VULNERABILITY IN GRAPHS–A COMPARATIVE SURVEY

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ABSTRACT

In assessing the "vulnerability" of a graph one determines the extent to which the graph retains certain properties after the removal of a number of vertices and/or edges. Four measures of vulnerability to vertex removal are compared for classes of graphs with edge densities ranging from that of trees to that of the complete graph.

1. INTRODUCTION

Connectivity, though certainly the most studied, is but one measure of the vulnerability of a graph. We use "vulnerability" in a generic sense. If we think of the graph as modeling a network, the vulnerability measures the resistance of the network to disruption of operation after the failure of certain stations or communication links. In this survey we consider vertex removal only and focus on the following four measures of vulnerability.

i) $\kappa(G)$: The *connectivity* of a graph G is defined by

$$\kappa(G) = \min_{S \subseteq V(G)} |S|$$

for which $G - S$ is disconnected or trivial.

ii) $t(G)$: The *toughness* of a graph G is defined by

$$t(G) = \min_{S} \frac{|S|}{\omega(G-S)}$$

where S is a vertex cut of G and $\omega(G-S)$ is the number of components of $G - S$.

iii) $b(G)$: The *binding number* of a graph G is defined by

$$b(G) = \min_{S} \frac{|N(S)|}{|S|}$$

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where $\phi \neq S \subseteq V(G)$ and $N(S) \neq V(G)$. The *neighborhood*, $N(S)$, of S consists of *all* vertices of G adjacent to at least one vertex of S.

iv) $I(G)$: The integrity of a graph G is given by

$$I(G) = \min_{S \subseteq V(G)} (|S| + m(G - S))$$

where $m(G - S)$ is the maximum number of vertices in a component of $G - S$.

The toughness of a graph was introduced by Chvátal [2] in connection with the hamiltonicity of a graph. Since K_n has no vertex cut, $t(K_n)$ is not defined in ii). Chvátal defined $t(K_n) = \infty$ but if we redefine $t(K_n)$ to be $(n-1)/2$ then the following result of Chvátal holds without excepting K_n.

Theorem A (Chvátal [2]). For all graphs G,

$$\frac{\kappa(G)}{\alpha(G)} \leq t(G) \leq \frac{1}{2}\kappa(G)$$

where $\alpha(G)$ is the independence number of G.

Woodall [5] defined the binding number of a graph and showed its relation to the existence of sets of independent edges and, also, to the hamiltonicity of the graph. He also proved the following relations of binding number to connectivity and toughness.

Theorem B (Woodall [5]). For all graphs G,

$$b(G) \leq \frac{n + \kappa(G)}{n - \kappa(G)} .$$

Theorem C (Woodall [5]). For all graphs G,

$$b(G) \leq t(G) + 1 .$$

The authors [1] introduced the integrity of a graph as an alternative measure of vulnerability. The reasons for our particular choice will be discussed in Section 3.

In the next section we will calculate the four measures of vulnerability for the complete graph K_n, the complete bipartite graph $K_{k,n-k}$, $k \leq n - k$, powers C_n^k of the n-cycle, $2 \leq 2k \leq n - 2$, and the graphs $G_{n,k}$ and $T_{n,k}$ pictured in Figure 1.

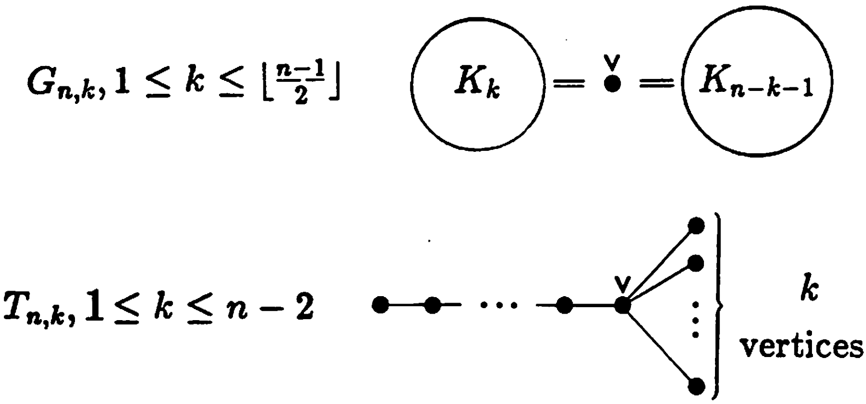

Figure 1.

These graphs were purposefully chosen, first because they exhibit the widest possible range of edge density and second to illustrate where the different measures of vulnerability differ in their effectiveness in measuring important structural characteristics of graphs. Throughout the next section we will use $\lfloor x \rfloor$ and $\lceil x \rceil$ to denote the largest integer not more than x and the least integer not less than x, respectively.

2. VULNERABILITY CALCULATIONS

The values of $\kappa(G)$ given in Table 2 are all obvious except, perhaps, for $\kappa(C_n^k)$. We give a derivation of $\kappa(C_n^k)$ for later comparison with the derivations of $t(C_n^k)$ and $b(C_n^k)$.

Fact 1. $\kappa(C_n^k) = 2k$ for $2 \leq 2k \leq 2n-2$.

Proof. Let C_n be the n-cycle with vertices $v_0, \ldots, v_{n-1}$ labeled so that $v_iv_j \in E(C_n)$ iff $|j-i| = 1$ (indices are read modulo n). Then $v_iv_j \in E(C_n^k)$ iff $1 \leq |i-j| \leq k$ so that, since $2k \leq n-2$, $\kappa(C_n^k) \leq 2k$. It remains to exhibit $2k$ internally disjoint $v_0 - v_j$ paths for $1 \leq j \leq n-k-1$. We consider two cases.

i) $k+1 \leq j \leq n-k-1$. For $0 \leq i \leq k-1$ let P_i be the path with vertex set $\{v_0, v_i, v_{k+i}, \cdots, v_{\lfloor (j-i)/k \rfloor k+i}, v_j\}$. Then $\{P_i | 0 \leq i \leq k-1\}$ is a set of pairwise internally disjoint $v_0 - v_j$ paths using only vertices v_r with $0 \leq r \leq j$. Since a set of $v_j - v_0$ paths using only vertices v_r, $j \leq r \leq n-1$ and v_0 may be similarly constructed, C_n^k contains $2k$ internally disjoint $v_0 - v_j$ paths in this case.

ii) $1 \leq j \leq k$. The paths $(v_0v_1v_j), \ldots, (v_0v_{j-1}v_j), (v_0v_j), (v_0v_{j+1}v_j), \ldots,$

$(v_0v_kv_j)$ and $(v_0v_{n-1}v_j)\cdots(v_ov_{n-k+j}v_j)$ form a set of $2k-j$ pairwise internally disjoint v_0-v_j paths using only vertices v_r, $0\le r\le k$ and $n-k+j\le r\le n-1$. For $1\le i\le j$ let P_i be the path with vertex set $\{v_j, v_{k+i}, \ldots, v_{\lfloor(n-k-1)/k\rfloor k+i}, v_{n-k+i-1}, v_0\}$. Since these paths are pairwise internally disjoint and use only vertices v_r with $k+1\le r\le n-k+j-1$ and v_0 and v_j, C_n^k contains $2k$ internally disjoint v_0-v_j paths.

We conclude that $\kappa(C_n^k)=2k$.

We have previously redefined $t(K_n)=(n-1)/2$. To show that $t(G_{n,k})=\frac{1}{2}$ we simply note that for any vertex cut S we have $\omega(G-S)\le 2$ and $|S|\ge 1$ and that the equalities may be simultaneously achieved. Chvátal [2] has shown $t(K_{k,n-k})=k/(n-k)$ and remarked that $t(C_n^k)=k$. We give a proof of the latter equality.

Fact 2. $t(C_n^k)=k$.

Proof. Let S be a vertex cut of C_n^k labeled as in the proof of Fact 1 and choose one vertex v_{i_j} from each of the $\omega(G-S)$ components of $G-S$. We may assume these vertices are labeled so that $i_1<i_2<\cdots<i_{\omega(G-S)}$. Set $V_j=\{v_i|i_j\le i\le i_{j+1}\}$ for $1\le j\le\omega(G-S)$ (we take $v_{i_{\omega(G-S)+1}}=v_{i_1}$). Since, as shown in the proof of Fact 1, there are k internally disjoint $v_{i_j}-v_{i_{j+1}}$ paths in C_n^k using only vertices of V_j we must have $|S\cap V_j|\ge k$. Consequently $|S|\ge k\omega(G-S)$. On the other hand, if we set $S=\{v_i|1\le i\le k \text{ or } n-k\le i\le n-1\}$ then $|S|=2k$ and $\omega(G-S)=2$. Consequently $t(C_n^k)=k$.

To determine $t(T_{n,k})$ we let S be a vertex cut of $T_{n,k}$ and note that to minimize $|S|/\omega(G-S)$, S must be an independent set of vertices with degree at least 2. If S is such a set and $v\notin S$ then

$$\frac{|S|}{\omega(G-S)}=\frac{|S|}{|S|+1}$$

whereas if $v\in S$ then

$$\frac{|S|}{\omega(G-S)}=\frac{|S|}{|S|+k}.$$

Consequently we take $S=\{v\}$ and have $t(T_{n,k})=\frac{1}{k+1}$.

We now proceed to calculate $b(G)$ for the various graphs G. In particular, if $G=K_n$ then, because $N(S)\ne V(G)$, S must contain only one vertex of G so that $b(K_n)=n-1$. If $G=G_{n,k}$ then, since $N(S)\ne V(G)$, we have only the following cases to consider.

i) $|S| = 1$. If $S = \{v\}$ then $|N(S)| = n-1$ and if $S \neq \{v\}$ then $|S| = k$ or $n-k-1$.

ii) $|S| > 1$. Here we note that $S \neq \{v\}$ implies $v \notin S$. In Table 1 we determine $|N(S)|$ for each of the possible choices of S.

$\lvert S \cap V(K_k)\rvert$	1	>1	1	>1	0
$\lvert S \cap V(K_{n-k-1})\rvert$	1	1	>1	0	>1
$\lvert N(S)\rvert$	$n-2$	$n-1$	$n-1$	$k+1$	$n-k$

Table 1.

From Table 1 it is easily determined that

i) If $k = 1$ we take $|S \cap V(K_k)| = 1$ and have $b(G_{n,k}) = 1$,

ii) If $k = 2$ we take $|S \cap V(K_k)| = 1$, $|S \cap V(K_{n-k-1})| = n-k-1$ and have $b(G_{n,k}) = \frac{n-1}{n-2}$ and

iii) If $k \geq 3$ we take $S = V(K_{n-k-1})$ and have $b(G_{n,k}) = \frac{n-k}{n-k-1}$.

Woodall [5] has shown $b(K_{k,n-k}) = \frac{k}{n-k}$. We determine $b(C_n^k)$ in the following.

Fact 3. $b(C_n^k) = \begin{cases} 1 & , k = 1, 2|n \\ \frac{n}{2} - 1 & , 2k = n-2 \\ \frac{n-1}{n-2k} & , \text{otherwise} \end{cases}$.

Proof. Woodall [5] has shown that

$$b(C_n) = \begin{cases} 1 & , 2|n \\ \frac{n-1}{n-2} & , 2 \nmid n \end{cases}$$

so that, since C_n is a spanning subgraph of C_n^k, we have $b(C_n^k) \geq 1$. Let $S = \{v_{i_1}, \ldots, v_{i_{|S|}}$ be a subset of $V(C_n^k)$ for which $b(C_n^k) = |N(S)|/|S|$. We assume the vertices of C_n are labeled as in the proof of Fact 1. We may assume further that $i_1 < i_2 < \cdots < i_{|S|}$ and that $k \geq 2$. We then have the following (in which $i_{|S|+1}$ is taken to be i_1 and i_0 to be $i_{|S|}$.

i) If for some r, $1 \leq r \leq |S|$, we have $i_{r-1} + 1 = i_r < i_{r+1} - 1$ then, setting $S' = S \cup \{v_{i_r+1}\}$ we have

$$\frac{|N(S')|}{|S'|} \leq \frac{|N(S)|+1}{|S|+1} \leq \frac{|N(S)|}{|S|}$$

and $N(S') = V(G)$ iff $N(S) = V(G) - \{v_{i_r+1+k}\}$.

ii) If for some r, $1 \le r \le |S|$, we have $i_r + k \ge v_{i_r+1}$ then, setting $S' = S \cup \{v_{i_r+1}\}$ we have

$$\frac{|N(S')|}{|S'|} = \frac{|N(S)|}{|S|+1} \le \frac{|N(S)|}{|S|}$$

and $N(S') \ne V(G)$.

If $N(S) = V(G) - \{v\}$ for some $v \in V(G)$ we take $S = \{u \in V(G) | d(v,u) > k\} \cup \{v\}$ and have

$$\frac{|N(S)|}{|S|} = \frac{n-1}{n-2k}.$$

On the other hand if $|N(S)| < n-1$ then from i) and ii) we have $i_r + k < i_{r+1}$ for $1 \le r \le |S|$ so that $|N(S)| \ge k|S|$. Now

$$\frac{n-1}{n-2k} > k \iff 2k^2 - nk + n - 1 > 0 \iff (n-2k-2)(1-k) = 2k^2 - nk + n - 2 \ge 0 .$$

The equality of Fact 3 immediately follows.

Fact 4. $b(T_{n,k}) = \begin{cases} \frac{n-1}{n+1} & , k = 1, 2 \nmid n \\ \frac{1}{k} & , \text{otherwise} \end{cases}$.

Proof. The case $k = 1$ is a result of Woodall [5]. We assume, then, that $k \ge 2$ and note that, in this cae,

$$b(T_{n,k}) \le \frac{N(S')}{|S'|} = \frac{1}{2}$$

where S' consists of two of the end vertices of $T_{n,k}$ adjacent to v. We also may assume $k \le n-3$ since the equality of Fact 4 obviously holds in this case.

Let v_1 be the end vertex of $T_{n,k}$ not adjacent to v and label the vertices of the $v_1 - v$ path $v_1, v_2, \ldots, v_{n-k-1}$, $v_{n-k} = v$ so that v_i is adjacent to v_{i+1} for $1 \le i \le n-k-1$. Let S be a subset of $V(G)$ for which $b(T_{n,k}) = |N(S)|/|S|$ and $N(S) \ne V(G)$. If $v_i \notin S$ for $1 \le i \le v_{n-k}$ then obviously $b(T_{n,k}) = \frac{1}{k}$. Otherwise let r be the least index such that $v_r \in S$.

If $r = 1$ and $v_2 \in S$ (or $v_2 \notin S$ but $v_3 \in S$) we set $S' = S - \{v_1, v_2\}$ (or $S' = S - \{v_1, v_3\}$, respectively) and have

$$\frac{|N(S')|}{|S'|} \le \frac{|N(S)|-1}{|S|-2} \le \frac{|N(S)|}{|S|}.$$

If $r = 1$ and $\{v_2, v_3\} \cap S = \phi$ or if $r \ge 2$ we set $S' = S - \{v_r\}$ and have

$$\frac{|N(S')|}{|S'|} \le \frac{|N(S)|-1}{|S|-1} \le \frac{|N(S)|}{|S|}.$$

In any case we may assume $v_i \notin S$ for $1 \le i \le n-k$ and have $b(T_{n,k}) = \frac{1}{k}$.

We now turn to $I(G)$; obviously $I(K_n) = n$. If $G = G_{n,k}$ we have two cases to consider.

i) If $v \notin S$ then, since $m(G-S) = |V(G)| - |S|$, we have $|S| + m(G-S) = n$.

ii) If $v \in S$ we set $|S \cap V(K_k)| = a$ and $|S \cap V(K_{n-k-1})| = b$ so that

$$|S| + m(G-S) = a + b + 1 + \max(k-a, n-k-b-1) = 1 + \max(k+b, n-k+a-1)\,.$$

This last expression is minimized by taking $a = b = 0$ so we have $I(G_{n,k}) = n - k$.

It is easily seen that $I(K_{k,n-k}) = k+1$. The integrity of C_n^k is calculated in [1] and shown to be $\min_m(m + k\left\lfloor\frac{n}{m+k}\right\rfloor + \epsilon_m)$ where

$$\epsilon_m = \begin{cases} 0, & (m+k)|n \\ 1, & (m+k)\nmid n \end{cases}.$$

In particular if $k=1$, i.e., $G = C_n$ we have $I(C_n) = \lceil 2\sqrt{n}\rceil - 1$.

G	Measure of vulnerability of G			
	$\kappa(G)$	$t(G)$	$b(G)$	$I(G)$
K_n	$n-1$	$\frac{n-1}{2}$	$n-1$	n
$G_{n,k}$	1	$\frac{1}{2}$	$\begin{cases} 1 & ,k=1 \\ \frac{n-1}{n-2} & ,k=2 \\ \frac{n-k}{n-k-1} & ,k\geq 3 \end{cases}$	$n-k$
$K_{k,n-k}$	k	$\frac{k}{n-k}$	$\frac{k}{n-k}$	$k+1$
C_n^k	$2k$	k	$\begin{cases} 1 & ,k=1,2\vert n \\ \frac{n}{2}-1 & ,2k=n-2 \\ \frac{n-1}{n-2k} & ,\text{otherwise} \end{cases}$	$\min_m(m+k\left\lfloor\frac{n}{m+k}\right\rfloor+\epsilon_m)$ where $\epsilon_m = \begin{cases} 0 & ,(m+k)\vert n \\ 1 & ,(m+k)\nmid n \end{cases}$
$T_{n,k}$	1	$\frac{1}{k+1}$	$\begin{cases} \frac{n-1}{n+1} & ,k=1,2\nmid n \\ \frac{1}{k} & ,\text{otherwise} \end{cases}$	$\begin{cases} \lceil 2\sqrt{n+1}\rceil-2, 1\leq k\leq\sqrt{n+1}-\frac{5}{4} \\ \lceil 2\sqrt{n-k}\rceil-1, \sqrt{n+1}-\frac{5}{4}\leq k\leq n-2 \end{cases}$

Table 2.

Fact 5. $I(T_{n,k}) = \begin{cases} \lceil 2\sqrt{n+1}\rceil - 2 & ,1 \leq k \leq \sqrt{n+1} - \frac{5}{4} \\ \lceil 2\sqrt{n-k}\rceil - 1 & ,\sqrt{n+1} - \frac{5}{4} \leq k \leq n-2 \end{cases}.$

Proof. Let S be a subset of $V(G)$ for which $I(T_{n,k}) = |S| + m(G-S)$. As we note below, we may assume $k \geq 2$ and it is easy to see that we may assume S was chosen so as to not contain any of the k end vertices adjacent to v. Let $P = (v_0 v_1 \cdots v_{n-k-1})$ be the path from $v(= v_0)$ to the end vertex v_{n-k-1} of $T_{n,k}$ furthest from v and let r be the least index for which $v_r \in S$.

The authors [1] showed that

$$I(T_{n,1}) = I(P_n) = \min_m(m - 1 + \left\lceil \frac{n+1}{m+1} \right\rceil) = \lceil 2\sqrt{n+1} \rceil - 2 .$$

We use these equalities in the following.

i) If $r = 0$ then, since $n \leq k + m(G - S) + |S|$, we have $|S| \geq (n - k)/(m + 1)$ (where $m = m(G - S)$) so that

$$I(T_{n,k}) \geq \min_m \left(m + \left\lceil \frac{n-k}{m+1} \right\rceil \right) = \lceil 2\sqrt{n-k} \rceil - 1 .$$

ii) If $r > 0$ then, since v and the end vertices adjacent to v all lie in the same component of $G - S$ we have

$$I(T_{n,k}) = I(P_n) = \lceil 2\sqrt{n+1} \rceil - 2 .$$

Thus

$$I(T_{n,k}) \geq \min(\lceil 2\sqrt{n-k} \rceil - 1, \lceil 2\sqrt{n+1} \rceil - 2) .$$

Now, for $k \geq 0$, we have

$$\begin{aligned} \lceil 2\sqrt{n-k} \rceil - 1 \leq \lceil 2\sqrt{n+1} \rceil - 2 &\iff \lceil 2\sqrt{n-k} \rceil \leq \lceil 2\sqrt{n+1} - 1 \rceil \\ &\iff 2\sqrt{n+1} - 2\sqrt{n-k} \geq 1 \iff k \geq \sqrt{n+1} - \tfrac{5}{4} . \end{aligned}$$

Thus

$$I(T_{n,k}) \geq \begin{cases} \lceil 2\sqrt{n+1} \rceil - 2, 1 \leq k \leq \sqrt{n+1} - \frac{5}{4} \\ \lceil 2\sqrt{n-k} \rceil - 1, \sqrt{n+1} - \frac{5}{4} \leq k \leq n-2 \end{cases} .$$

Easy constructions (space the members of S equally on path P) show that equality holds and the proof of Fact 5 is complete. This entry also completes Table 2.

3. DISCUSSION

If a system such as a communication network is modeled by a graph G and a function $f(G)$ is designed to measure the ability of the system to operate after one or more stations are disabled it seems reasonable to expct $f(G)$ to satisfy the following criteria.

i) If H is a spanning subgraph of G then we should have $f(H) \leq f(G)$.

ii) $f(G)$ achieves its maximum value only at $G = K_n$ and its minimum value only at

$G = \overline{K_n}$ where $\overline{K_n}$ is the complement of K_n.

These two criteria are included in the next.

iii) If H is a proper subgraph of G then $f(H) < f(G)$.

It is easily verified that $\kappa(G)$, $t(G)$, $b(G)$ and $I(G)$ all satisfy i) but that ii) is satisfied only by $I(G)$ since both $\kappa(G)$ and $t(G)$ are 0 iff G is not connected ($S = \phi$ is allowed in the calculation of $t(G)$) and $b(G) = 0$ iff G has an isolated vertex.

Criterion iii) is not satisfied by any of the four measures of vulnerability since it requires such a measure to assume at least $\binom{n}{2}$ different values and $I(G)$, being integer valued, cannot do that.

We now turn our discussion to the entries of Table 2. The graphs $G_{n,k}$ perhaps best illustrate the inability of connectivity to provide a realistic measure of the vulnerability of graphs. Certainly disabling a station located at vertex v is less damaging to the operation of the remaining system when $k = 1$ than when $k = \lfloor (n-1)/2 \rfloor$. Yet neither $\kappa(G_{n,k})$ nor $t(K_{n,k})$ reflect this. Also, $b(G_{n,k})$ is quite insensitive to the value of k. On the other hand, $I(G_{n,k})$ provides a significant indication of the change in the nature of the structure of the system for $1 \le k \le (n-1)/2$.

For $G = T_{n,k}$ $\kappa(G)$ is totally insensitive to the value of k while $I(G)$ is a non-constant function of k only for limited values of k. On the other hand, both $t(G)$ and $b(G)$ reliably reflect the weakness of the system in terms of the degree of v.

4. CONCLUDING REMARKS

The edge analog of (vertex) connectivity is well studied. Chvátal [2] defined an edge analog of toughness but showed that its value for any graph G was one half of the edge connectivity of G. The authors [1] have defined the edge integrity I' of a graph by $I'(G) = \min_S(S + m(G-S))$ where $S \subseteq E(G)$. As with connectivity and edge connectivity we have $I(G) \le I'(G)$ for all graphs. However, in contrast to connectivity and edge connectivity, integrity and edge integrity have quite different properties for certain classes of graphs. For example we have shown that of all trees on n vertices the path P_n has the largest integrity while the star $K_{1,n-1}$ has the smallest but that $K_{1,n-1}$ has the largest edge integrity and P_n has the least.

Further measures of edge vulnerability have been devised by Lipman and Pippert [3,4].

They have studied the parameter $\lambda_i(G)$ which denotes the minimum number of edges which must be removed from G so as to separate at least i vertices from the remaining vertices. Also, in this connection, Robin Dawes has informed us that Kockay and Skillicorn are studying the function $v(G,i)$ which denotes the minimum cardinality of a subset S of $V(G)$ for which $G-S$ has at least two components each with at least i vertices.

We close with the observation that, except for binding number, all measures of vulnerability discussed depend on the concept of a graph being connected.

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