# SOME ACHIEVABLE DEFECT GRAPHS FOR PAIR-PACKINGS ON SEVENTEEN POINTS

R.G. Stanton
Department of Computer Science
University of Manitoba
Winnipeg, Canada R3T 2N2

Anne Penfold Street
Department of Mathematics
University of Queensland
St. Lucia, Queensland, Australia 4067

## 1. Introduction.

If we are given a set V of v elements, and if we select a family of k-sets so that no two k-sets have t elements in common, then these k sets form a (t,k,v) packing design. The largest possible number of elements in such a design is called the packing number, D(t,k,v).

It is known that D(2,4,17) = 20 (see, for example, [1]); consequently, of the 136 possible pairs on 17 elements, only 20(6) = 120 actually occur in the design. A non-adjacency graph, or defect graph, G, can be associated with the design; the seventeen vertices of G represent the elements of V, and the sixteen edges of G represent the pairs that are missing from the design.

In this paper, we discuss some of the possible defect graphs G and indicate how they are related to the exact packing number $g^{(4)}(17)$, as defined in [2]; cf. also Section 5.

## 2. Possible Configurations for G.

Each occurrence of an element in a quadruple of the design accounts for three pairs containing that element. Hence, each element occurs in 0,3,6,9,12, or 15 pairs; consequently, in the complementary defect graph, each element has valence 16,13,10, 7, or 1. Let $a_i$ be the number of vertices of valence i in G. Counting vertices gives us the equation

$$a_1 + a_4 + a_7 + a_{10} + a_{13} + a_{16} = 17.$$

Counting valencies gives the equation

$$a_1 + 4a_4 + 7a_7 + 10a_{10} + 13a_{13} + 16a_{16} = 32.$$

We thus obtain the equation

$$a_4 + 2a_7 + 3a_{10} + 4a_{13} + 5a_{16} = 5,$$

and deduce seven possible solutions for the $a_i$s.

| | $a_{16}$ | $a_{13}$ | $a_{10}$ | $a_7$ | $a_4$ | $a_1$ |
|---|---|---|---|---|---|---|
| I | 1 | 0 | 0 | 0 | 0 | 16 |
| II | | 1 | 0 | 0 | 1 | 15 |
| III | | | 1 | 0 | 2 | 14 |
| IV | | | 1 | 1 | 0 | 15 |
| V | | | | 2 | 1 | 14 |
| VI | | | | 1 | 3 | 13 |
| VII | | | | | 5 | 12 |

We consider each of these seven cases in turn, to determine the ways in which they can occur. We use the following copy of affine plane of order 4 as the starting-point for constructing the designs, and denote the elements of each design by A,...,P,∞.

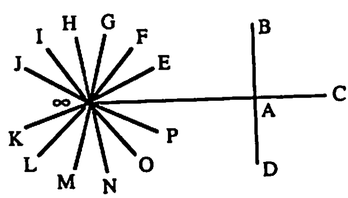
| | | | | |
|---|---|---|---|---|
| ABCD | AEIM | AHJO | AGLN | AFKP |
| EFGH | BFJN | BGIP | BHKM | BELO |
| IJKL | CGKO | CFLM | CEJP | CHIN |
| MNOP | DHLP | DEKN | DFIO | DGJM |

In all cases, we use G to denote the complement (that is, the defect graph) of the design with 20 quadruples.

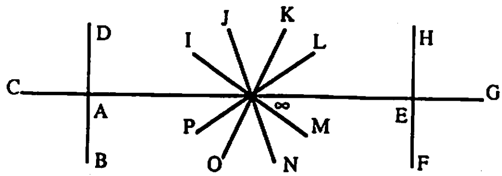
## 3. Determination of Defect Graphs.

**Case I.** G is the star $K_{1,16}$; the blocks of the design are those shown in the previous section. The centre of the star is the point ∞. This is the only defect graph G that we shall not illustrate.

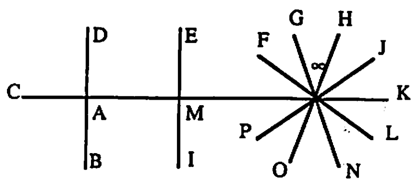
**Case II.** Replace the block ABCD by the block ∞BCD; the other blocks are unaltered, and G is as shown.



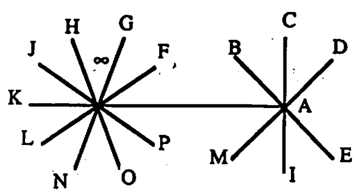
**Case III.** The two possible defect graphs are as shown. In each case (and in all succeeding cases), any blocks that are not mentioned remain as they were in the original plane. In Case III(a), we replace ABCD and EFGH by ∞BCD and ∞FGH.



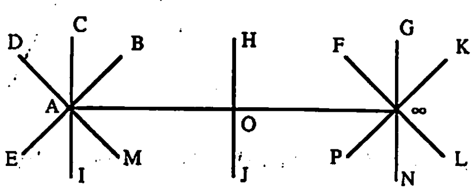

In Case III(b), we replace ABCD and AEIM by ∞BCD and AEI∞.



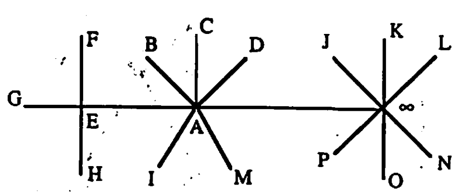

**Case IV.** We replace ABCD and AEIM by ∞BCD and ∞EIM.




**Case V.** Again, two graphs are possible. In V(a), we replace ABCD, AEIM, and AHJO by ∞BCD, ∞EIM, and AHJ∞.




In V(b), we replace ABCD, EFGH, and AEIM by ∞BCD, ∞FGH, and ∞EIM.



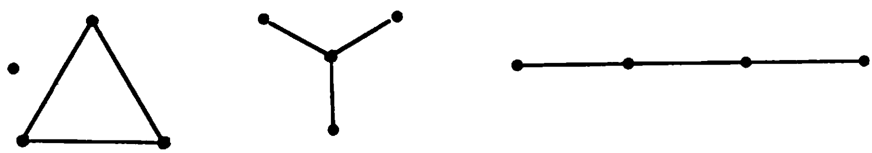
Case VI. Let s be the vertex of valence 7, and let $f_i$ (i = 1,2,3) be the vertices of valence 4 in G. The remaining 13 vertices all have valence 1. Let $b_7$ denote the number of vertices of valence 4 that are adjacent to s, and let $b_{4i}$ denote the number of vertices of valence either 7 or 4 that are adjacent to $f_i$ (i = 1,2,3). Then the total number of vertices in G is

$$17 = (7 - b_7) + (4 - b_{41}) + (4 - b_{42}) + (4 - b_{43}) + 4,$$

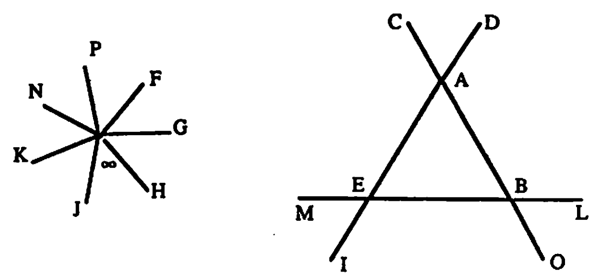
so that

$$b_{41} + b_{42} + b_{43} + b_7 = 6,$$

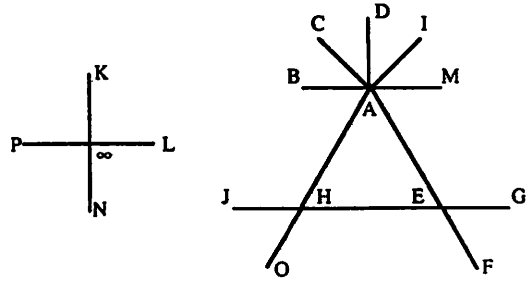
where each of the $b_i$ lies between 0 and 3 inclusive. Hence the possible subgraphs of G induced by the vertices of valencies 7 and 4 have valence sequences 2,2,2,0; or 3,1,1,1; or 2,2,1,1. These give rise to the subgraphs $G_A$, $G_B$, and $G_C$, respectively, as shown.



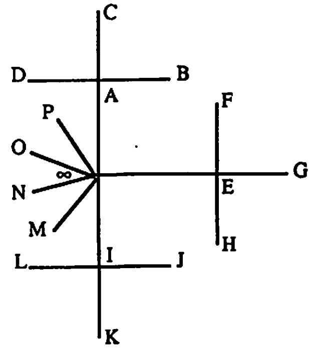
Each of these subgraphs leads to two possible structures for G. In Case VI(a), we replace ABCD, BELO, and AEIM by ∞BCD, ∞ELO, and A∞IM.



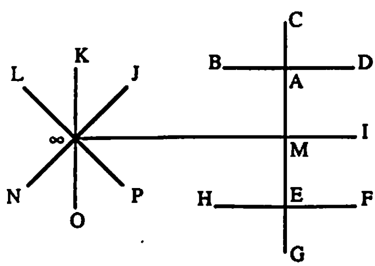

In Case VI(b), we replace ABCD, AEIM, AHJO, EFGH, by ∞BCD, ∞EIM, A∞JO, and ∞FGH.



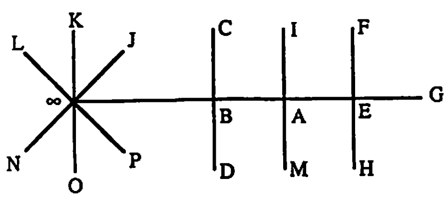
In Case VI(c), we replace ABCD, EFGH, and IJKL, by ∞BCD, ∞FGH, and ∞JKL.



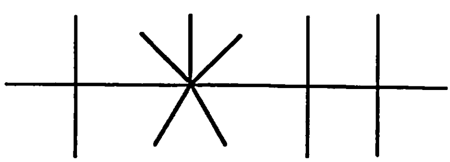

In Case VI(d), we replace ABCD, EFGH, AEIM, by ∞BCD, ∞FGH, AEI∞.




In Case VI(e), we replace ABCD, EFGH, AEIM, by A∞CD, ∞FGH, ∞EIM.



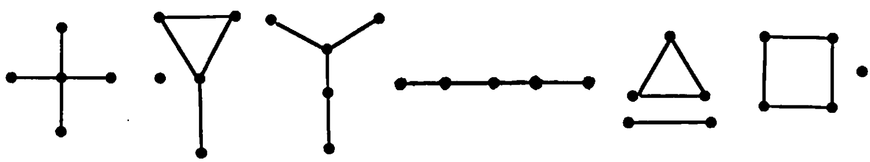

We have not achieved the graph in VI(f).



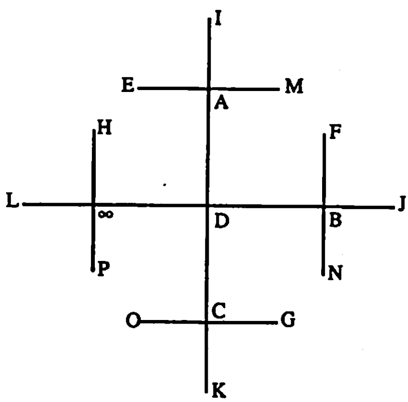
Case VII. Let $f_i$ ($i = 1,2,3,4,5$) be the vertices of valence 4 in G; the remaining 12 vertices have valence 1. Let $b_i$ denote the number of vertices of valence 4 adjacent to $f_i$. Then the total number of vertices in G is

$$17 = (4 - b_1) + \dots + (4 - b_5) + 5.$$

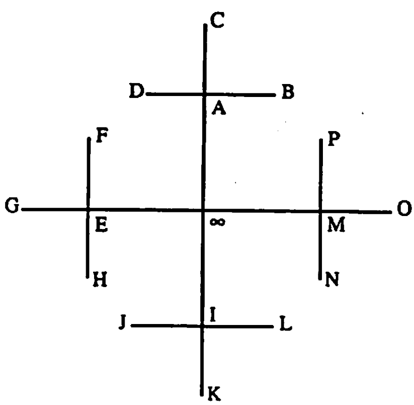
Hence, the $b_i$ sum to 8, and each $b_i$ is between 0 and 4, inclusive. Hence, the possible subgraphs of G induced by the vertices of valence 4 are the six graphs shown in the following diagrams; the valence sequences are 4,1,1,1,1; or 3,2,2,1,0; or 3,2,1,1,1; or 2,2,2,1,1 (two graphs); or 2,2,2,2,0. Each of these subgraphs leads to one possible structure for G.



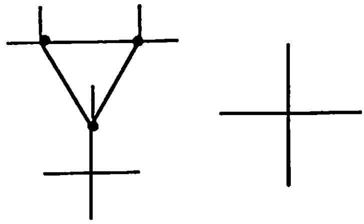
Case VII(a). Replace ABCD, AEIM, BFJN, CGKO, by ABC∞, ∞EIM, ∞FJN, ∞GKO.



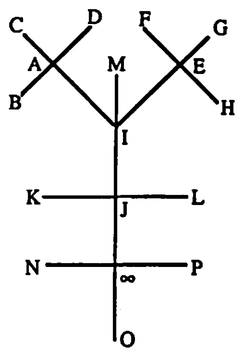

Or, replace ABCD, EFGH, IJKL, MNOP, by ∞BCD, ∞FGH, ∞JKL, ∞NOP.



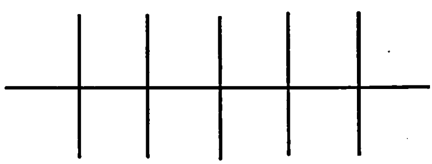
Case VII(b). We have not achieved this graph.

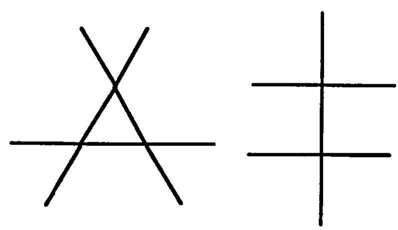


Case VII(c). Replace ABCD, EFGH, IJKL, AEIM, by ∞BCD, ∞FGH, I∞KL, AE∞M.

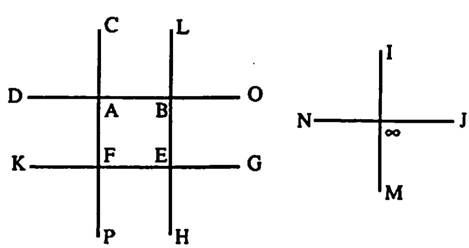



Cases VII(d) and VII(e) have not been achieved.





Case VII(f). Replace ABCD, EFGH, AFKP, BELO, by ∞BCD, ∞FGH, A∞KP, ∞ELO.



We are thus led to two conclusions. First, there are 19 possible defect graphs; of these, 15 can be obtained from the star graph formed as the complement of the affine plane on 16 points. Secondly, we note that all the defect graphs are characterized by the fact that they contain at most one triangle. This point has significance in the next section.

## 5. Connection with the Exact Covering Number.

The exact covering number $g^{(4)}(17)$ is defined in [2]; it is the minimum number of sets covering all 136 pairs with no duplication and with no sets of size greater than 4. The numbers $g^{(4)}(v)$ were determined in [3] for all v except for 17, 18, 19. We note here that it is not possible to achieve the value $g^{(4)}(17)$ using 20 quadruples.

The proof of this is immediate. Let $g_i$ (i = 2,3,4) be the number of i-sets in an exact covering. Then

$$g_2 + g_3 + 20 = g, \quad g_2 + 3g_3 = 16.$$

It follows that $g = 36 - 2g_3$. However, none of the defect graphs with $g_4 = 20$ has more than one triangle in it. We thus have proved that, using 20 quadruples, it is not possible to obtain an exact covering in fewer than 34 blocks ( it is possible to do so in 34 blocks, since a defect graph with one triangle is achievable). Now Stinson and Seah [4] have exhibited an exact covering in 31 blocks; hence, the optimal covering has 31 or fewer blocks, and thus does not contain 20 quadruples.

For reference, we list the Stinson-Seah covering as follows (17 quadruples, 10 triples, 4 pairs).

| | | | |
|---|---|---|---|
| 1 8 C $X_4$ | A 2 7 8 | $X_1$ $X_2$ 3 | $X_1$ A |
| 6 9 C $X_1$ | A 3 4 6 | $X_1$ $X_4$ 5 | $X_2$ B |
| 2 4 C $X_2$ | 2 5 6 D | $X_1$ $X_3$ 7 | $X_3$ C |
| D 7 9 $X_2$ | B 3 8 9 | $X_2$ 5 8 | $X_4$ D |
| 1 2 B $X_1$ | 3 5 7 C | $X_4$ 2 3 | |
| 1 3 D $X_3$ | 1 A 5 9 | $X_4$ 4 9 | |
| 4 D 8 $X_1$ | ABCD | $X_3$ 6 8 | |
| B 4 5 $X_3$ | | $X_2$ 1 6 | |
| B 6 7 $X_4$ | | $X_3$ 2 9 | |
| A $X_2$ $X_3$ $X_4$ | | 1 4 7 | |

## REFERENCES


[1] A.E. Brouwer, *Optimal Packings of $K_4$'s into a $K_n$, J.* Combinatorial Theory A, 26-3 (1979), 278-297.

[2] R.G. Stanton, J.L. Allston, and D.D. Cowan, *Pair-coverings with Restricted Largest Block Length*, Ars Combinatoria 11 (1981), 85-98.

[3] R.G. Stanton and D.R. Stinson, *Perfect Pair-coverings with Block Sizes Two, Three, and Four*, J. Combinatorics, Information, and System Sciences, 8-1 (1983),21-25.

[4] D.R. Stinson and E. Seah, personal communication.