SOME FURTHER RESULTS ON ONE-FACTORIZATIONS OF CARTESIAN PRODUCTS

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We use standard graph-theoretic ideas. Our notations include K_n and C_n for the complete graph and the cycle on n vertices; P_n denotes a path with n edges. A graph P_1 is also denoted E. Given graphs G and H, the cartesian product G x H is formed by replacing each vertex of G by a labelled copy of H; if two vertices of G were adjacent, then each vertex in one of the copies of H is joined to the corresponding vertex of the other copy of H. In other words, if H has h vertices, G is replaced by h copies of G and the h copies of each vertex of G are connected up as a copy of H.

If G is any graph, an edge-coloring of G into K colors is a way of labelling the edges of G with k labels, called colors, such that no vertex lies on two edges of the same color. A one-factor is a spanning subgraph with every vertex of degree 1, and a one-factorization is a decomposition of the edge-set of a graph into pairwise edge-disjoint one-factors; so only regular graphs have one-factorizations, and a one-factorization of a regular graph G of degree k is precisely an edge-coloring of G into k colors.

We are interested in the following questions, posed by Kotzig [1]: if G is a bridgeless cubic graph, does G x K_3 necessarily have a one-factorization?

Since we shall be discussing the Cartesian products of various graphs with K_3, we shall establish some notation. If G has vertex-set X, then G x K_3 has vertex-set $\{u_x, v_x, w_x : x \in X\}$, with the edges

$$u_x v_x,\ u_x w_x,\ v_x w_x:\ x \in X,$$

$$u_x u_y,\ v_x v_y,\ w_x w_y:\ xy \text{ an edge of } G.$$

In other words, the vertices with the same superscript form a copy of G and the vertices with the same subscript form a K_3. We also note an easy but useful decomposition: if F is a spanning subgraph of G, then

$$G \times K_3 = F \times K_3 \cup (G-F) \times \overline{K}_3$$

and the union is disjoint.

$G \times K_3$ clearly has a one-factorization if G does: we write

$$G \times K_3 = (F_1 \times K_3) + (F_2 \times \overline{K}_3) + (F_3 \times \overline{K}_3)$$

where F_1, F_2 and F_3 are the factors of G, and observe that the second and third terms are one-factors themselves while $F_1 \times K_3$ is the union of disjoint copies of the triangular prism $E \times K_3$, where E is a single edge; and these triangular prisms can be one-factorized -- see Figure 1. So we are interested in the case where G is a bridgeless cubic graph which does not have a one-factorization.

In discussing the triangular prism $E \times K_3$ we shall refer to the edges on the two triangular faces as end edges and the other edges, which constitute a copy of $E \times \overline{K}_3$, as side edges. In the same way, the graph $H_1 = P_{2k+1} \times K_3$, where P_{2k+1} is a path with 2k+1 edges, consists of two end triangles (containing end edges), 2k inner triangles and 2k+1 sets of three side edges, each set being an $E \times \overline{K}_3$ corresponding to an edge of P_{2k+1}.

The following results were proven in [2].

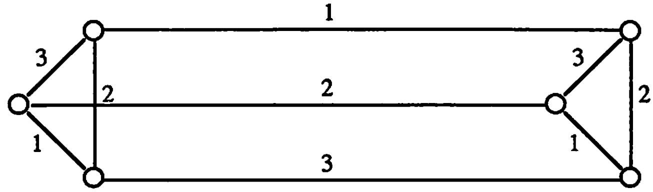

Figure 1

LEMMA 1 ([2], Lemma 2). Suppose G is a graph with a subgraph H isomorphic to $E \times K_3$, and G_1 is G with H replaced by $H_1 = P_{2h+1} \times K_3$. Suppose G is colored in t colors, $t \geq 5$, in such a way that the side edges of H are in colors chosen from {1, 2, 3, 4}. Then G_1 can be colored so that the end edges of H_1, and all edges not in H_1, are the same as in the coloring of G, and the side edges and inner triangles of H_1 contain only colors from {1, 2, 3, 4}.

LEMMA 2 ([2], Lemma 3). If C_{2k+1} has two distinguished vertices x and y, then there is a way of coloring $C_{2k+1} \times K_3$ such that:

(i) no edge of color 1 touches u_x;

(ii) no edge of color 2 touches v_x;

(iii) no edge of color 3 touches u_y;

(iv) no edge of color 4 touches v_y;

(v) edges $u_x v_x$ and $u_y v_y$ are of color 0 and all other edges are colored from {1, 2, 3, 4}.

LEMMA 3 ([2], Lemma 4). If C_{2k} has two distinguished vertices x and y, then there is a way of coloring $C_{2k} \times K_3$ such that:

(i) no edge of color 1 touches u_x or u_y;

(ii) no edge of color 2 touches v_x or v_y;

(iii) edges $u_x v_x$ and $u_y v_y$ are of color 0 and all other edges are colored from {1, 2, 3, 4}.

Suppose G is any bridgeless cubic graph. Then G decomposes into a one-factor F_1 and a two-factor F_2. In [2] we associated with G a graph G^*, the *cycle graph* of G (with regard to the decomposition $F_1 \cup F_2$), formed by contracting each cycle to a point, and a vertex of G^* is called an odd (even) cycle point if it comes from an odd (even) cycle. The main result of [2] was that if G has a cycle graph G^* which possesses a two-factor in which no component cycle contained an odd number of odd cycle points, then $G \times K_3$ has a one-factorization. It is in fact easy to see that one can slightly weaken the hypothesis of that theorem, to obtain

THEOREM 1. Suppose G is a bridgeless cubic graph which has a cycle graph G^* with the following property: G^* has a 2-regular subgraph G_2 which contains all the odd cycle points, and no component cycle of this 2-regular graph contains an odd number of odd cycle points. The $G \times K_3$ has a one-factorization.

The proof is almost identical to that of the theorem in [2].

We shall generalize Theorem 1. We first prove several lemmas about special colorings of $C \times K_3$.

LEMMA 4. Suppose x, y, z and t are four vertices of an odd cycle C_{2k+1}. Then one can 5-color the edges of $C_{2k+1} \times K_3$ in such a way that:

(i) no edge of color 1 touches u_x, u_z or u_t;

(ii) no edge of color 2 touches v_x, v_z or v_t;

(iii) no edge of color 3 touches u_y;

(iv) no edge of color 4 touches v_y;

(v) the edges u_xv_x, u_yv_y, u_zv_z and u_tv_t are of color 0 and all others are colored from {1,2,3,4}.

<u>Proof</u>. For convenience, suppose that as one follows the C_{2k+1} the four special vertices occur in the order x, y, z, t. Then the C_{2k+1} is the union of four paths which are disjoint except at their endpoints: an (x,y)-path, a (y,z)-path a (z,t)-path and a (t,x)-path. Two cases must be distinguished, according as one or three of those paths have odd length; each of these divides according to whether the two paths which contain y have the case or different parity. Figure 2 shows four examples, corresponding to these four cases: Figures 2(a) and 2(b) are examples with one odd path, in the "same parity" and "different parity" cases, respectively; Figures 2(c) and 2(d) cover three odd paths.

In each case, the example shows paths of length 1 for the odd paths and paths of length 2 for the even paths. These can be generalized to arbitrary lengths using Lemma 1.

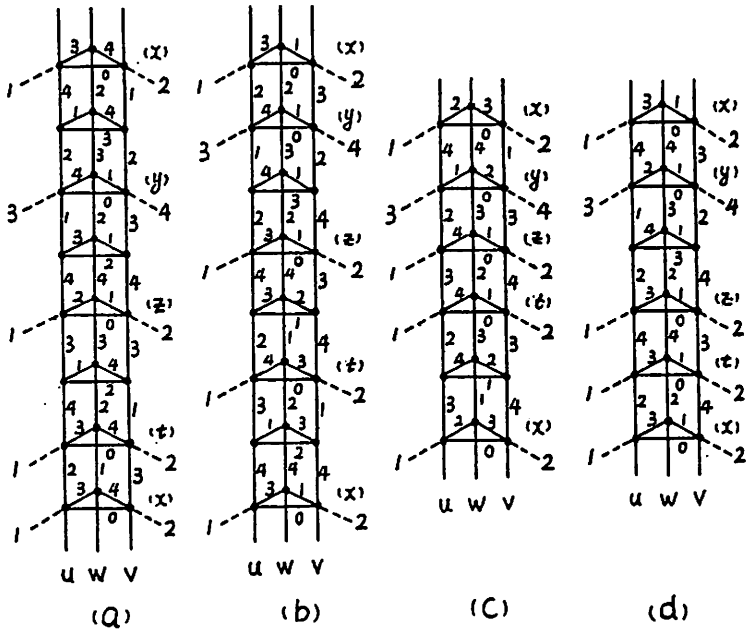

FIGURE 2

It is easy to see that the coloring in Lemma 4 is still valid if a permutation (13)(24) is applied to the color set.

LEMMA 5. Suppose x, y, z, t are four points of the even cycle C_{2k}. Then there is a 5-coloring of $C_{2k} \times K_3$ with the properties:

(i) no edge of color 1 touches u_x, u_y, u_z or u_t;

(ii) no edge of color 2 touches v_x, v_y, v_z or v_t;

(iii) edges u_xv_x, u_yv_y, u_zv_z and u_tv_t are colored 0, and all other edges are colored from {1,2,3,4}.

Proof. Again, assume that x, y, z and t occur in the C_{2k} in that order, and consider the four paths (x,y), (y,z), (z,t) and (t,x) which are disjoint except at their endpoints. Four cases arise: (a) all four paths can be of even length; (b) there can be two adjacent even paths and two adjoint odd paths; (c) there can be two odd paths and two even paths, with paths whose lengths have the same parity being inadjacent; (d) all four can be odd. Four appropriate examples are shown in Figure 3; each can be extended to arbitrary length paths using Lemma 1.

Again, the colorings in Lemma 5 remain valid under the permutation (13)(24).

LEMMA 6. Suppose x, y, z and t are four vertices of C_{2k} with the property that neither x nor z is adjacent to y or t. Then there is a way of 5-coloring $C_{2k} \times K_3$ such that:

(i) no edge of color 1 touches u_x or u_z;

(ii) no edge of color 2 touches v_x or v_z;

(iii) no edge of color 3 touches u_y or u_t;

(iv) no edge of color 4 touches v_y or v_t;

(v) edges $u_x v_x$, $u_y v_y$, $u_z v_z$, $u_t v_t$ are colored 0, and all other edges are colored from {1, 2, 3, 4}.

<u>Proof</u>. In this case there are two different orders in which the four distinguished vertices may occur around the

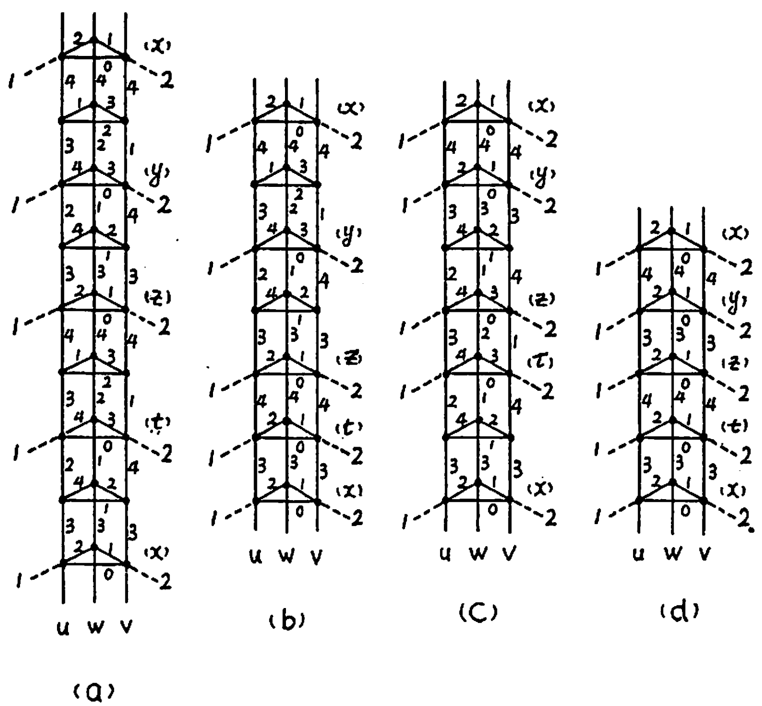

FIGURE 3

cycle C_{2k}: x-z-y-t-x is essentially different from x-y-z-t-x.

Suppose first that the ordering is x-z-y-t-x. Again the four vertices partition C_{2k} into four paths which are internally disjoint; say the (x,z)-path, (z,y)-path, (y,t)-path, (t,x)-path are called P_1, P_2, P_3 and P_4, respectively. There are five different cases:

(a) all are of even length;

(b) P_1 and P_2 are even, P_3 and P_4 are odd;

(c) P_1 and P_3 are even, P_2 and P_4 are odd;

(d) P_2 and P_4 are even, P_1 and P_3 are odd;

(e) all are odd.

Figure 4 shows a suitable coloring in every case, with paths of minimum length; again, they can be generalized using

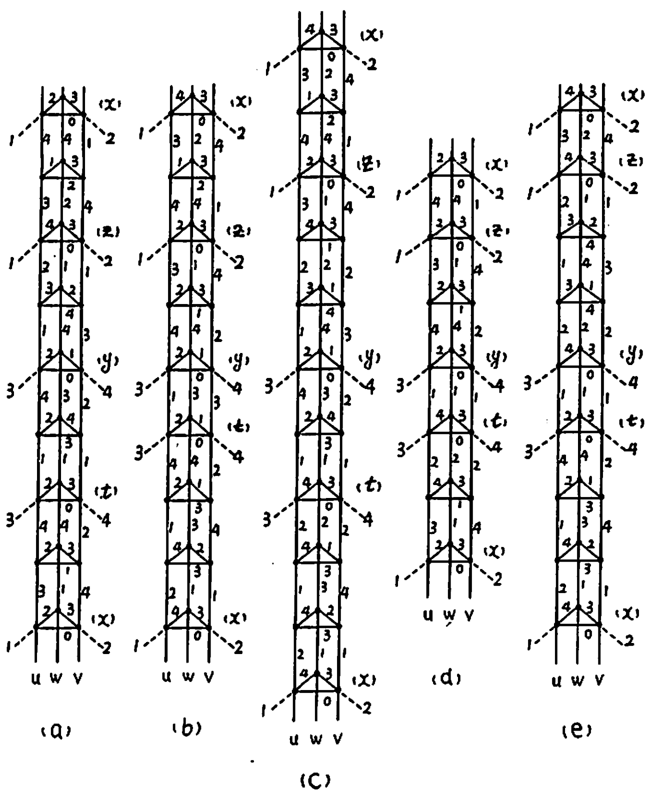

FIGURE 4

Lemma 1. (In the case where P_2 has odd length, for example, the minimum length is 3, since z and y are not adjacent.)

If the ordering is x-y-z-t-x, four subcases occur. If we write P_1, P_2, P_3, P_4 for the (x,y)-, (y,z)-, (z,t)- and (t,x)-paths respectively, they are:

(a) all paths are of even length;

(b) P_1 and P_2 are even, P_3 and P_4 are odd;

(c) P_1 and P_3 are even, P_2 and P_4 are odd;

(d) all are odd.

Suitable examples are shown in Figure 5.

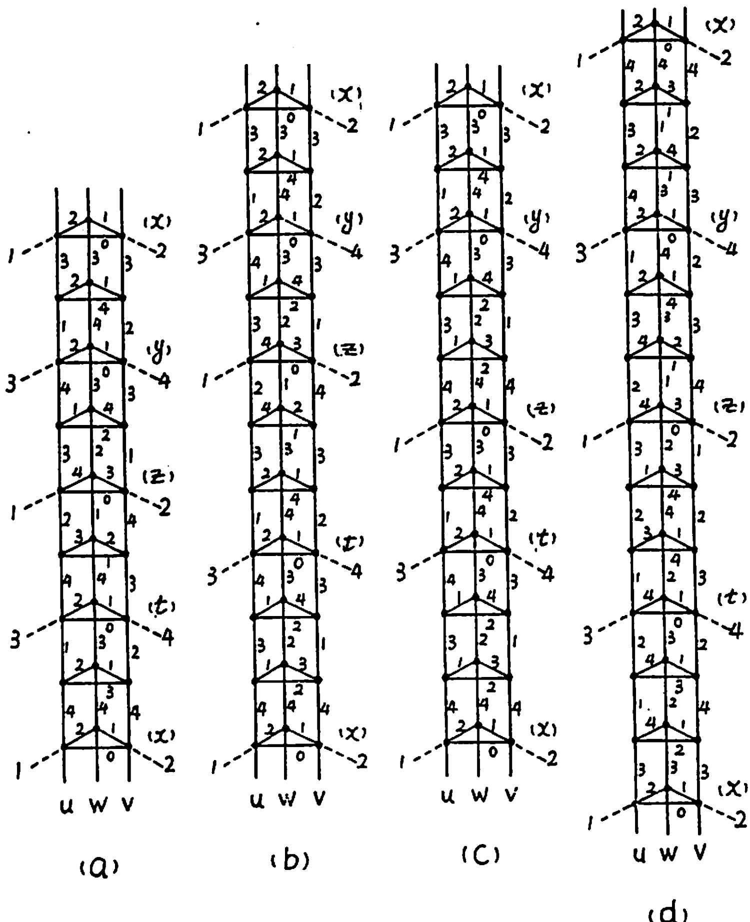

FIGURE 5

In order to generalize Theorem 1 using the above Lemmas, we first examine the proof of Theorem 1. Suppose G is a bridgeless cubic graph which decomposes as $F_1 \cup F_2$, where F_1 is a one-factor and F_2 a two-factor, and suppose G^* is the corresponding cycle graph. If $C^* = (g_1, g_2, \ldots, g_k)$ is a cycle in G^*, then each vertex g_i corresponds to a cycle C_i in F_2, and the edge $g_i g_{i+1}$ corresponds to some edge e_i in F_1 which joins a vertex of C_i to a vertex of C_{i+1}. Assume G satisfies the conditions of Theorem 1; let G_2 be the 2-regular subgraph specified in the theorem, and denote by U the set of all edges of F_1 corresponding to edges of G_2. To construct the one-factorization of $G \times K_3$, we label the endpoints of the edges in U as follows. Two vertices in C_i are labelled x_i and y_i: e_i intersects C_i at x_i; if C_i is an odd cycle, e_i is $x_i x_{i+1}$ or $y_i y_{i+1}$; if C_i is an even cycle, e_i is $x_i y_{i+1}$ or $x_{i+1} y_i$ (where subscripts are reduced modulo k if necessary). For details, see [2]. Observe that the consistency is guaranteed because G_2 contains no cycle with an odd number of odd cycle points, and that when labelling we are concerned only with the symbols x and y, and not with the subscripts. We now generalize this labelling procedure.

Suppose the bridgeless cubic graph G has a decomposition $F_1 \cup F_2$ into a one-factor and a two-factor for which the corresponding cycle graph G^* contains a collection of edge-disjoint (but not necessarily vertex disjoint) cycles which between them contain all the odd cycle points of G^*. Further, suppose that no cycle point is a vertex of more than two of these cycles. We shall call such a set a *proper collection* of cycles. If P is a proper collection, we write E(P) for the collection of edges of F_1 which are used to give edges in the members of C. A common vertex between two members of C will be called a *common* odd or even cycle point, according as it is an odd or even cycle point of G^*.

In order to label the endpoints of the edges in E(P) we first reclassify the common cycle points. A common odd cycle point is considered as an odd cycle point in one of the cycles and an even cycle point in the other. A common even cycle point is either classified as class I and considered as an even cycle point in both cycles, or classified as class II and considered odd in both cycles. Then the same scheme is used for labelling the endpoints of the edges of E(P) as was used for the endpoints of the edges of U previously. Such a labelling is consistent whenever in each cycle the number of odd cycle points (after reclassification) is even. In each cycle in F_2 corresponding to a common cycle point there will be two vertices labelled with x and two vertices labelled with y. To be precise and avoid this multiple labelling we let x and y denote the endpoints of edges in E(P) which appear in one cycle of G^*, while x' and y' denote the endpoints of edges in E(P) which appear in the other cycle of G^*. Such a labelling we shall call a *consistent labelling* of E(P).

THEOREM 2. Suppose G is a bridgeless cubic graph admitting a cycle Graph G^* which contains a proper collection P with the properties:

(i) each cycle of P contains an even number of odd cycle points after some reclassification;

(ii) after that reclassification there is a consistent labelling of E(P) in which no vertex labelled x or x' in an even cycle of F_2 corresponding to a common even cycle point is adjacent to any vertex labelled y or y' in the same cycle.

Then $G \times K_3$ has a one-factorization.

Proof. We color $G \times K_3$ as follows. The edges of $(F_1 \setminus E(P)) \times K_3$ have color 0. For the edges e in E(P), the $e \times K_3$ are colored as in the proof of Theorem 1. Finally, for all cycles C in F_2, $C \times K_3$ is colored as follows: if C corresponds to a common cycle point then $C \times K_3$ is colored by Lemma 4, Lemma 5 or Lemma 6 according as the common cycle point is class I or class II (with the substitution of x' for z and y' for t); for other cycles in F_2 the coloring is done using Lemma 2 or Lemma 3, permuting the colors via (13)(24) if necessary. ∎

Notice that Lemma 5 does not require non-adjacency in C_{2k} of the vertices $\{x,z\}$ and $\{y,t\}$, and therefore does not require non-adjacency between $\{x,x'\}$ and $\{y,y'\}$ in Theorem 2. It is easy to see that the "non-adjacency" condition in part (ii) of the theorem is not required for the class I common even cycle points in some cycle if the cycle contains only one such point, or if it contains several such points between any two of which there is an even number of odd cycle points including those obtained from common cycle points.

EXAMPLE. In Figure 6(a), G is a bridgeless cubic graph. It can be expressed as a decomposition $F_1 \cup F_2$, F_1 being a one-factor with black line and F_2 being a two-factor with broken line.

In G^*, which is shown in Figure 6(b), one can find three edge-disjoint cycles containing all odd cycle points $C_1^* = (g_1g_2g_3g_4)$, $C_2^* = (g_4g_5g_6g_7g_8)$ and $C_3^* = (g_8g_9g_5g_{10}g_{11}g_{12})$. Each point in G^* is a common vertex of at most two cycles. g_4 can be considered as an even cycle point in C_1^* and odd cycle point in C_2^*; g_5 an even one in C_2^* and odd one in C_3^*; g_8 an even one in both C_2^* and C_3^*. So $P = \{C_1^*, C_2^*, C_3^*\}$ is a proper collection of cycles. On the other hand, C_1^* has two odd cycle points g_1 and g_2; C_2^* has two odd cycle points g_4 and g_6; C_3^* contains four odd cycle points g_9, g_5, g_{10} and g_{12}. So condition (i) of Theorem 2 is satisfied. Since there is only one class I even cycle point in G^*, condition (ii) is not required. From the theorem we know that $G \times K_3$ has a one-factorization.

A suitable labelling to the endpoints of edges in $E(P)$ is shown in Figure 6(a), where for the common even cycle point g_8 the vertex of C_8 labelled x or x' is not adjacent to the vertex of C_8 labelled y or y'.

Therefore, in order to determine whether for any given bridgeless cubic graph G the product $G \times K_3$ has a

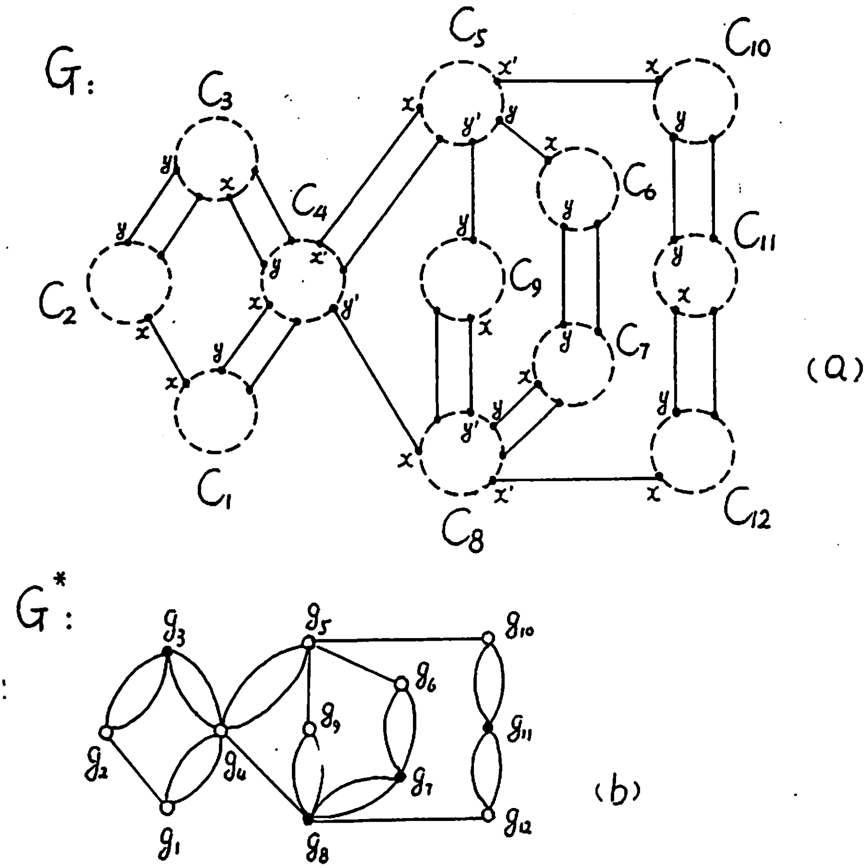

FIGURE 6

one-factorization, one can first try G itself and see if G has a one-factorization. If not, try the cycle graph G^* and see if Theorem 1 is applicable. If any 2-regular subgraph of G^* always contains some cycle with odd number of of odd cycle points one can then try Theorem 2 before trying a new decomposition $F_1 \cup F_2$ and a new cycle graph G^*. Here the cycles in G^* which contain all odd cycle points are not necessarily disjoint and form a 2-regular subgraph; they can have something in common (common points), but not too much (they are edge-disjoint and each point can be a common point of at most two cycles of G^*). From this improvement we have

<u>COROLLARY</u>. If a bridgeless cubic graph G has a two-factor with at most three components, then $G \times K_3$ has a one-factorization.

<u>Proof</u>. Suppose a bridgeless cubic graph G has a one-factor F_1 and a two-factor F_2 such that $G = F_1 \cup F_2$. If the component number of F_2 satisfies $w(F_2) \leq 2$, the cycle graph G^* is a Hamilton graph since G is bridgeless, and Theorem 1 implies that $G \times K_3$ has a one-factorization. We then suppose $w(F_2) = 3$. If F_2 has three even cycles, apparently $G \times K_3$ has a one-factorization. We further suppose that F_2 has two odd cycles and one even cycle.

Since G is a bridgeless cubic graph, for either odd cycle C_i in F_2 $(i = 1,2)$ there are at least three edges in F_1 with one endpoint in C_i and another endpoint in other cycles of F_2. There are four possible cases:

(i) there are at least three edges in F_1 each linking one vertex in C_1 and one vertex in C_2;

(ii) there are exactly two edges in F_1 each linking one vertex in C_1 and one vertex in C_2;

(iii) there is exactly one edge in F_1 linking one vertex in C_1 and one vertex in C_2;

(iv) none of edges in F_1 can link one vertex in C_1 and one vertex in C_2.

Theorem 1 is applicable to the first three cases. There remains only case (iv).

Suppose the end vertices in C_3 of three edges linking C_1 and C_3 in F_1 are a_1, a_2, a_3; the end vertices in C_3 of three edges linking C_2 and C_3 in F_1 are b_1, b_2, b_3. Then, $a_1, a_2, a_3, b_1, b_2, b_3$ are six distinct vertices in C_3. Their order in C_3 (clockwise or counterclockwise) fits one of the three following patterns:

aabbba, aabbab, ababab.

In each case there always exists a set of four points x, y, x', y' such that $x, y \in \{a_1, a_2, a_3\}$ and $x', y' \in \{b_1, b_2, b_3\}$, and such that neither x nor x' is adjacent to y or y': for example, in the above listings one can take the second and third points in order as x and x', and the fifth and sixth as y and y'. In the cycle graph G^* of G there are two cycles $C_1^* = (g_1, g_3)$ and $C_2^* = (g_2, g_3)$, where g_3 is a class II common even cycle point. Theorem 2 is now applicable and $G \times K_3$ has a one-factorization.

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[2] W. D. Wallis and Wang Zhi-jian, On one-factorizations of cartesian products. Congressus Num. 49(1985), 237-245.