Stirling Polynomials of the Second Kind

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1. INTRODUCTION

In this paper the (signless) Stirling numbers of the second kind are denoted $S(n,k)$; they are defined combinatorially as the number of partitions of the set $\{1,\ldots,n\}$ into k non-empty disjoint subsets. Hence $S(n,0) = \delta_{n0}$, where δ_{nj} denotes the Kronecker symbol. The r-Stirling numbers of the second kind represent a certain generalization of the regular Stirling numbers $S(n,k)$. These are denoted by $S_r(n,k)$ and defined combinatorially as the number of partitions of the set $\{1,\ldots,n\}$ into k non-empty disjoint subsets, such that the numbers, $1,\ldots,r$ are in distinct subsets (see, e.g., [1]). It is easy to see that $S_0(n,k) = S_1(n,k) = S(n,k)$.

Following L. Carlitz ([2]) we define the Stirling polynomials of the second kind in the following manner

$$R(n,k;x) = \sum_{m=0}^{n} \binom{n}{m} S(m,k)x^{n-m} \quad (x \in \mathbb{R}).$$

A. Broder ([1]) has shown

$$R(n,k;r) = S_r(n+r,k+r)$$
$$(r,k,n = 0,1,\ldots,; \quad k \le n).$$

The purpose of this paper is to study some properties of the polynomials $R(n,k;x)$. Integral representation formulas are established in Section 3. New recurrence formulas as well as some inequalities that hold for these polynomials are given in Sections 4 and 5, respectively.

Complete proofs of all the theorems presented below will be published elsewhere.

2. PRELIMINARIES

Let $t_0 < \ldots, < t_k$ $(k > 0)$ be given real numbers. Further, let f be a real-valued function defined on $[t_0, t_k]$. A k-th order divided difference of f at the points $t_0, \ldots, t_k$ may be defined recursively by

$$[t_i]f = f(t_i) \qquad (i = 0, 1, \ldots, k)$$

and

$$[t_0, \ldots, t_k]f = ([t_1, \ldots, t_k]f - [t_0, \ldots, t_{k-1}]f)/(t_k - t_0).$$

The number $[t_0, \ldots, t_k]f$ is independent of the order of the points $t_0, \ldots, t_k$.

The following material on B-splines has its origin in the paper [3] of Curry and Schoenberg.

For fixed t let $M(t;x) = k(x-t)_+^{k-1}$, defined to be $k(x-t)^{k-1}$ if $x \geq t$ and zero otherwise. The function

$$M_k(t) = [t_0, \ldots, t_k]M(t;\cdot)$$

(k-th divided difference of $M(t;x)$ with respect to x at $t_0, \ldots, t_k$) is commonly referred to as a B-spline of degree $k-1$ (order k) and has the following elementary properties:

(i) $M_k(t) > 0$ if $t \in (t_0, t_k)$ and $M_k(t) = 0$ otherwise.

(ii) In each interval $[t_i, t_{i+1}]$ $(i = 0, 1, \ldots, k-1)$ M_k coincides with an algebraic polynomial of degree $k-1$ or less.

(iii) $M_k \in C^{k-2}(\mathbb{R})$.

(iv) If f has a continuous k-th derivative in (t_0, t_k), then

$$[t_0, \ldots, t_k]f = \frac{1}{k!} \int_{t_0}^{t_k} M_k(t) f^{(k)}(t)dt.$$

For our further purposes we would like to mention that the function

$$C_r(t_0,\dots,t_k) \equiv C_r = \sum_{i_0+\dots+i_k=r} t_0^{i_0} \cdot \dots \cdot t_k^{i_k}$$

($i_0,\dots,i_k \in \{0,1,\dots,r\}$) is referred to as a complete symmetric function of order r in variables $t_0,\dots,t_k$ (see, e.g., [4]).

3. REPRESENTATION FORMULAS FOR R(n,k;x)

The following identity

$$(3.1) \qquad R(n,k;x) = [x,x+1,\dots,x+k]t^n \qquad (n,k = 0,1,\dots,;\quad k \le n;\quad x \in \mathbb{R})$$

is known (see [1] and [2]). Making use of (3.1) and some facts presented in Section 2 we can prove the following theorems.

Theorem 3.1. Let $0 \le k \le n$ and let $x \in \mathbb{R}$. Then

$$(3.2) \qquad R(n,k;x) = \begin{bmatrix} n \\ k \end{bmatrix} \int_x^{x+k} M_k(t)t^{n-k}dt,$$

where M_k denotes the B-spline with knots at $x,x+1,\dots,x+k$.□

Let $S^k = \{(\lambda_1,\dots,\lambda_k) \in \mathbb{R}^k\colon \lambda_i \ge 0 \text{ (all i)}, \sum_{i=1}^{k} \lambda_i \le 1\}$ denote a simplex in $\mathbb{R}^k$. The following theorem provides another integral representation for Stirling polynomials of the second kind. We have

Theorem 3.2. Let $\lambda_0 = 1 - \sum_{i=1}^{n} \lambda_i$. Then under the assumptions of Theorem 3.1 the following identity

$$(3.3) \qquad R(n,k;x) = n(n-1)\cdot\ldots\cdot(n-k+1) \cdot \int_{S^k} \dots \int [\lambda_0 x + \dots + \lambda_k(x+k)]^{n-k} d\lambda_1 \dots d\lambda_k$$

Theorem 3.3. If $0 \le k \le n$, then

$$R(n,k;x) = c_{n-k}(x, x+1, \dots, x+k) \quad (x \in \mathbb{R}). \tag{3.4}$$

4. THE RECURRENCE FORMULAS FOR STIRLING POLYNOMIALS OF THE SECOND KIND

The following recurrence formula

$$R(n,k;x) = (x+k)R(n-1,k;x) + R(n-1,k-1;x) \tag{4.1}$$
$$(0 < k \le n; \quad x \in \mathbb{R})$$

was given by L. Carlitz [2]. This formula has a remarkable property. Namely, if $x \ge 0$, then the algorithm based on (4.1) is numerically stable.

Below we give more recurrence formulas for Stirling polynomials of the second kind.

Theorem 4.1. If $0 \le k \le n$, then

$$R(n,k;x+1) = (k+1)R(n,k+1;x) + R(n,k;x). \tag{4.2}$$ □

Theorem 4.2. Let $1 \le k \le n$. Then the following formula

$$(k-1)k\,R(n,k;x) = n(k-1)R(n-1,k-1;x) + \sum_{j=0}^{n-k-1} \begin{bmatrix} n \\ j+k \end{bmatrix} [(x+k-1)(j+k)R(j+k-1,k-1;x) - (j+1)R(j+k,k-1;x)] \tag{4.3}$$

is valid. □

Theorem 4.3. Assume $x \ne 0, -1, \dots, -k$. Then for any k and n $(0 \le k \le n)$ we have

$$R(n,k;x) = \sum_{j=0}^{k} \omega_j R(n+1,k-j;x), \tag{4.4}$$

where $\omega_j = (-1)^j / \prod_{i=0}^{j} (x+k-i) \quad (j = 0, 1, \dots, k).$ □

5. INEQUALITIES INVOLVING $R(n,k;x)$

All results presented below hold true provided $x \geq 0$. We are now prepared to state

Theorem 5.1. Let $0 \leq k \leq n$. Then

$$(5.1) \qquad \frac{(x+k)^n - k(x+k-1)^n}{k!} \leq R(n,k;x) \leq \frac{(x+k)^n}{k!}$$

with equalities if and only if $k = 0$. □

Theorem 5.2. For any $k, m, n \geq 0$ and any α, where

$$\alpha \in \begin{cases} (-\infty, \infty) & m \text{ even} \\ [x+k, \infty) & m \text{ odd} \end{cases}$$

the following inequality

$$(5.2) \qquad \sum_{\ell=0}^{m} (-1)^{m-\ell} \begin{bmatrix} m \\ \ell \end{bmatrix} \begin{bmatrix} p-\ell \\ k \end{bmatrix}^{-1} \alpha^{\ell} R(p-\ell,k;x) \geq 0 \qquad (p = n+m+k)$$

holds. If $k > 0$ then strict inequality holds. □

Theorem 5.3. Let $0 \leq k < n$. Then

$$(5.3) \qquad R(n+1,k;x)R(n-1,k;x) \leq R(n,k;x)^2 \leq \frac{n}{n-k}\,\frac{n+1-k}{n+1}\,R(n+1,k;x)R(n-1,k;x).$$

If $k \neq 0$ then strict inequalities hold. □

The first inequality tells us the sequence $(R(\cdot,k;x))$ is logarithmically concave.

Our next result reads as follows.

Theorem 5.4. For $k \geq 0$

$$(5.4) \qquad R(k+1,k;x) \geq R(k+2,k;x)^{\frac{1}{2}} \geq R(k+3,k;x)^{\frac{1}{3}} \geq \cdots .$$

Moreover,

$$\text{(5.5)} \qquad \begin{bmatrix} k+1 \\ k \end{bmatrix}^{-1} R(k+1,k;x) \le \left[\begin{bmatrix} k+2 \\ k \end{bmatrix}^{-1} R(k+2,k;x) \right]^{\frac{1}{2}} \le \dots .$$

Above inequalities become equalities only if $k = 0$. □

Theorem 5.5. If $1 \le k \le n$, then

$$\text{(5.6)} \qquad k\, R(n,k;x) \ge n\, R(n-1,k-1;x).$$

If $k < n$ then strict inequality holds. □

Theorem 5.6. For any non-negative integers n and p with $n \le p$ the following inequality

$$\text{(5.7)} \qquad R(n+k,k;x) \le \frac{(n+1)\cdot \dots \cdot (n+k)}{(p+1)\cdot \dots \cdot (p+k)} R(p+k,k;x)$$

holds true. Equality holds in (5.6) if and only if $n = k$. □

Theorem 5.7. Let $1 < k < n$. Then

$$\text{(5.7)} \qquad x\, R(n-1,k;x) < \frac{n-k}{n} R(n,k;x)$$
$$< (x+k) R(n-1,k;x) < R(n,k;x). \qquad □$$

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