of which are associated with some diagonal element of a square of order $6t$, or from some $STS(9) \in N_0$ (see Remark, Lemma 2.2.5). The last block of the parallel class is the additional block. Thus, the parallel class contains the remaining block.

Lemma 3.4. *If* $6t+3 \in N_i$, $i = 0,1$, *or 2, then* $18t+7 \in N_1$.

Proof. The case $t = 0$ is discussed in Lemma 2.2.2.1. For $t > 1$, we use an incomplete $GD(18t+6;6t+2;3)$ with i holes and add one point to each group. For each group, use $6t+3 \in N_i$ and fill in each of i holes with an $STS(9) \in N_0$. One remaining block comes from the decomposition of the incomplete GD, since $6t+2 \in C_1$ (Lemma 2.2.5). Thus $18t+7 \in N_1$. Note that when $t = 1$, $6t+3 = 9 \in N_0$, and we have a $GD(24;8;3)$ with no holes, and so we get $25 \in N_1$.

Lemma 3.5. *If* $6t+7 \in N_i$, $i = 1,2$ *or* 3, *then* $18t+15 \in N_2$.

Proof. Write $18t+15 = 3(6t+4)+3$. We use an incomplete $GD(18t+12;6t+4;3)$ with $i-1$ holes (and $t \geq 1$) and add three points to each group. Let those three points form a remaining block in the decomposition of $6t+7 \in N_i$. Fill in each of the $i-1$ holes with an $STS(9) \in N_0$. Then we know $18t+15 \in N_2$, where exactly two blocks are remaining, one from the incomplete GD and the other formed by the three new points. Note that when $t = 0$, $7 \in N_1$ and we use a $GD(12;4;3)$ with no holes and get $15 \in N_2$.

Remark. The $STS(18t+15)$ in Lemma 3.5 contains a parallel class of blocks, that is, $6t+5$ disjoint blocks, $6t+4$ of which are associated with some diagonal element of a square of order $6t+4$, or from some $STS(9) \in N_0$ (see Remark, Lemma 2.2.5). The last block of the parallel class is the additional block. Therefore the parallel class contains the two blocks remaining.

The "$v \rightarrow 2v+1$ construction for STS is very old; one account of it can be found in [1, p.329]. Let (S,T) be any triple system, ∞ an element not in $\{1,2\}\times S$, and $K = \{\infty\} \cup (\{1,2\}\times S)$. Then (K,B) is an $STS(2v+1)$ where B contains the following blocks:

(1) the $|S|$ blocks $(\infty,(1,x),(2,x)) \in B$ for every $x \in S$.

(2) For every triple $(x,y,z) \in T$ define a copy of a triple system of order 7 on the set $\{\infty\} \cup \{(1,2)\times(x,y,z)\}$ such that the blocks $((1,x),(2,y),(2,z))$, $((1,y),(2,x),(2,z))$, $((1,z),(2,x),(2,y))$, and $((1,x),(1,y),(1,z))$ belong to B.

Lemma 3.6. *If* $18t+15 \in N_2$ *and the* $STS(18t+15)$ *has a parallel class of blocks, then* $36t+31 \in N_2$.

Proof. First we know that, if $18t+15 \in N_2$, then there exists an $STS(18t+15)$ which has a parallel class of blocks for all t, from the remark in Lemma 3.5. Now write $36t+31 = 2(18+15)+1$ and use the "$v \rightarrow 2v+1$" construction. Suppose that $STS(18t+15)$ is such a system (S,T) with a parallel class T_0 of blocks in B such that the first three blocks in (2) form a triangle in $(x,y,z) \in T\setminus T_0$, but for $(x,y,z) \in T_0$ we form two triangles $((1,y),(2,x),(2,z))$; $((1,x),(2,x),\infty)$; $((1,y),(2,y),\infty))$; and $((1,z),(2,y),(2,x))$; $((1,x),(2,y),(2,z))$; $((1,z),(2,z),\infty)$. What is remaining in B is exactly the blocks $((1,x),(1,y),(1,z))$, where $(x,y,z) \in T$. Since $18t+15 \in N_2$, we then know that $36t+31 \in N_2$.

Lemma 3.7. *If* $6t+3 \in N_i$, $i = 1,2$ *or* 3 *and the* $STS(6t+3)$ *has a parallel class of blocks containing the* i *disjoint blocks remaining in the decomposition, then there is an* $STS(12t+7)$ *with a sub-*$STS(7)$ *such that*

$12t+7 \in N_i$ and the decomposition contains a sub-decomposition on the STS(7). By this we mean than an STS(7) (two triangles and one remaining block - Figure 1) comprise a subset of the triangulation of STS($12t+7$).

Proof. First we note that if (for all t) $6t+3 \in N_0 \cap N_3$ or $6t+3 \in N_i$, $i = 1$ or 2, then each STS($6t+3$) contains a parallel class of blocks containing the i ($i = 1,2$, or 3) disjoint remaining blocks in the decomposition (Lemmas 2.2.2.3, 2.2.6, 3.2, 3.3, 3.5 - Remarks). Now write $12t+7 = 2(6t+3)+1$. We do a construction similar to that of Lemma 3.6. For a block (x,y,z) remaining in the decomposition of STS($6t+3$), we have $(x,y,z) \in T_0$ and then the block $((1,x),(1,y),(1,z))$ and two corresponding triangles form a sub-decomposition on a sub-STS(7).

From [2, p.96], we have an indirect product construction. Suppose that there exist the following ingredient designs:

(1) STS(u),

(2) STS(v) missing sub-STS(w),

(3) $GD(3(v-a);v-a;3)$ missing a sub-$GD(3(w-a);w-a;3)$,

(4) STS($u(w-a)+a$).

Then, there is an STS($u(v-a)+a$).

Lemma 3.8. *If $12t+7 \in N_i$, $i = 1,2$, or 3, and the decomposition contains a sub-decomposition on a sub-STS(7), then $36t+13 \in N_2$.*

Proof. Write $36t+13 = 3(12t+3)+4$ and let $u = 3$, $v = 12t+7$, $w = 7$, and $a = 4$. From the indirect product, we have an STS($36t+13$). We will further look at its decomposition. First, we have (assume $t \geq 1$) a $GD(3(12t+3);12t+3;3)$ with i holes whose blocks are decomposed into triangles. Distinguish one hole and fill in each of the $i-1$ holes with an STS(9) $\in N_0$. Add four new points to each group. For each group we use a decomposition of STS($12t+7$) with a sub-decomposition on a sub-STS(7) missing so that the three points corresponding to the distinguished hole all belong to the missing sub-STS(7). Now all the blocks are decomposed into triangles except those in the set formed by the nine points corresponding to the distinguished hole and the four new points. Since $13 \in N_2$, from Lemma 2.2.2.2, we use such an STS(13) on the set of 13 points. Thus we get an STS($36t+13$) such that $36t+13 \in N_2$. Note that, when $t = 0$, we have already shown that $13 \in N_2$ (Figure 2).

4. Summary.

The foregoing results are brought together below.

Theorem 4.1. *For every positive integer v, $v \equiv 1$ or $3 (mod\, 6)$, there is an $STS(v)$ with its blocks decomposable into triangles such that no two triangles have common blocks and there are i disjoint blocks remaining, where $i = 0,1,$ or 2, as appropriate.*

Proof. From Lemmas 2.2.2.1, 2.2.2.2, 2.2.2.3, and 2.2.6, we know that $3 \in N_1$, $7 \in N_1$, $9 \in N_0 \cap N_3$, and $13 \in N_2$. Also, since $7 \in N_1$, then from Lemma 3.5, $15 \in N_2$. The results follows by induction.

We still need to prove the sufficiency of the last necessary condition in 2.2.1, that is, $v \equiv 13,15 (mod\, 18)$, implies that $v \in M_2$.

5. Lemmas needed to prove sufficiency of 2.2.1.4.

Lemma 5.1. *If $18t + 15 \in M_2$ and the $STS(18t+15)$ has a parallel class of blocks, then $36t + 31 \in M_2$..*

Proof. Assume that $18t+15 \in M_2$ for all $t \geq 0$. From Lemma 3.6, using the "$v \to 2v+1$" construction, we would have the blocks of $STS(36t+31)$ in triangles except for $((1,x),(1,y),(1,z))$, where (x,y,z) is a block in an STS$(18t+15)$. Therefore, if $18t+15 \in M_2$, then $36t+31 \in M_2$.

Lemma 5.2. $36t+13 \in M_2$.

Proof. By examining Lemma 3.8, which is used to show that $36t + 13 \in N_2$ for every $t \geq 1$ (Theorem 4.1), we see that, if $13 \in M_2$, then $36t+13 \in M_2$ for $t \geq 0$. Now $13 \in M_2$ as shown in Figure 17.

1	2	5	6	7	10	2	3	6
5	6	9	10	11	1	2	7	9
1	6	8	7	12	1	9	1	3
3	4	7	4	5	8	9	10	13
4	9	11	8	9	12	12	13	3
11	3	5	12	4	6	3	8	10
13	1	4	7	8	11	11	12	2
8	13	2	6	11	13	5	10	12
10	2	4	13	5	7			

Figure 17.

The two remaining blocks are (11,12,2) and (5,10,12), which intersect.

Therefore all that is left to prove is that $18t+15 \in M_2$.

Lemma 5.3. *$12t+1 \in N_i$, $i = 2,3$, or 4 (for $t \geq 2$), and there exists an STS$(12t+1)$ which contains a sub-decomposition on a sub-STS(7).*

Proof. In Figure 18, the elements of an *STS*$(2v+7)$ are shown, where $v = 2s+1$ and $v \equiv 1,3 (mod\, 6)$.

These are the points of the *STS*$(2v+7)$: $(0,i)$, $(1,i)$, $i = 0,1,...,v-1$;

$\infty_0, \infty_1, \ldots, \infty_6$.

These are the blocks of the *STS*$(2v+7)$:

(1) $(\infty_0,(0,0),(1,0))$, $(\infty_1,(0,0),(1,1))$ } $mod(-,2s+1)$

$(\infty_2,(0,0),(1,-1))$, $(\infty_3,(0,0),(1,2))$, $(\infty_4,(0,0),(1,-2))$, $(\infty_5,(0,0),(1,3))$, $(\infty_6,(0,0),(1,-3))$ } $mod(-,2s+1)$

$((0,0),(1,\alpha),(1,-\alpha))$ . . . for every $\alpha = 4,5,...,s$; $((1,0),(1,2),(1,6))$ } $mod(-,2s+1)$

(2) blocks of an *STS*(7) on the elements ∞_i;

(3) blocks of an *STS*(v) on the elements $(0,i)$.

Figure 18

Note that the above construction does not hold for $v = 3$; so we let $v = 6t-3$ for $t \geq 2$ (that is, $s = 3t-2$). We get $2v+7 = 2(6t-3)+7 = 12t+1$ (for $t \geq 2$). Some of the blocks of an *STS*$(12t+1)$, for $t \geq 2$, are shown in Figure 19.

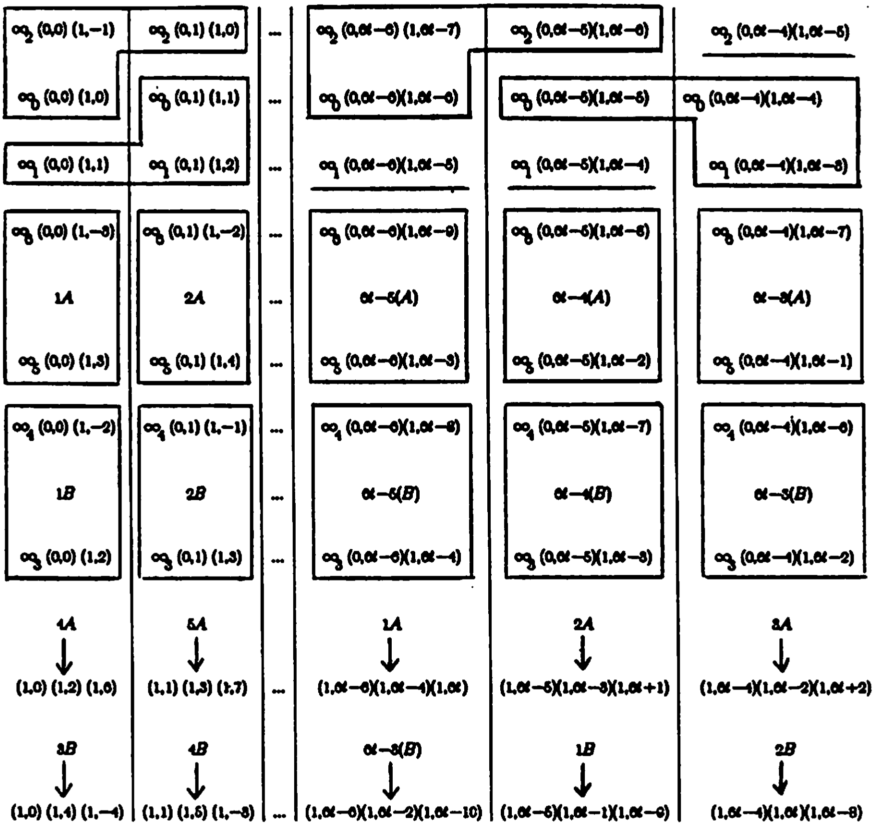

Figure 19

In Figure 19, triangulations are illustrated by the blocks circled, as well as the pairs of blocks circled with an associated number linking them with a third block in one of the bottom two rows. Note that there are $6t-10$ columns (an even number) in the space where there are three dots. They can be triangulated like the pair of columns on the left. The three remaining blocks are underlined; they are $(\infty_1,(0,6t-6),(1,6t-5))$, $(\infty_1,(0,6t-5),(1,6t-4))$, and $(\infty_2,(0,6t-4),(1,6t-5))$.

Thus we are left with the 3 remaining blocks and:

(1) $((0,0),(1,\alpha),(1,-\alpha))$ for every $\alpha = 5,6,\ldots,s$, $mod(-,2s+1)$ (only if $s > 4$, that is, $t \geq 3$);

(2) blocks of an $STS(v)$ on the elements $(0,i)$;

(3) blocks of an $STS(7)$ on the elements ∞_i.

Now remembering that $s = 3t-2$ (or $s \equiv 1(mod\,3)$), we write the blocks $((0,0),(1,\alpha),(1,-\alpha))$ for $\alpha = 5,6,\ldots,s$ in Figure 20.

$$\frac{s-4}{3}A\ ((1,0),(1,5),(1,-5))\ \cdots$$

$$\frac{s-7}{3}A\ ((1,0),(1,6),(1,-6))\ \cdots$$

$$\vdots$$

$$3A\ ((1,0),(1,\frac{s+2}{3}),(1,-\frac{s+2}{3}))\ \cdots$$

$$2A\ ((1,0),(1,\frac{s+5}{3}),(1,-\frac{s+5}{3}))\ \cdots$$

$$1A\ ((1,0),(1,\frac{s+8}{3}),(1,-\frac{s+8}{3}))\ \cdots$$

$$1B\ ((1,0),(1,\frac{s+11}{3}),(1,-\frac{s+11}{3}))\ \cdots$$

$$2B\ ((1,0),(1,\frac{s+14}{3}),(1,-\frac{s+14}{3}))\ \cdots$$

$$3B\ ((1,0),(1,\frac{s+17}{3}),(1,-\frac{s+17}{3}))\ \cdots$$

$$\vdots$$

$$3C\ ((1,0),(1,s-2),(1,-s+2))\ \cdots$$

$$2C\ ((1,0),(1,s-1),(1,-s+1))\ \cdots$$

$$1C\ ((1,0),(1,s),(1,-s))\ \cdots$$

$\frac{s-7}{3}$ (rows $\frac{s-7}{3}A$ to $2A$)

2:1 ratio

$\frac{2s-14}{3}$ (rows $2B$ to $2C$)

mod $(-,2s+1)$

Figure 20

Group the rows according to the numbers on the left (there are $\frac{s-4}{3}$ "groups" of rows) where the group of rows KA, KB, and KC can be put into sets of triangles. We can put them into triangles since the second parts of the second element of the blocks of row KB are greater than the corresponding parts in the second element of blocks in row KA by $2K-1$. But in row KC the second parts of the third element of each block are greater than the second parts of the second element of each block by $2K-1$. Thus since in each row we have the blocks $mod(-,2s+1)$, each difference of $2K-1$ (between KA and KB) corresponding to each of the $2s+1$ blocks (in rows KA and KB) would appear once in KC.

Example: $K = 1$

$s = 10$

$A = ((1,0),(1,6),(1,6))$ · · ·

$B = ((1,0),(1,7),(1,-7))$ · · ·

⋮

$C = ((1,0),(1,10),(1,-10))$, · · · $((1,17),(1,6),(1,7))$ ...

Figure 21

Another Example: $K = 3$

$s = 13$

$3A = ((1,0),(1,5),(1,-5))$ · · ·

$3B = ((1,0),(1,10),(1,-10))$ · · ·

⋮

$3C = ((1,0),(1,11),(1,-11))$,... $((1,21),(1,5),(1,10))$, · · ·

Figure 22

Thus, in this way (see Figures 21 and 22), all the rows can be triangulated.

We now have remaining:

(1) the 3 remaining blocks in Figure 19

(2) blocks of an $STS(v)$ on the elements $(0,i)$

(3) blocks of an $STS(7)$ on the elements ∞_i; $i = 0,...,6$.

Now since $v = 6t-3$, from Lemma 2.2.6, and the fact that Lemmas 3.2, 3.3, and 3.5 hold for $t \geq 0$ (Theorem 4.1), we know that $v \in N_i$ if b, the number of blocks congruent to 1 or 2 $((mod\, 3)$, respectively and $v \in N_0 \cap N_3$ otherwise. (If $v \in N_0 \cap N_3$ we examine the case $v \in N_3$.) So assume one of the remaining blocks is: $((0,6t-6),(0,6t-5),(0,6t-4))$. Note that we can always get this result by permuting the symbols. We note that this block forms a triangle with $(\infty_1,(0,6t-6),(1,6t-5))$ and $(\infty_1,(0,6t-5),(1,6t-4))$. Therefore what we have remaining is:

$$i-1 \quad \text{(i.e. 0,1, or 2)} \quad \begin{cases} ((0,j),(0,k),(0,\ell)) \\ ((0,m),(0,n),(0,p)) \end{cases}$$

$$(\infty_2,(0,6t-4),(1,6t-5)$$

$$(\infty_1,\infty_3,\infty_4)$$

Now we know that the top $i-1$ blocks are disjoint from $((0,6t-6),(0,6t-5),(0,6t-4))$ and that we can always get $(\infty_1,\infty_3,\infty_4)$ as the remaining block in a triangulation of an *STS*(7) by permuting the symbols. Thus, we observe that all of the blocks above are disjoint. So the result we have is that $12t+1 \in N_i$, $i = 2,3$, or 4 for all $t \geq 2$, and the *STS*$(12t+1)$ contains a sub-decomposition on a sub-*STS*(7) (which is, in our construction, the *STS*(7) on ∞_i, $i = 0,\ldots,6$).

Lemma 5.4. *$18t+15 \in M_2$ for $t = 0$ and $t \geq 2$, and there exists such an STS$(18t+15)$ which contains a parallel class of blocks.*

Proof. Now we know $GD(3(6t+4);6t+4;3) \in C_1$ from Lemma 2.2.5. Now assume that $t \geq 2$. If we take the transversal to be the diagonal elements of a square of order $2t+1$ (compare with Lemma 2.2.5) with which we use to replace the squares of order 3 (of a square of order $6t+4$) by the upper left parts of some squares of order 4 (see Figure 8), we have the configuration shown in Figure 23.

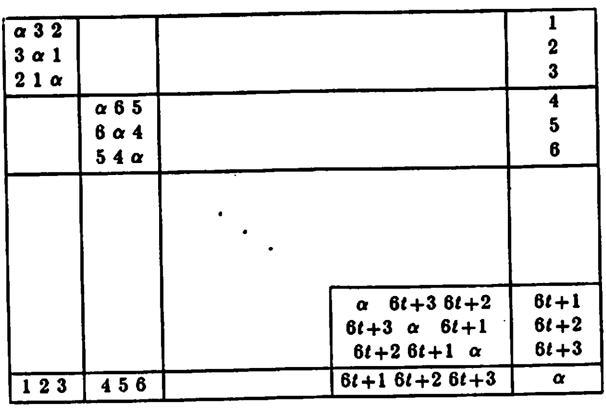

Figure 23

Observe that $GD(3(6t+4);6t+4;3)-GD(12;3;4) \in C_0$, where the $GD(12;3;4)$ is associated with the square of order 4 on the bottom right corner (remembering that $t \geq 2$).

Now if we use a tranversal (off diagonal) of the square of order $2t+1$ we can find 0,1,2, or 3 "holes" (3×3) that do not involve the bottom four rows or the four right-most columns. This is true because we have assumed that $t \geq 2$ (thus $6t+4 \geq 16$), and the square of order 4 on the bottom right only eliminates two possible holes; but there are 5 holes (since $(16-1)/3 = 5$). Note that we get an incomplete GD even if we remove an *off-diagonal* 3×3 cell, since we can permute the rows and columns of the square of order $6t + 4$.

Now we know from Lemma 3.7 that $12t+7 \in N_i$, $i = 1,2$, or 3, and contains a sub-decomposition on a sub-$STS(7)$, and from Lemma 5.3 that $12t+1 \in N_i$, $i = 2,3$, or 4, and contains a sub-decomposition on a sub-$STS(7)$. Therefore, for t even, $6t+7 \in N_i$, $i = 1,2$, or 3; for t odd, $6t + 7 \in N_i$, $i = 2,3$, or 4; in either case it contains a sub-decomposition on a sub-$STS(7)$. So we now take an incomplete $GD(3(6t+4);6t+4;3)$ with $i-1$ holes *off the diagonal cells*, and add three points to each group. Let those three points form a remaining block in the decomposition of an $STS(6t+7)$, $t \geq 2$. Thus we would have the incomplete GD in Figure 24.

		$i-1$ disjoint blocks	one remaining block
group 1	$x_{6t+1}\ x_{6t+2}\ x_{6t+3}\ x_{6t+4}$	$x_1\ x_2 \cdots x_{6t}$	$x_a\ x_b\ x_c$
group 2	$y_{6t+1}\ y_{6t+2}\ y_{6t+3}\ y_{6t+4}$	$y_1\ y_2 \cdots y_{6t}$	$x_a\ x_b\ x_c$
group 3	$z_{6t+1}\ z_{6t+2}\ z_{6t+3}\ z_{6t+4}$	$z_1\ z_2 \cdots z_{6t}$	$x_a\ x_b\ x_c$
	$6t+7$		

Figure 24

We fill in each of the $i-1$ holes with an $STS(9) \in N_0$. For each group we use a decomposition of $STS(6t+7)$ with a sub-decomposition on a sub-$STS(7)$ such that the four points corresponding to the distinguished square (the square of order 4 on the bottom right of Figure 23) as well as the points x_a, x_b, and x_c all belong to the sub-$STS(7)$. Now since:

(1) $(6t+4) \in C_1 \quad t \geq 0$

(2) $((6t+4);3) \in C_1 \quad t \geq 1$

(3) $((6t+4);3;3) \in C_1 \quad t \geq 1$

(4) $((6t+4);3;3;3) \in C_1 \quad t \geq 1$

by Lemma 2.2.5, and since each of (1), (2), (3), and (4), minus $GD(12;4;3)$ is in C_0 (for $t \geq 2$), we are left with fifteen points : x_{6t+1}, x_{6t+2}, x_{6t+3}, x_{6t+4}, y_{6t+1}, y_{6t+2}, y_{6t+3}, y_{6t+4}, z_{6t+1}, z_{6t+2}, z_{6t+3}, z_{6t+4}, x_a, x_b, and x_c. These can be triangulated by triangulating the $GD(12;4;3)$ (as in to Figure 8). Note that the block $(x_{6t+4}, y_{6t+4}, z_{6t+4})$ remains from triangulating the $GD(12;4;3)$. We include a triangulated $STS(7)$ on the points x_{6t+1}, x_{6t+2}, x_{6t+3}, x_{6t+4}, x_a, x_b, and x_c, with remaining block (x_a, x_b, x_c); plus a triangulated $STS(7)$ on the points y_{6t+1}, y_{6t+2}, y_{6t+3}, y_{6t+4}, x_a, x_b, and x_c, with remaining block (x_a,x_b,x_c); plus a triangulated *STS*(7) on the points z_{6t+1}, z_{6t+2}, z_{6t+3}, z_{6t+4}, x_a, x_b, and x_c, which contains the block (x_a,x_b,x_c) and has the block (z_{6t+4},x_a,z_{6t+1}) remaining. So if we remove the two blocks of (x_a,x_b,x_c) in the first two *STS*(7)'s (since this block appears in the third *STS*(7)), we are left with a triangulated *STS*(15) with remaining blocks $(x_{6t+4},y_{6t+4},z_{6t+4})$ and (z_{6t+4},x_a,z_{6t+1}) which intersect (Figure 25). Therefore $18t + 15 \in M_2$ for $t = 0$ and $t \geq 2$.

x_b	x_{6t+1}	x_{6t+2}	x_{6t+1}	x_c	x_{6t+3}	
x_c	x_{6t+2}	x_{6t+4}	x_{6t+2}	x_{6t+3}	x_a	$x_a x_b x_c$
x_{6t+3}	x_{6t+4}	x_b	x_{6t+4}	x_a	x_{6t+1}	(remove)
x_b	y_{6t+1}	y_{6t+2}	y_{6t+1}	x_c	y_{6t+3}	
x_c	y_{6t+2}	y_{6t+4}	y_{6t+2}	x_{6t+3}	x_a	$x_a x_b x_c$
y_{6t+3}	y_{6t+4}	x_b	y_{6t+4}	x_a	y_{6t+1}	(remove)
x_a	x_b	x_c	x_b	z_{6t+1}	z_{6t+2}	
z_{6t+1}	x_c	z_{6t+3}	x_c	z_{6t+2}	z_{6t+4}	$z_{6t+4}\ x_a\ z_{6t+1}$
z_{6t+2}	z_{6t+3}	x_a	z_{6t+3}	z_{6t+4}	x_b	

Figure 25

Remarks. The *STS*$(18t+15)$ in Lemma 5.4 ($t \geq 2$) contains a parallel class of blocks, that is, $6t+3$ disjoint blocks each of which is associated with diagonal elements of off-diagonal 3×3 cells (that is, the product of the square of order 3 with an off-diagonal transversal of the square of order $2t+1$ (which exists for $t \geq 1$) or from some *STS*(9), plus the block $(x_{6t+4},y_{6t+4},z_{6t+4})$ (as explained in Lemma 5.4, this block is in the specified *GD*(12;4;3)), plus the additional block (x_a,x_b,x_c). The *STS*(15) contains a parallel class of blocks: $(x_{6t+1},y_{6t+2},z_{6t+3})$, $(x_{6t+2},y_{6t+3},z_{6t+1})$, $(x_{6t+3},y_{6t+1},z_{6t+2})$, $(x_{6t+4},y_{6t+4},z_{6t+4})$, and (x_a,x_b,x_c).

Note that if we wish to construct $18t+15 \in M_2$ (for $t \geq 2$), we need to apply Lemmas 3.1 through 3.8 recursively (Theorem 4.1) until we obtain, for t odd, $6t' - 3 \in N_i$ where $i = 1$ or 2 if b is congruent to 1 or $2 (mod\, 3)$ respectively, and $6t'-3 \in N_0 \cap N_3$ otherwise (for $t = 2t' - 1$) or we obtain a construction for an *STS*$(6t+7)$, for t even, such that it contains a sub-decomposition on a sub-*STS*(7) (Lemma 3.7).

Lemma 5.5. $18t+15 \in M_2$ *for* $t = 1$, *that is,* $33 \in M_2$ *such that the STS*(33) *has a parallel class of blocks.*

Proof. We take a *GD*(30;10;3) which we know is in C_1 (Lemma 2.2.5). If we take the transversal to be the diagonal elements of a square of order 9 (a similar procedure to that used in Lemma 5.4) which we use to replace the squares of order 3 (of a square of order 10) by the upper left parts of some squares of order 4 (see Figure 8), we have the configuration shown in Figure 26.

α	2	3	7	9	8	4	6	5	1
3	α	1	9	8	7	6	5	4	2
2	1	α	8	7	9	5	4	6	3
7	9	8	α	6	5	1	3	2	4
9	8	7	6	α	4	3	2	1	5
8	7	9	5	4	α	2	1	3	6
4	6	5	1	3	2	α	9	8	7
6	5	4	3	2	1	9	α	7	8
5	4	6	2	1	3	8	7	α	9
1	2	3	4	5	6	7	8	9	α

Figure 26

The diagonal cells can be triangulated with the bottom row and right-most column. Therefore, if we remove the $GD(12;4;3)$ on the bottom right, we have an incomplete GD which is in C_0. Now let us look at the three groups and add three new points to each group (Figure 27).

$$\begin{array}{l} \text{group 1} \\ \text{group 2} \\ \text{group 3} \end{array} \left[\begin{array}{cccc} x_7 & x_8 & x_9 & x_{10} \\ y_7 & y_8 & y_9 & y_{10} \\ z_7 & z_8 & z_9 & z_{10} \end{array}\right] \begin{array}{l} x_1\ x_2 \cdots x_6\ x_{11}\ x_{12}\ x_{13} \\ y_1\ y_2 \cdots y_6\ x_{11}\ x_{12}\ x_{13} \\ z_1\ z_2 \cdots z_6\ x_{11}\ x_{12}\ x_{13} \end{array}$$

Figure 27.

Now we know $13 \in M_2$ (Figure 17); so we group the elements by row, as shown above. We let one of the remaining blocks in each row be (x_{11},x_{12},x_{13}), and let the other be (x_8,x_9,x_{11}), (y_8,y_9,x_{11}), and (z_8,z_9,x_{11}), respectively. (We can guarantee this by permutation of the symbols.) Thus the blocks remaining in $STS(33)$ that are not yet in triangles are the four blocks just mentioned and the $GD(12;4;3)$. These blocks can be triangulated as shown in Figure 28.

x_8	x_9	x_{11}	z_8	z_9	x_{11}	x_7	y_7	z_{10}
x_8	y_7	z_9	x_7	y_8	z_9	x_7	y_{10}	z_7
x_9	y_7	z_8	x_7	y_9	z_8	x_{10}	y_7	z_7
x_8	y_9	z_7	x_8	y_8	z_{10}	x_9	y_{10}	z_9
x_9	y_8	z_7	x_8	y_{10}	z_8	x_{10}	y_9	z_9
x_9	y_9	z_{10}	x_{10}	y_8	z_8	x_{10}	y_{10}	z_{10}

remaining blocks: (x_{11}, x_{12}, x_{13}) and (y_8, y_9, x_{11})

Figure 28

Therefore, since the two remaining blocks (x_{11}, x_{12}, x_{13}) and (y_8, y_9, x_{11}) intersect, we have that $33 \in M_2$.

Remark. The STS(33) in Lemma 5.5 contains a parallel class of blocks, that is, the 10 disjoint blocks each of which comes from an off-diagonal transversal element of the square of order 10 and the block (x_{10}, y_{10}, z_{10}), and the additional block (x_{11}, x_{12}, x_{13}).

6. Summary.

The foregoing results are brought together below.

Theorem 6.1. *For every positive integer v, $v \equiv 1$ or $3 (mod\, 6)$, where an $STS(v)$ has $3q+2$ blocks, there is an $STS(v)$ with its blocks decomposable into triangles such that no two triangles have common blocks and there are two remaining blocks which intersect at a point.*

Proof. This follows directly from the results of Lemmas 5.1, 5.2, 5.4, and 5.5. Therefore we have proved the sufficiency of the necessary condition stated in 2.2.1.4.

7. Lemmas needed to prove Theorem 7.1.

Lemma 7.1 *If $v = 18t + 15$ $(t \geq 0)$ and there exists a $TSTS(v)$ of type I and type II whose underlying $STS(v)$'s are isomorphic and have a parallel class of blocks, then for $w = 36t + 31$, there exists a $TSTS(w)$ of type I and type II whose underlying $STS(w)$'s are isomorphic.*

Proof. This result follows directly from Lemmas 13 and 16.

Lemma 7.2. *For all $v = 36t + 13$, there exists a TSTS(v) of type I and type II whose underlying STS(v)'s are isomorphic.*

Proof. Since the *STS*(13)'s used in Lemmas 3.8 and 5.2 (Figures 2 and 17 respectively) are isomorphic (in fact identical), the result follows.

Lemma 7.3. *If $v = 18t + 15$ ($t = 0$ and $t \geq 2$), then there exists a TSTS(v) of type I and type II whose underlying STS(v)'s are isomorphic, and such that the STS(v)'s have a parallel class of blocks.*

Proof. We know from Lemmas 3.7 and 5.3 respectively, that for even nonnegative t, $6t+7 \in N_i$, $i = 1,2$, or 3, as appropriate; for odd ($t \geq 3$) $6t + 7 \in N_i$, where $i = 2,3$, or 4, as appropriate. In either case, there exists a corresponding *TSTS*$(6t + 7)$ which contains a sub-decomposition on a sub-*STS*(7). So we now use the square of order $6t + 4$ used in Lemma 5.4 (Figure 23) to prove a modified version of Lemma 3.5. Lemma 3.5 taking an incomplete $GD(3(6t+4);6t+4;3)$ with $i - 1$ holes *off the diagonal cells* (these exist for $t \geq 2$, as discussed in Lemma 5.5; for $t = 0$, use 0 holes), and add three points to each group. (Note that we added three points to each group originally.) We let those three points form a remaining block in the decomposition of an *STS*$(6t+7)$. We fill in each of the $i - 1$ holes with an *STS*$(9) \in N_0$. As in Lemma 5.4, for each group we use a decomposition on a sub-*STS*(7) such that the four points (for each group) corresponding to the distinguished square (the square of order 4 on the bottom right of Figure 23) as well as the points x_a,x_b, and x_c all belong to the sub-*STS*(7).

We now need only to show that the *STS*(15) corresponding to the points of the square of order 4 on the bottom right of Figure 23 and the points x_a,x_b, and x_c in this case (the new Lemma 3.5) and the case of Lemma 5.4 are isomorphic. The square of order 4 and the points x_a,x_b, and x_c are triangulated like the $GD(12;4;3)$ (Figure 8) and with three *STS*(7)'s as shown in Figure 29.

x_b	x_{6t+1}	x_{6t+2}	x_{6t+1}	x_c	x_{6t+3}	
x_c	x_{6t+2}	x_{6t+4}	x_{6t+2}	x_{6t+3}	x_a	$x_a x_b x_c$
x_{6t+3}	x_{6t+4}	x_b	x_{6t+4}	x_a	x_{6t+1}	(remove)
x_b	y_{6t+1}	y_{6t+2}	y_{6t+1}	x_c	y_{6t+3}	
x_c	y_{6t+2}	y_{6t+4}	y_{6t+2}	y_{6t+3}	x_a	$x_a x_b x_c$
y_{6t+3}	y_{6t+4}	x_b	y_{6t+4}	x_a	y_{6t+1}	(remove)
x_b	z_{6t+1}	z_{6t+2}	z_{6t+1}	x_c	z_{6t+3}	
x_c	z_{6t+2}	z_{6t+4}	z_{6t+2}	z_{6t+3}	x_a	$x_a x_b x_c$
z_{6t+3}	z_{6t+4}	x_b	z_{6t+4}	x_a	z_{6t+1}	

Figure 29

We remove two blocks of (x_a,x_b,x_c) and note that we have a triangulation with two remaining disjoint blocks: $(x_{6t+4},y_{6t+4},z_{6t+4})$ and (x_a,x_b,x_c). Since we are triangulating the same $GD(12;4;3)$ as in Lemma 5.4 and since the blocks in Figure 29 are identical with those of Figure 25, we have the following result.

From Lemma 5.4 and the new Lemma 3.5, there exists a $TSTS(18t+15)$ ($t = 0$ and $t \geq 2$) of type I and type II whose underlying $STS(v)$'s are isomorphic. Also, the $STS(18t+15)$'s have a parallel class of blocks as discussed in the remark in Lemma 5.4.

Lemma 7.4. *There exists a $TSTS(33)$ of type I and type II whose underlying $STS(33)$'s are isomorphic and have a parallel class of blocks.*

Proof. We start with the square of order 10 in Figure 26 (Lemma 5.5), and we remove a $GD(9;3;3)$ by removing the 3×3 cell in the centre part of the top row, as shown in Figure 30.

α	2	3				4	6	5	1
3	α	1				6	5	4	2
2	1	α				5	4	6	3
7	9	8	α	6	5	1	3	2	4
9	8	7	6	α	4	3	2	1	5
8	7	9	5	4	α	2	1	3	6
4	6	5	1	3	2	α	9	8	7
6	5	4	3	2	1	9	α	7	8
5	4	6	2	1	3	8	7	α	9
1	2	3	4	5	6	7	8	9	α

Figure 30

Our stategy is to repeat, in a different way, the case in Lemma 3.5 to show that $33 \in N_2$. Now we know that $13 \in N_2$ and $13 \in M_2$, where the $STS(13)$'s are isomorphic (Figures 2 and 17). So we use the incomplete $GD(30;10;3)$ (Figure 30) and take an $STS(13)$ on $\{x_1,\ldots,x_{13}\}$ such that the blocks (x_{11},x_{12},x_{13}) and (x_1,x_2,x_3) remain. We also take an $STS(13)$ on $\{_1,\ldots,y_{10},x_a,x_b,x_c\}$ such that (x_{11},x_{12},x_{13}) and (y_4,y_5,y_6) remain. Finally we take an $STS(13)$ on $\{z_1,\ldots,z_{10},x_{11},x_{12},x_{13}\}$ such that (x_{11},x_{12},x_{13}) and

(z_7,z_8,z_9) remain. Note that these three *STS*(13)'s are isomorphic to the three *STS*(13)'s used in Lemma 5.5; in fact we can easily choose our *STS*(13)'s in Lemma 5.5 to be identical with these three *STS*(13)'s. So we now fill in the hole in the *GD*(30;10;3) with an *STS*(9) $\in N_0$ on the elements $x_1,x_2,x_3,y_4,y_5,y_6,z_7,z_8,z_9$. After removing two blocks of (x_{11},x_{12},x_{13}), we have remaining the blocks (x_{11},x_{12},x_{13}) and (x_{10},y_{10},z_{10}) which are disjoint.

Since we have filled in the hole with an *STS*(9) with the same symbols that appeared in the centre cell in the top row of Figure 26, and since the three *STS*(13)'s we just used are identical with those used in Lemma 5.5, therefore the *STS*(33) we constructed in Lemma 5.5 is identical with this new *STS*(33). Note that, if we only use the fact that the two sets of *STS*(13)'s are *isomorphic,* we would have to permute the symbols $(x_1,\ldots,x_{10})$, and/or the symbols $(y_1,\ldots,y_{10})$, and/or the symbols $(z_1,\ldots,z_{10})$, and then we would get the two *GD*(30;10;3)'s to be isomorphic; so then the *STS*(33)'s would be isomorphic. Therefore there exists a *TSTS*(33) of type I and type II whose underlying *STS*(33)'s are isomorphic, and have a parallel class of blocks (as discussed in Lemma 5.5).

Theorem 7.1. *For every positive $v \equiv 1$ or 3 (mod 6) there exists a TSTS(v) of type I and type II whose underlying STS(v)'s are isomorphic.*

Proof. Let *STS*(v) have $3q + s$ blocks. If $s \equiv 0$ or 1, any triangulation is of both type I and type II. Therefore Theorem 7.1 holds as a result of Theorem 4.1. If $s = 2$, then the result follows directly from Lemmas 7.1, 7.2, 7.3, and 7.4.

References

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