THE GENERALIZED MOORE GRAPHS ON ELEVEN VERTICES

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1. Introduction

The topological design of computer communications networks led to the analysis and development of generalized Moore graphs. For a consideration of network cost, delay, and reliability, we refer to [3]; these graphs also satisfy a criterion of minimal average path length, as discussed in [6]. A discussion of the existence or non-existence of the graphs for valence 3 in [2], [4], [5], [7], [8], and [9] provides a fairly complete census of the trivalent generalized Moore graphs. Initial work on a census of tetravalent generalized Moore graphs was begun in [1], where we discussed graphs having 10 or fewer vertices; in the current paper, we discuss the most complex small case, that of 11 vertices. The behavior of graphs having 12 to 20 vertices is treated in a parallel paper.

As in [1], we abbreviate "Generalized Moore Graph" to "GM graph", and assume that all graphs under discussion are regular and of valence 4 at every vertex. For completeness, we recall the definition of a GM graph (of valence 4). If we select a particular node as the root node, then there are 4 nodes at distance 1 from it, 12 nodes at distance 2 from it, 36 nodes at distance 3 from it, etc. There may, of course, be incomplete levels; a GM graph with 61 nodes would have exactly 8 nodes situated on the fourth level (at distance 4 from the root node). The essential feature of a GM graph is that the distance property holds, no matter what node is selected as the root node. In short, any vertex has the same "distance distribution" as any other vertex, and any node has the maximum possible number of nodes at distances 1,2,3, etc.

We also find it useful to refer to the girth of a GM graph; this is the length of the smallest cycle in the graph. If the graph happens to be complete (that is, if the m'th level is full), then the girth of the graph is 2m+1; on the other hand, if the m'th level is not complete, then there can be an edge of the graph that joins two points in the (m-1)th level, and so the girth of the graph is 2m-1.

In general, if we have a GM graph on N nodes which is regular and of degree V, then we denote it by the symbol M(N,V). In an earlier paper,we have discussed the graphs M(N,4) for N less than eleven; in this paper, we give the results for N = 11. As before, we abbreviate M(N,4) to M(N). We also mention that we have again used Kocay's isomorphism testing method throughout the paper, as described in [1].

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2. The Graphs M(11,4); Case (9,6,3).

We use the notation of [1]. Start with the three equations

$$a+b+c = 18, \quad 2a+b = 24, \quad b+2c = 12,$$

where a represents the number of joins at level 2, c represents the number of joins at level 1, and b represents the number of edges joining level-1 vertices to level-2 vertices. Since the six level-2 vertices must reach node 1 in two steps, there is a further restriction that b be at least 6.

The solutions for (a,b,c) are (6,12,0), (7,10,1), (8,8,2), and (9,6,3).

The (9,6,3) case will be discussed in this section. The only way to join the top 6 vertices with 9 lines and a valence of 3 at each node is to form M(6,3), for which there are only two graphs. There are, however, three ways for level 1 to be configured; thus we must discuss a total of 6 cases.

Case 2.1. The first case is set up in Figure 1.

Begin by joining vertices 2 and 6 by an edge. In order that 6 may reach both 4 and 5, join 7 to 4 and 9 to 5. To permit 7 to reach 2 in two steps, there is no loss in generality in making the join (10,2). To satisfy the distance property for vertex 9, (11,3) must be a join, leaving 5 to join to vertex 8. The completed graph and its Hamiltonian are presented in Figures 2A and 2B.

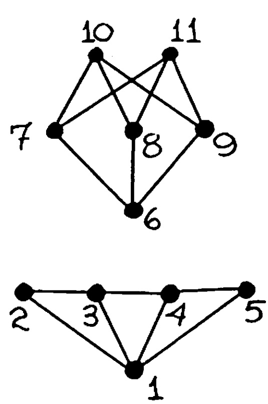

Figure 1

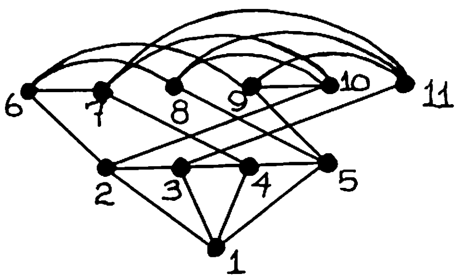

Figure 2A

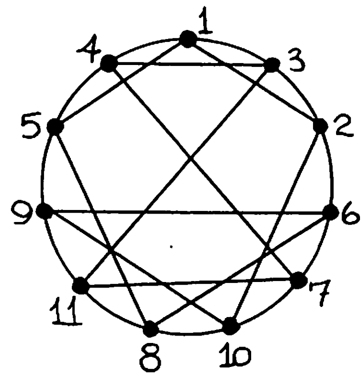

Figure 2B

We note that the M(6,3) graph illustrated as the top level in Figure 1 is bipartite with 3 vertices in each partition. Hence, the first join is completely arbitrary, and only one distinct graph graph can exist for this case.

Case 2.2. The next case has the same top level arrangement as the previous case, but the level-1 joins all emanate from a common vertex (Figure 3).

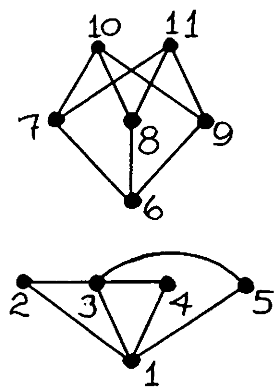

Figure 3

Since the top level is bipartite (the bipartition separates vertices 6, 10, and 11 from vertices 7, 8, and 9), the first join is arbitrary; take (6,2). There is no loss of generality in joining 7 to 4 and 9 to 5; this satisfies the distance criterion for 6. To permit 7 to reach 5 in two steps, select (10,5). Now, 11 must be joined to 4 so that 9 can reach 4 through at most 2 edges. Then an edge joins vertices 2 and 8. This graph is displayed in Figures 4A and 4B.

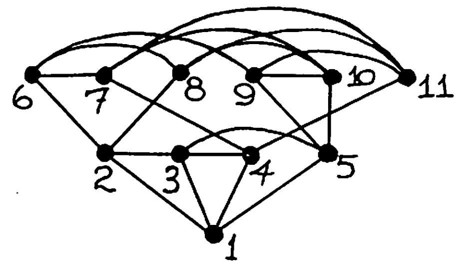

Figure 4A

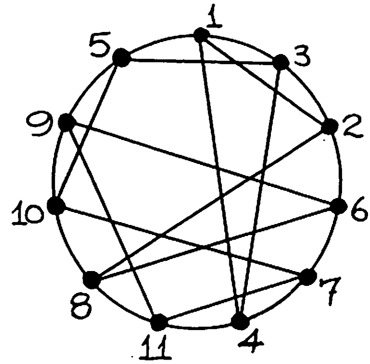

Figure 4B

Case 2.3. Figure 5 depicts a new level-1 configuration with the same top level configuration.

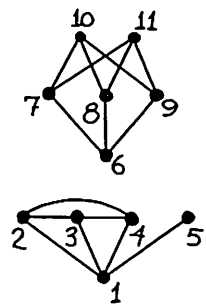

Figure 5

The first join is completely arbitrary; take (6,2). So that 6 may reach vertex 5 in two steps, select (5,8). To satisfy the distance criterion for vertex 8,we may join 10 to 3 and 11 to 4; then 5 is joined to both 7 and 9. Figures 6A and 6B show the completed graph.

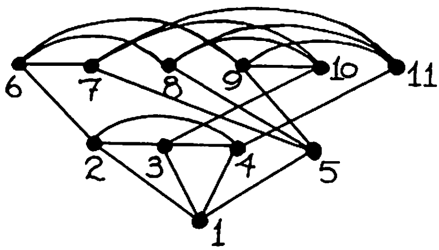

Figure 6A

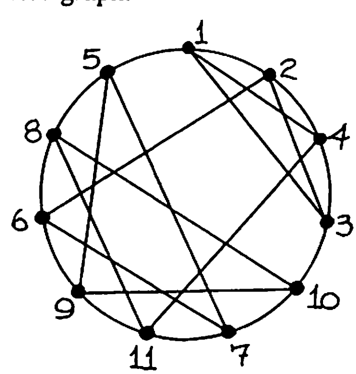

Figure 6B

Case 2.4. Figure 7 shows a new level-2 configuration, with a path spanning the level-1 vertices.

As a result of the symmetry of the top level,we may join 2 to 6. Now two possibilities exist: either vertex 8 or vertex 9 can be joined to 4.

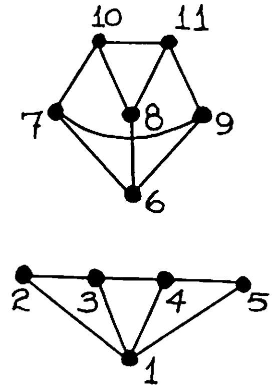

Figure 7

Let us make the join (9,4). If vertex 7 is joined to 3, then 3 can not reach both 8 and 11 in two steps; hence, 10 must be joined to 3. In order for 4 to reach 8, 5 must be joined to 8. For 2 to reach 11, the join (2,11) must be made, leaving 5 to join to 7. Figures 8A and 8B present the graph.

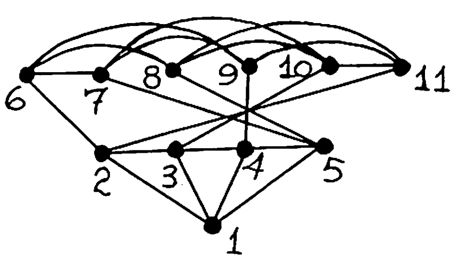

Figure 8A

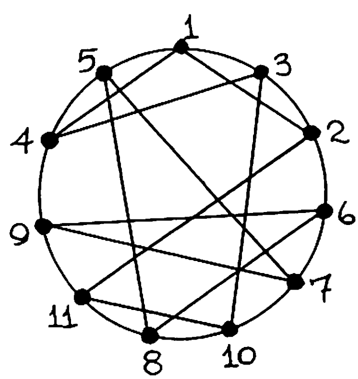

Figure 8B

On the other hand, if vertex 8 is joined to 4, we may take (9,5). Joining 7 to 3 gives a graph isomorphic to that of Figure 8A; so we join 11 to 3. Then 5 is joined to 10. The completed graph is shown in Figures 9A and 9B.

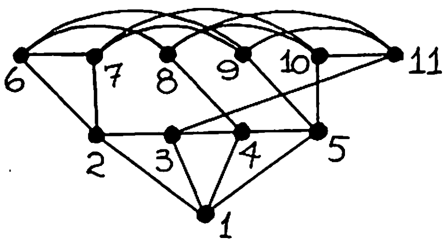

Figure 9A

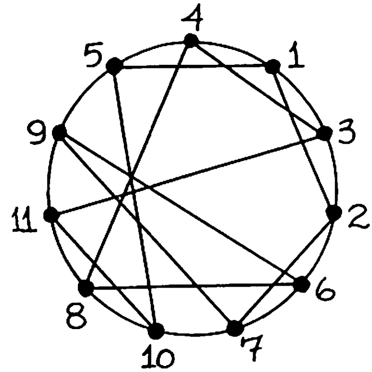

Figure 9B

Case 2.5. In Figure 10, all edges at level 1 emanate from a common vertex.

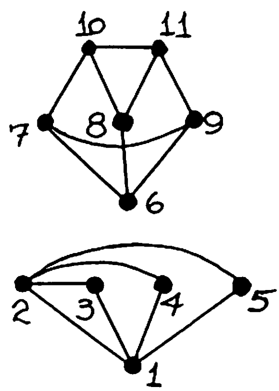

Figure 10

Without loss of generality, we make the joins (3,6) and (5,9) . Now, either 7 or 8 must be joined to 4.

Take the edge (8,4). So that 4 may be joined to both 7 and 9, take (7,4). If 3 is joined to 10, then 5 must be joined to 11, and we get the graph of Figures 11A and 11B. If 3 is joined to 11 (and thus 5 is joined to 10), we get the graph shown in Figures 12A and 12B.

Since all cases where both 7 and 8 are joined to 4 have been considered, the remaining cases must have 7 joined to 4 and 8 joined to some other vertex. The only new case occurs when 3 is joined to 8. To avoid a previous case, the join (4,10) must be made; then 5 is joined to 11. This graph is displayed in Figures 13A and 13B.

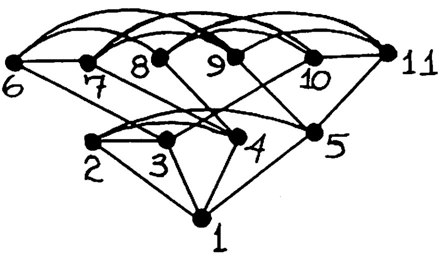

Figure 11A

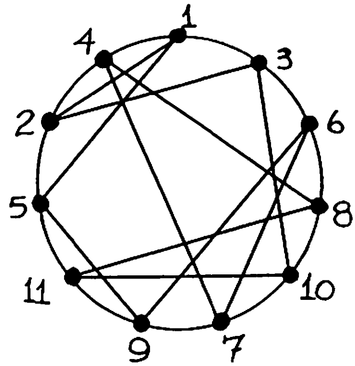

Figure 11B

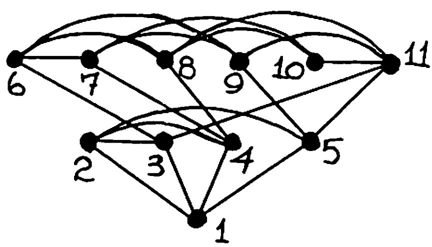

Figure 12A

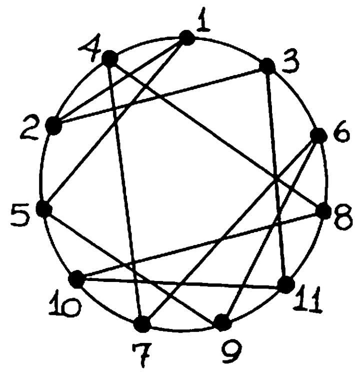

Figure 12B

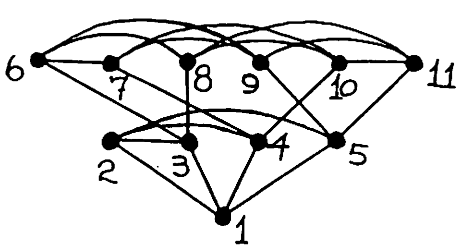

Figure 13A

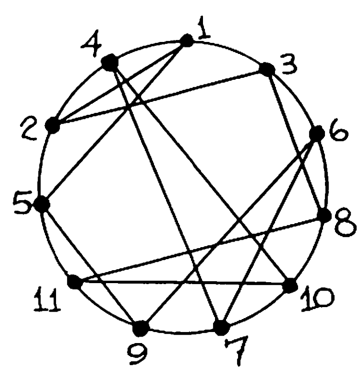

Figure 13B

Case 2.6. The last (9,6,3) subcase is depicted in Figure 14.

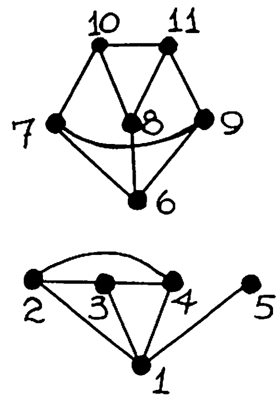

Figure 14

Without loss of generality, join 3 to 6. The distance property for 3 requires the joins (2,10) and (4,11). Then vertex 5 must be joined to 7, 8, and 9, and vertex 2 can not reach 9 through at most 2 edges.

We summarize our results so far in

Theorem 1. If (a,b,c) = (9,6,3), there are eight distinct tetravalent Moore graphs.

3. The Case (8,8,2).

If (a,b,c) = (8,8,2), there are two possible level-1 joins, and five level-2 configurations.

Case 3.1. The first of ten configurations is illustrated in Figure 15.

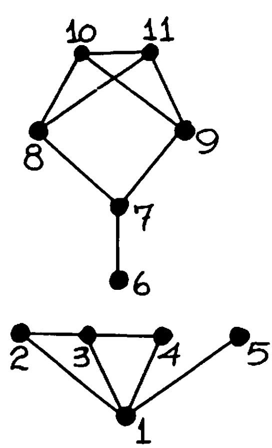

Figure 15

Since at most two 3-cycles may be formed through any vertex, we see that vertex 6 must be connected to 2, 4, and 5. Arbitrarily, we join 10 to 2, so that 6 can reach 10 through at most 2 edges; now, for 11 to reach 6 in two steps, it may be joined to either to 4 or to 5.

If 11 is joined to 4, then 3 must be joined to 7 in order to reach 7, 8, and 9 in two steps. Then 5 must be joined to both 8 and 9. Figures 16A and 16B illustrate the completed graph.

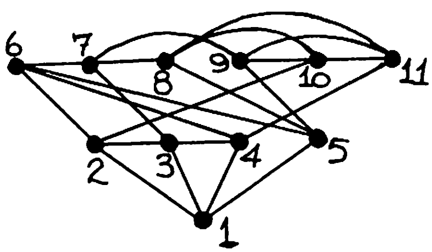

Figure 16A

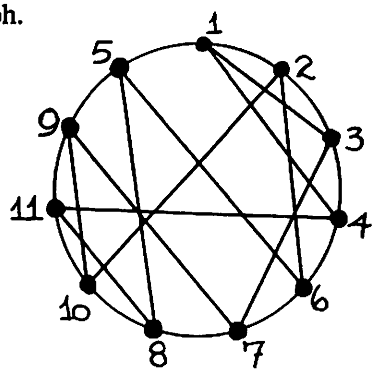

Figure 16B

If, on the other hand, 11 is joined to 5, then 10 must reach 4 by the edge (9,4). This forces the join (3,8); the remaining edge is (5,7). This graph is shown in Figures 17A and 17B.

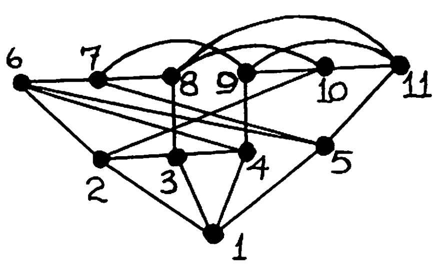

Figure 17A

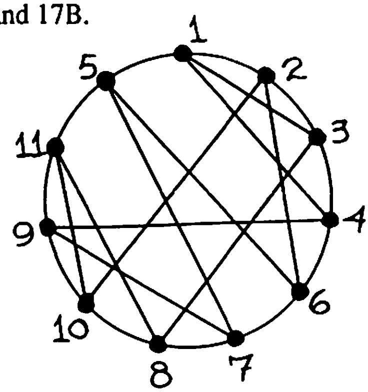

Figure 17B

Case 3.2. Figure 18 displays the next configuration .

Without loss of generality, 6 may be joined to 2, 3, and 4. Also, we may join 10 to 2. Now if 11 is joined to 3, then 10 must reach 4 and 5; so we take the joins (8,4) and (9,5). The final edge is (5,7); see Figures 19A and 19(B).

If 11 is joined to 4, then the joins (8,3) and (9,5) may be made; the final join is (5,7). This graph is displayed in Figures 20A and 20B.

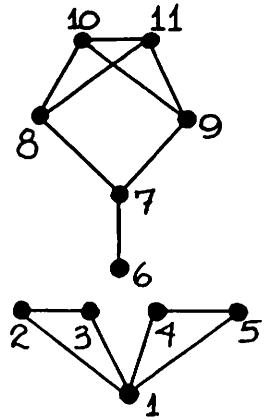

Figure 18

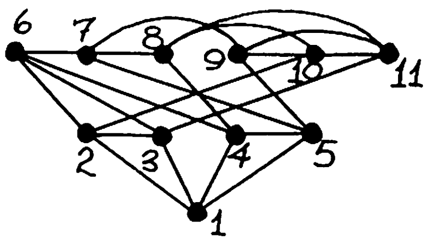

Figure 19A

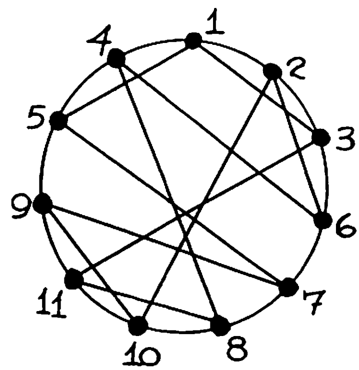

Figure 19B

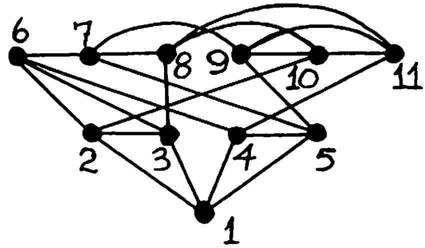

Figure 20A

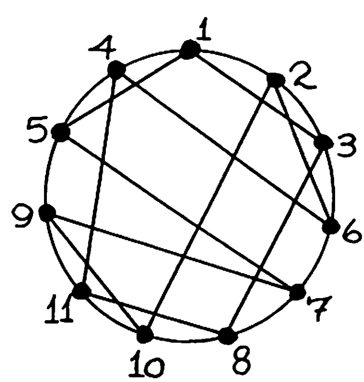

Figure 20B

Case 3.3. The third subcase starts out as illustrated in Figure 21.

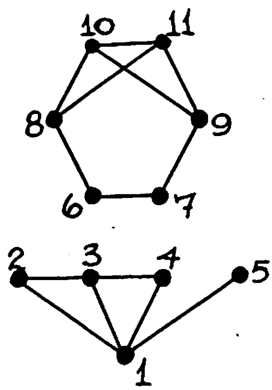

Figure 21

Vertex 6 can not be joined to both 3 and 5 (for then either 2 or 4 has to be joined to two of 9, 10, and 11, thus forming three triangles through either 10 or 11). Also, 6 can not be joined to both 2 and 4 (for then 5 would be joined to 7, 8, and 9); there is no loss of generality in joining 10 to 3, but then 11 can not reach both 2 and 4 in at most two steps.

Hence, 6 must be joined to both 2 and 5. If we apply the same argument to 7, we find that 7 must be joined to 2 and 5 or to 4 and 5; if 7 is joined to 2 and 5, then 2 can not reach both 10 and 11. Hence (7,4) and (7,5) are joins.

If 3 is joined to 8, then 2 may be joined to either 9 or 10. If (2,9) is an edge, then the joins (4,10) and (5,11) may be made without loss of generality. Figures 22A and 22B illustrate the completed graph. Figures 23A and 23B present the graph resulting from joining 2 to 10. The distance property requires the join (4,9); (5,11) is the final edge.

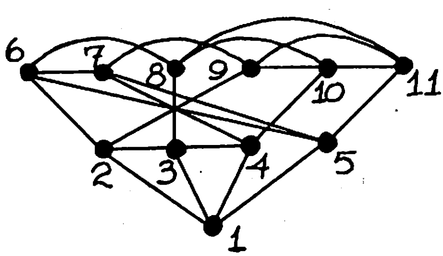

Figure 22A

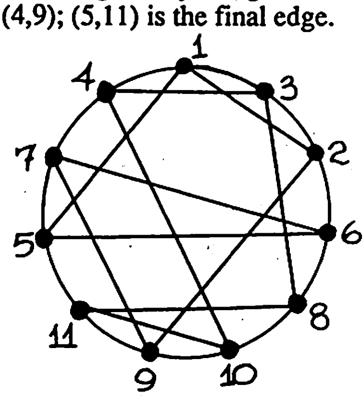

Figure 22B

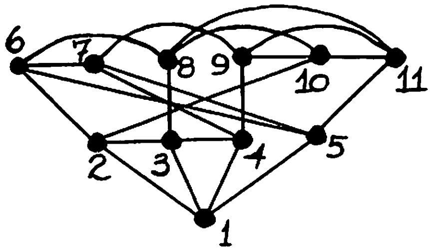

Figure 23A

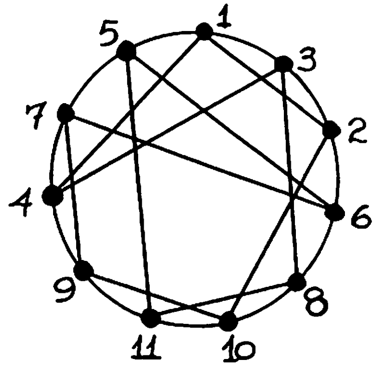

Figure 23B

Now consider the situation when 3 is joined to 10; 5 can be joined to either 9 or 11. The edge (5,9) forces (2,11) and (4,8); see Figures 24A and 24B.

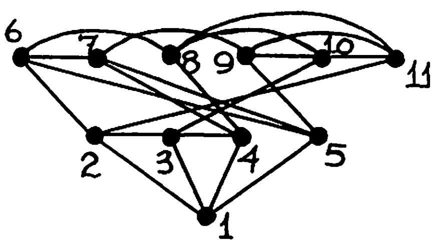

Figure 24A

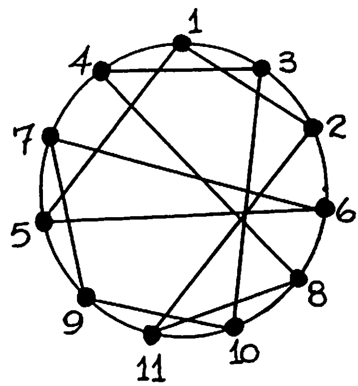

Figure 24B

If (5,11) is an edge, then (4,8) and (2,9) are forced; see Figures 25A and 25B.

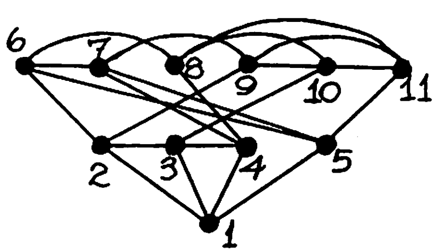

Figure 25A

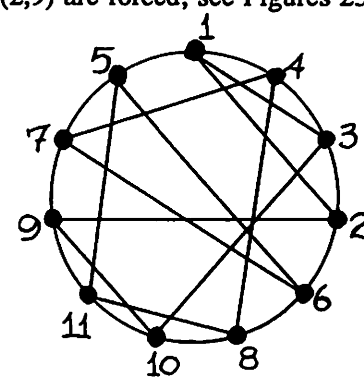

Figure 25B

Case 3.4. Figure 26 presents a configuration with different level-1 joins.

Vertex 2 can not be joined to 7 (the join (2,6) can be made with no loss of generality, and then 3 triangles are formed through 10 and 11). If 2 is joined to 6 and 8, 6 can not be joined to 3, but it may be joined to 4 without loss of generality. This forces edge (3,9). Next, the joins (10,4) and (11,5) may be made without loss of generality. Then 7 must be joined to both 3 and 5. Figures 27A and 27B illustrate the com - pleted graph.

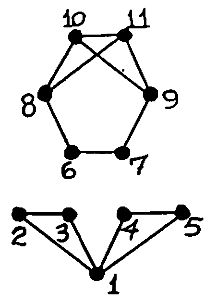

Figure 26

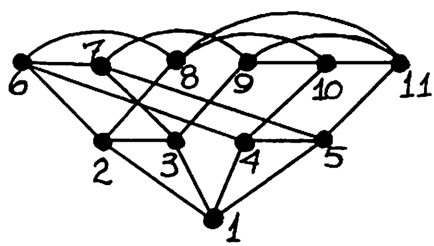

Figure 27A

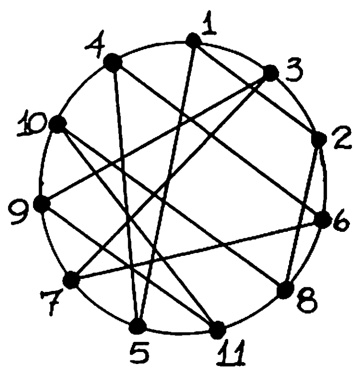

Figure 27B

If 2 is joined to 6 and 10, then 11 may be joined to either 3 or 5. First take (11,3). To avoid three triangles through 11, make the join (7,3). Now 6 may be joined to 4 without loss of generality; however, 7 can not joined to 4 (then 4 would not reach both 10 and 11 through 2 edges). So we select (7,5).

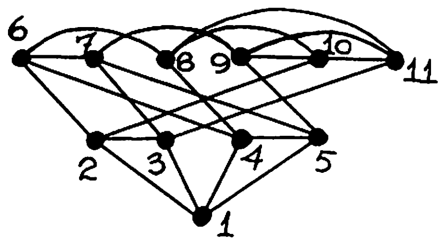

Figure 28

If (8,4) and (9,5) are joins, the graph (Figure 28) is isomorphic to the one in Figure 27. If (8,5) and (9,4) are joins, we obtain the graph in Figures 29A and 29B.

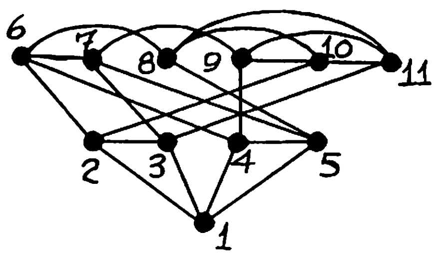

Figure 29A

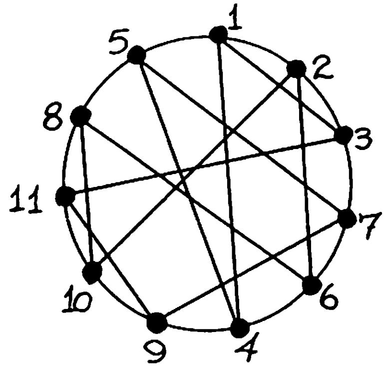

Figure 29B

If (11,5) is a join, (7,5) must also be. Now, if 6 is joined to 3, then 8 must be joined to 4. Then the join (7,4) is forced, as is (9,3). The completed graph (Figure 30) is isomorphic to the one of Figure 24.

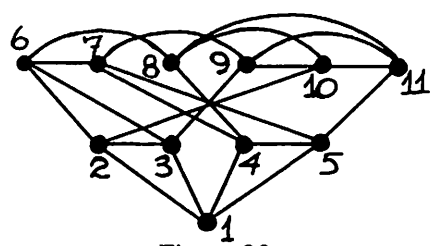

Figure 30

If (6,4) is a join, so must (7,3) be. The distance property requires (3,8) and (4,9). This case is represented in Figures 31A and 31B.

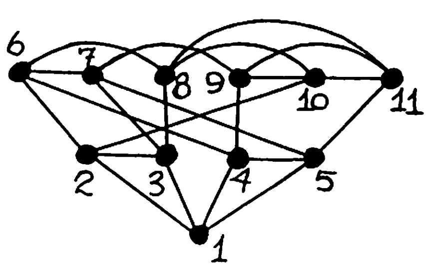

Figure 31A

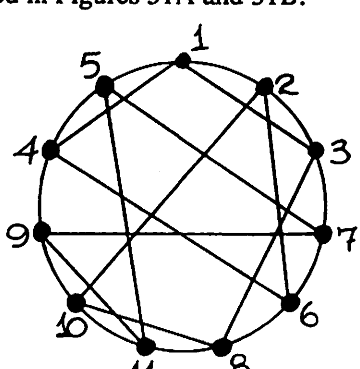

Figure31B

Case 3.5. Figure 32 shows another possible initial configuration.

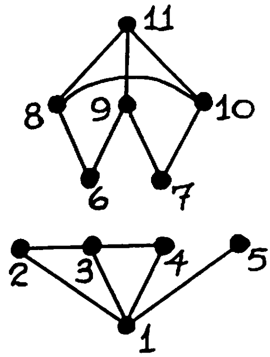

Figure 32

To begin, we join 6 to 2 and 4. If 2 and 4 are also joined to 7, the distance property for 2 and 4 requires (3,11); then 5 must be joined to 8, 9, and 10. The completed graph (Figure 33) is isomorphic to the graph in Figure 16.

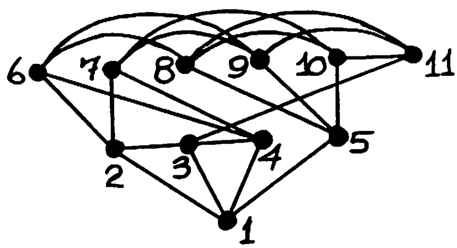

Figure 33

Let us now join 7 to 2 and 5. The distance property for 2 requires (3,11). In order for 4 to reach 7 and 10, it must be joined to10. Then 5 must be joined to 8 and 9. Figures 34A and 34B display the graph.

If 7 is joined to both 3 and 5, we take (8,2) so that 8 may reach 3 in two steps. Since 3 is connected to all but 11, we must take (4,11). Then 5 must be joined to 9 and 10 (Figure 35). The graph is isomorphic to the one in Figure 19.

Instead of joining 6 to 2 and 4, suppose we join 6 to 2 and 5. Three possible new cases result. If we also join 7 to 2 and 5, 3 must be joined to 11 to satisfy the distance property for vertex 2. To allow 5 to reach 11 in two steps, take (5,10); then 4 is joined to 8 and 9. The completed graph (Figure 37) is isomorphic to the graph of Figure 31.

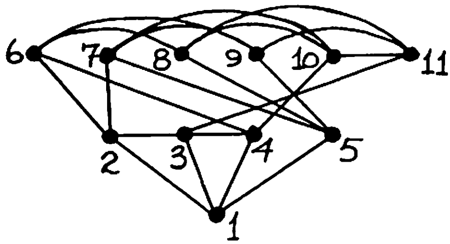

Figure 34A

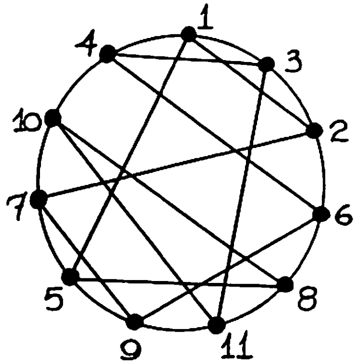

Figure 34B

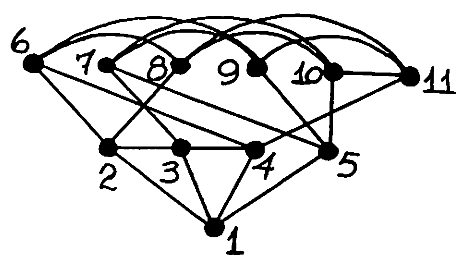

Figure 35

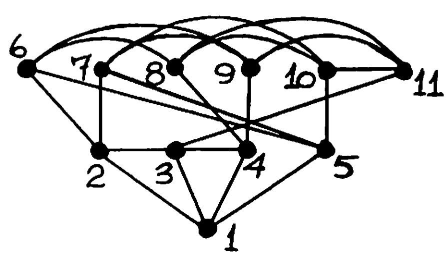

Figure 36

Now let 7 be joined to both 3 and 5. Since 8 must reach 3 in two steps, it must be joined to either 2 or 4. If 8 is joined to 2, then (4,11) is needed to satisfy the distance property for vertex 3. Since 4 must reach 6, we need (4,9) and (5,10). This graph (Figure 37) is isomorphic to that of Figure 29.

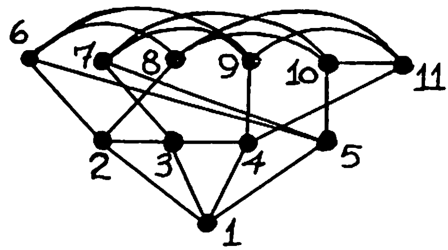

Figure 37

Now consider joining 6 to 2 and 5, and 7 to 4 and 5.

If 8 is joined to 4, the distance criterion for 2 requires (2,10). Figures 38A and 38B display the graph resulting from (9,3) and (11,5). Figure 39 displays the graph when (9,5) is a join; it is isomorphic to that in Figure 38.

If 9 is joined to 4, 10 must be joined to 2 to satisfy the distance cri - terion for 7. Figures 40A and 40B and 41A and 41B present the cases where (5,8) and (3,11) are joins, and when (5,11) and (3,8) are joins, respectively.

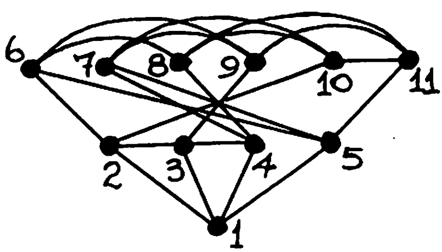

Figure 38A

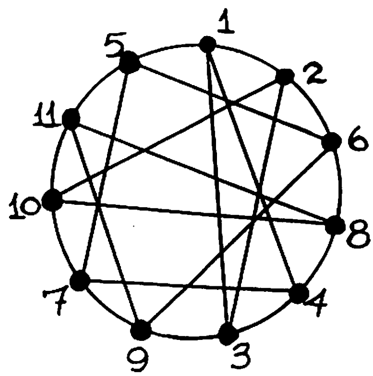

Figure 38B

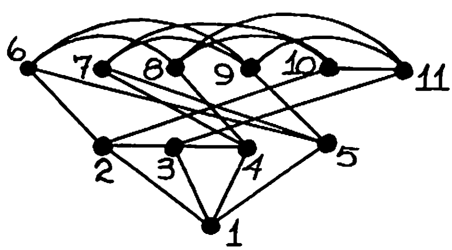

Figure 39

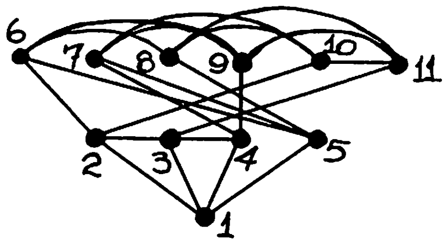

Figure 40A

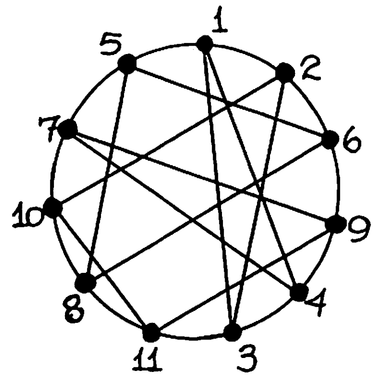

Figure 40B

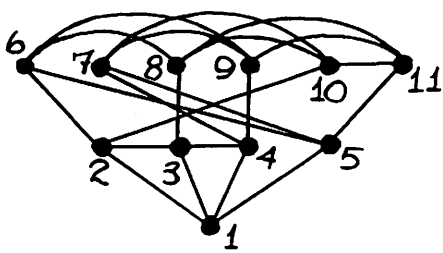

Figure 41A

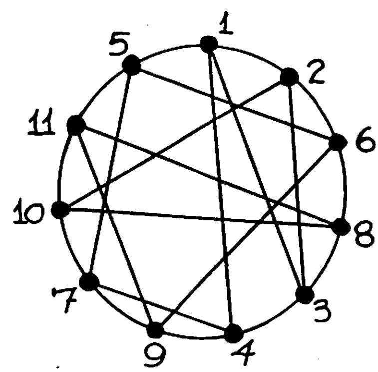

Figure 41B

Case 3.6. Figure 42 displays the next case in our discussion.

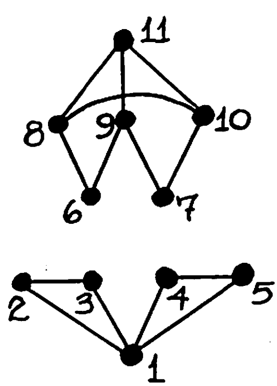

Figure 42

We commence by joining 6 to vertices 2 and 3. Since 6 must reach 4 and 5, there is no loss of generality if we form (8,4) and (9,5).

Now join 7 to 3. The distance property requires (2,11). If 7 is joined to 4, 5 is joined to10 (Figure 43); the graph is isomorphic to that in Figure 41 .

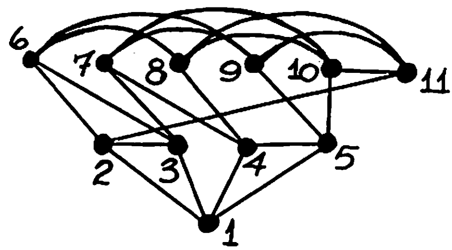

Figure 43

The graph that results from joining 7 to 5 (leaving 4 to join 10) is presented in Figure 44, and is isomorphic to the graph of Figure 23 .

Note that vertex 7 may not be joined to both vertices 4 and 5 (since three triangles would be formed through vertex 5).

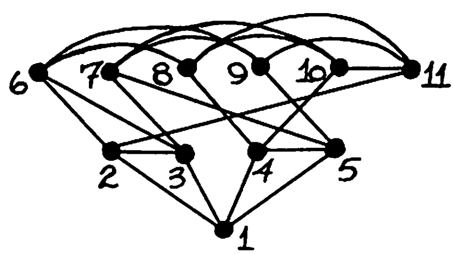

Figure 44

Consider now joining 6 to both 2 and 4. Vertex 7 can not be joined to both 2 and 4, since 2 and 4 would not be able to reach vertex 11. Thus, we first join 7 to both 2 and 5. To satisfy the distance property for 2, we must join 3 to 11. Now, either 8 or 10 may be joined to 3. Since the permutation (6 7)(4 5) leaves the graph unchanged, 6 and 7 are symmetric. Hence, take (8,3) without loss of generality. Since 4 must reach both 10 and 11, we require (4,10), and thus (5,11). Figures 45A and 45B display the graph.

Let us now join 7 to 3 and 5. Since (2,4) and (3,5) are symmetric pairs, 8 must be joined to either 2 or 3.

Suppose (8,2) is a join. If 3 is joined to 10, (4,11) is required by the distance property for 4; this forces (5,9) and the distance property for 5 is not satisfied. Hence, 3 must be joined to 11, and this forces (5,10), and then (4,9). Figures 46A and 46B depict the completed graph.

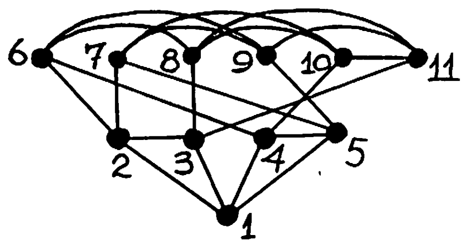

Figure 45A

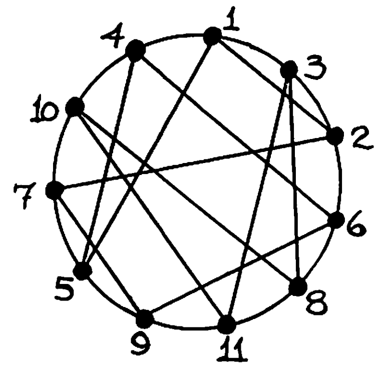

Figure 45B

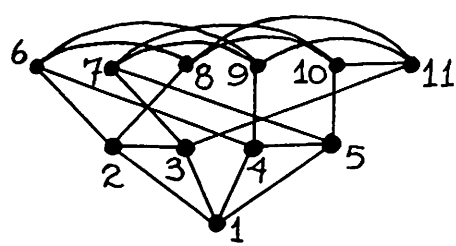

Figure 46A

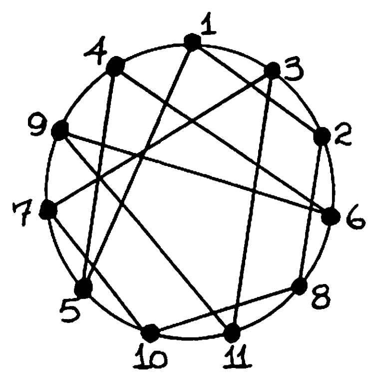

Figure 46B

We may also join 3 to 9. The only non-isomorphic case arises when 4 is joined to 10. The graph is shown in Figures 47A and 47B.

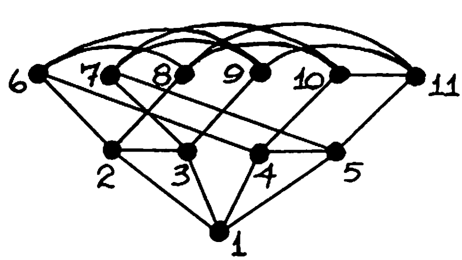

Figure 47A

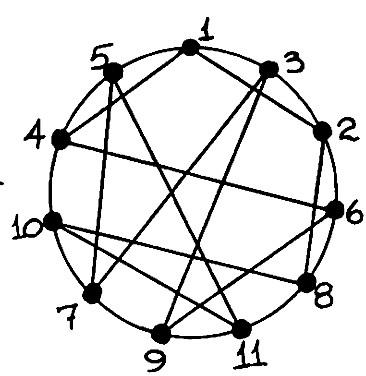

Figure 47B

Now, let us join 8 to 3 instead of to 2. We can not join 2 to 10 since the distance property would not be satisfied for 4 (or 5). If 2 is joined to 11, no new graph is obtained; hence we must join 2 to 9. The graph that results from joining 4 to 10, and 5 to 11, is shown in Figures 48A and 48B. Figures 49A and 49B illustrate the graph if (4,11) and (5,10) are edges.

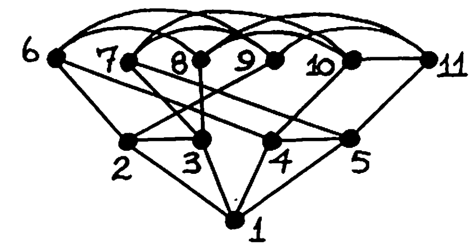

Figure 48A

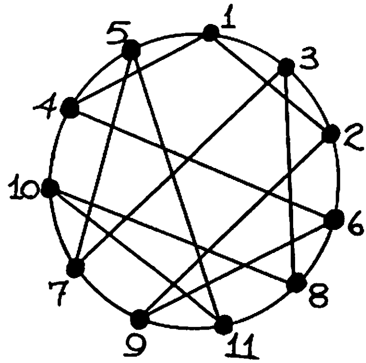

Figure 48B

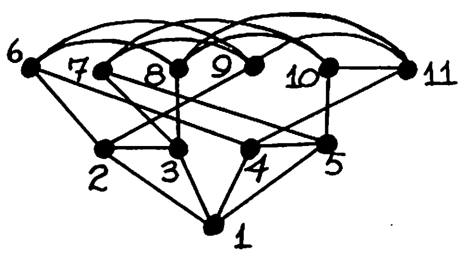

Figure 49A

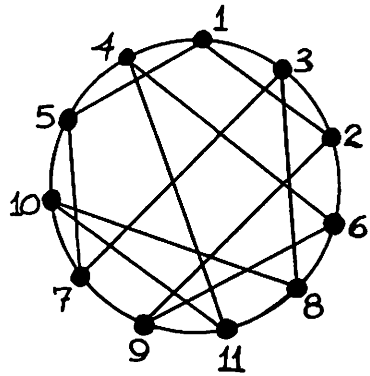

Figure 49B

Case 3.7. Figure 50 shows the next initial level-1 and level-2 configurations.

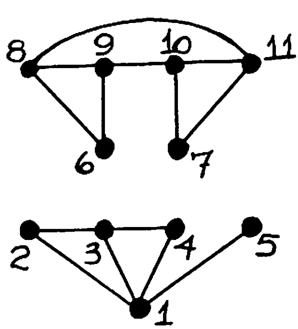

Figure 50

From symmetry, there are only three possibilities that avoid three triangles through a vertex. The first possibility is to join 6, 10, and 11 to 5. To avoid three triangles through a vertex, 7 must be joined to 2 and 4; but then, only one of 10 and 11 can reach 3 in two steps.

Next consider joining 5 to 6, 7, and 11. If (7,3) is a join, there is no loss of generality in taking (6,2). The distance criterion forces (2,10); so 4 is joined to 8 and 9. The graph (Figure 51) is isomorphic that in Figure 24.

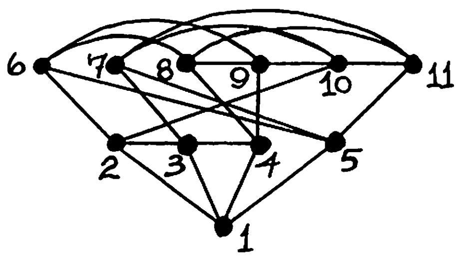

Figure 51

If 7 is joined to 4 rather than to 3, then (2,10) must be a join. So that 11 may reach 3, take (8,3). Then we either have (4,6) and (2,9); or we have (4,9) and (2,6). The respective graphs (Figures 52 and 53) are isomorphic to the graphs in Figures 23 and 22.

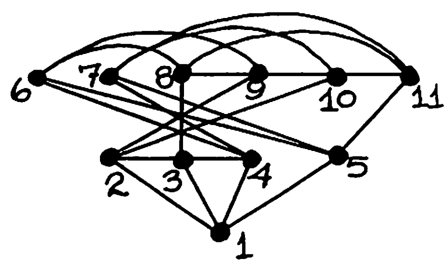

Figure 52

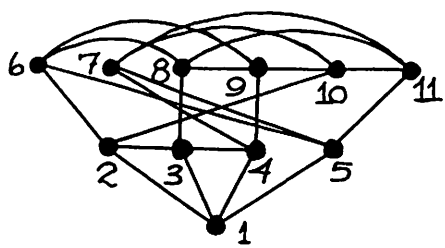

Figure 53

Finally, join 7, 9, and 11 to 5. To avoid three triangles through a vertex, join 6 to 2 and 4. If 7 were joined to 3, 10 could not reach 2 and 4 through two edges; so join 7 to 4. Then the distance property for 7 requires (2,10) and (8,3). The graph (Figure 54) is isomorphic to that in Figure 17.

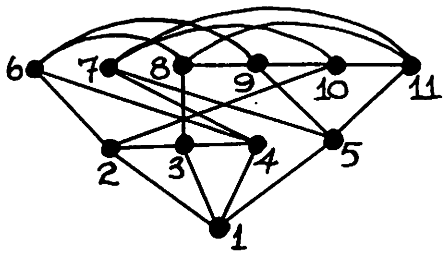

Figure 54

Case 3.8. Figure 55 displays another initial case.

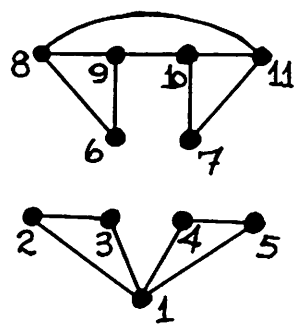

Figure 55

First, consider joining 6 to 2 and 3, and 7 to both 3 and 4. There is no loss of generality in joining 2 to 10, since it must reach both 10 and 11. Since 8 and 9 must be joined to 4 and 5 to satisfy the distance property for 6, we require (5,11). The joins (5,8) and (4,9) give a graph (Figure 56) that is isomorphic to that in Figure 27; the joins (5,9) and (4,8) give a new graph shown in Figures 57A and 57B.

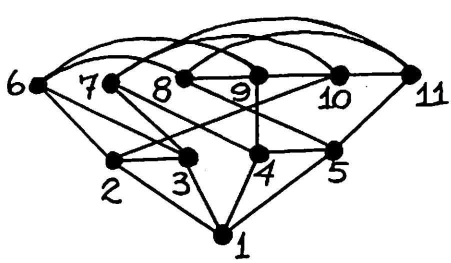

Figure 56

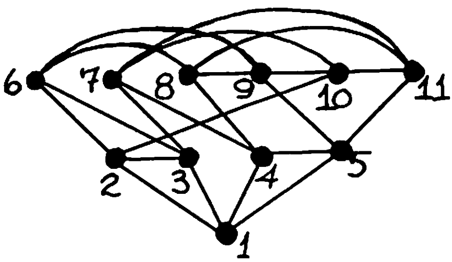

Figure 57A

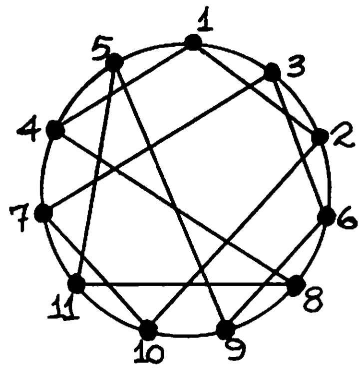

Figure 57B

Another possibility is to join 6 to both 2 and 3, 7 to both 4 and 5. To avoid forming three triangles through a vertex, we may, without loss of generality, take edges (8,4), (9,5), (10,2), and (11,3). See Figures 58A and 58B.

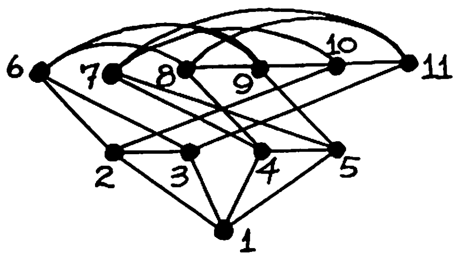

Figure 58A

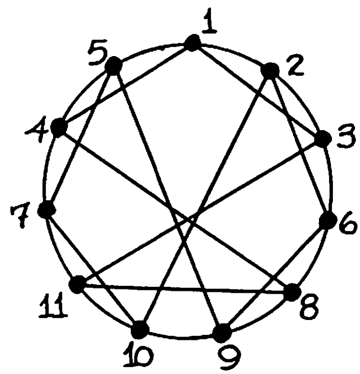

Figure 58B

Consider next joining both 6 and 7 to both 3 and 4. If both 10 and 11 are joined to 5, 8 and 9 must be joined to 2. The graph (Figure 59) is isomorphic to that in Figure 19.

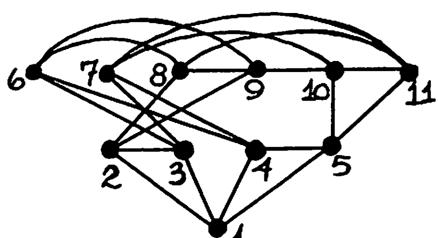

Figure 59

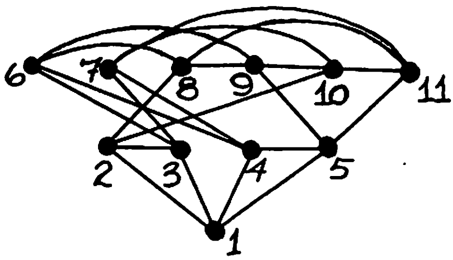

Figure 60A

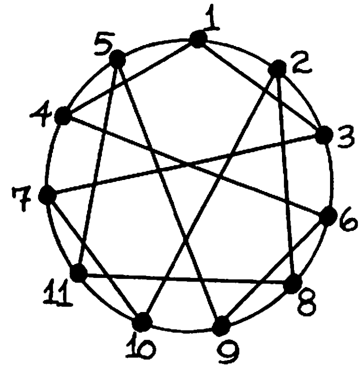

Figure 60B

Figures 60A and 60B illustrate the graph that results when both 11 and 9 are joined to 5, leaving 8 and 10 to be joined to 2.

Another graph results if 8 and 11 are joined to 5; then 9 and 10 are joined to 4. The graph (Figure 61) is isomorphic to that in Figure 46.

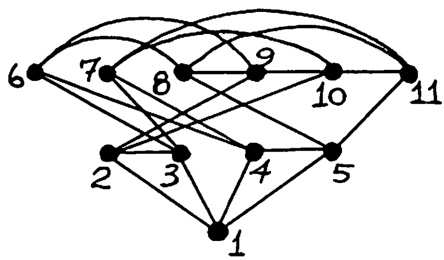

Figure 61

Now consider joining 6 to 2 and 4, while joining 7 to both 3 and 5. We first select (11,4). If (5,8) is a join, then we require (9,3) and (10,2). The graph is shown in Figures 62A and 62B.

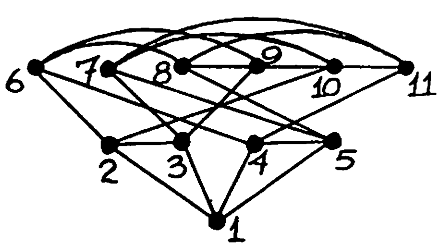

Figure 62A

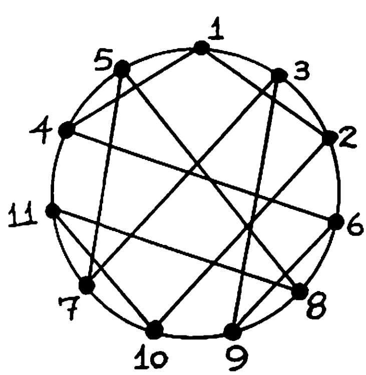

Figure 62B

If (5,9) is a join, then the edges (2,8) and (3,10) give the graph shown in Figures 63A and 63B. The joins (2,10) and (3,8) give the graph shown in Figures 64A and 64B.

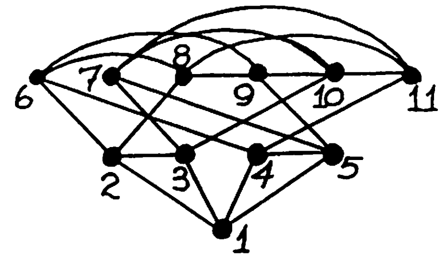

Figure 63A

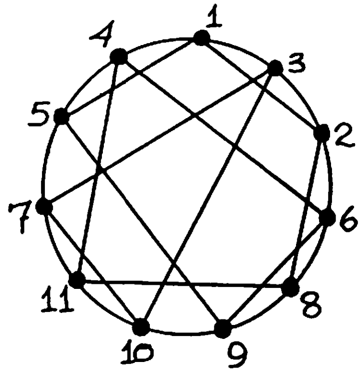

Figure 63B

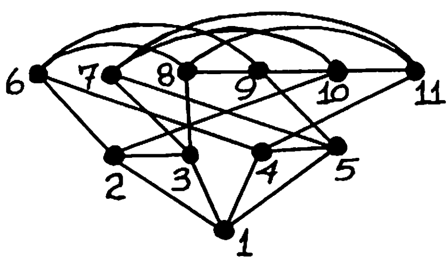

Figure 64A

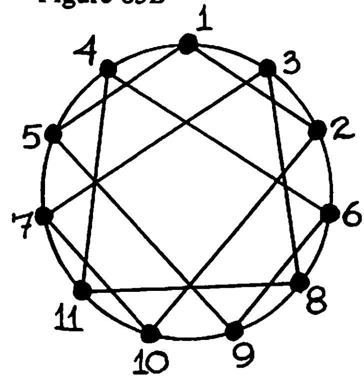

Figure 64B

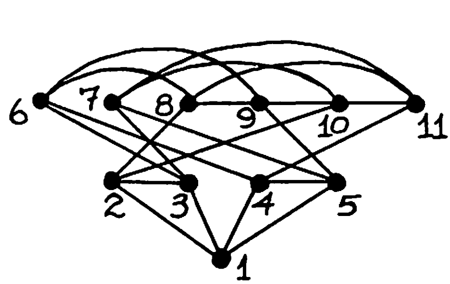

Figure 65A

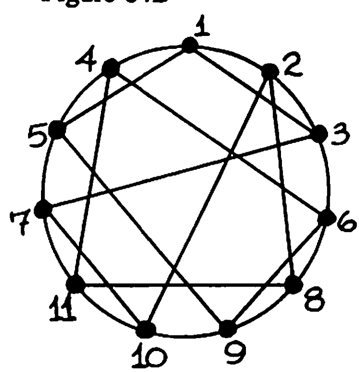

Figure 65B

The case of a join (5,11) need not be discussed, since we merely switch the roles of (2,3) and (4,5). Hence, we finally consider joining 6 to 3 and 4, and joining 7 to 3 and 5. Without loss of generality, we select

(2,8). So that 8 may reach 5 in two steps, we take (9,5). The selection (10,2) and (11,4) gives the graph shown in Figures 65A and 65B.

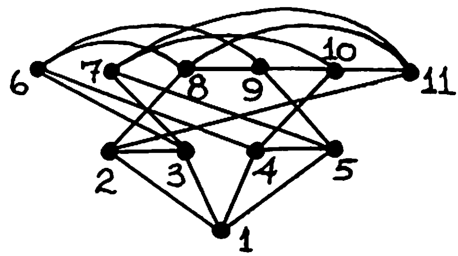

Figure 66

The selection (10,4) and (11,2) gives Figure 66, which is isomorphic to that in Figure 47. If (11,5) is a join, the distance property requires (9,4) and (10,2). The completed graph (Figure 67) is isomorphic to the graph of Figure 23.

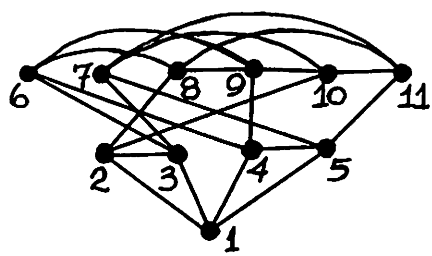

Figure 67

Case 3.9. The ninth (and penultimate) major subcase is shown in Figure 68.

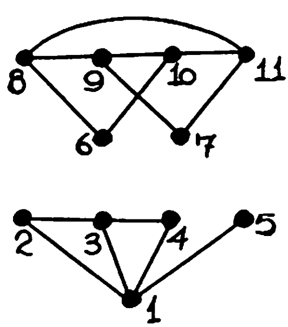

Figure 68

First, we note that both 6 and 7 can not be joined to both 2 and 4 (if they do, there is no loss of generality in joining 8 to 3; but then 10 can not reach 3 in two steps). If 6 is joined to 2 and 4, and 7 to 4 and 5, the distance property for 7 allows us to select (9,2) arbitrarily. Then the distance property for 3 requires (3,11); so 8 and 10 must both be joined to 5. Figures 69A and 69B display the completed graph.

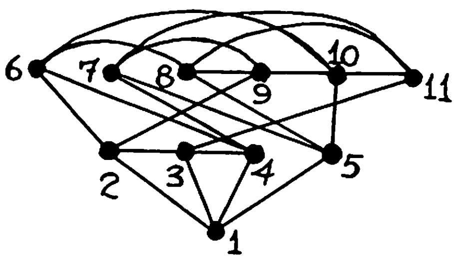

Figure 69A

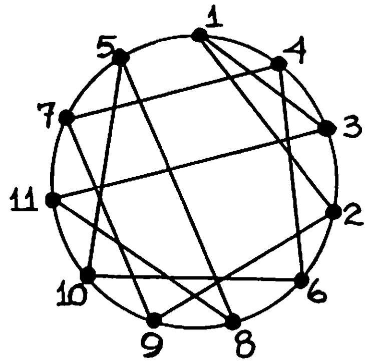

Figure 69B

If 6 is joined to both 2 and 4, and 7 to both 3 and 5, then 10 may be joined to 5 so that 6 can reach 5 in 2 steps; then 10 can not reach 3 in two steps.

If 6 is joined to 2 and 5 and 7 is joined to 3 and 5, the distance property for 6 allows us to select (10,4). Then we require (4,8), and (9,2) is needed as a direct join. Then the distance property for 11 can not be satisfied.

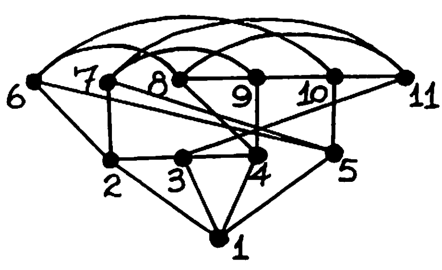

Figure 70

Thus we join both 6 and 7 to 2 and 5. So that 6 may reach 4 in two steps, take (9,4). Now 4 may not be joined to 11 (it would not reach 6 in two steps); hence, with no loss of generality, we choose (11,3) and (10,5). The final join is (8,4). The graph (Figure 70) is isomorphic to that in Figure 43.

Case 3.10. The final (8,8,2) subcase is illustrated in Figure 71.

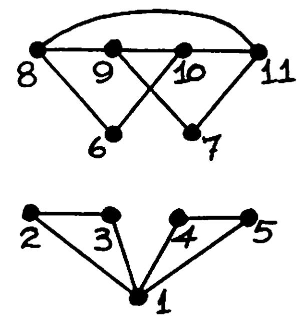

Figure 71

Both 6 and 7 can not be joined to both 2 and 3 (one may not have three triangles through a vertex). If 6 is joined to 2 and 3, and 7 to 3 and 4, then 6 must reach 4 and 5 in at most two steps. From symmetry, take (8,4) and (10,5); then vertices 9 and 11 can not reach 2 in two steps.

If we join 6 to 2 and 3, 7 to 4 and 5, the distance property for 6 allows us to take (8,4) and (10,5). The distance property for 7 requires (9,2) and (11,3). The graph is shown in Figures 72A and 72B.

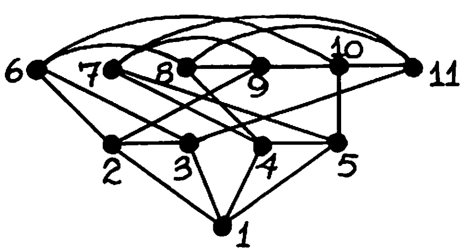

Figure 72A

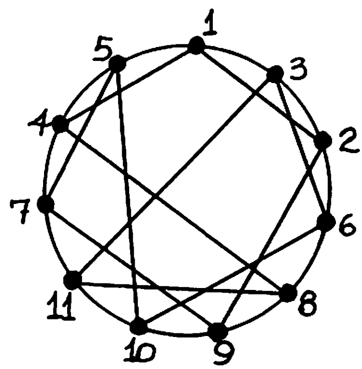

Figure 72B

Now join both 6 and 7 to 3 and 4. Without loss of generality, we select (11,5). Now, 5 may reach 9 by a join to 8, 9, or 10. Since (5,8) and (5,10) are symmetric, select (5,10). The other edges are (8,2) and (9,2), and the graph is shown in Figures 73A and 73B. If 5 is joined to 9, 8 and 10 must be joined to 2, and the graph is shown in Figures 74A and 74B.

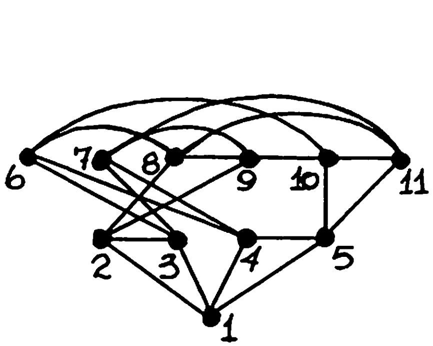

Figure 73A

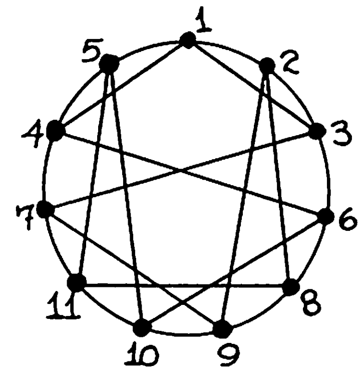

Figure 73B

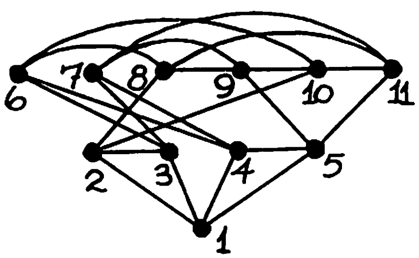

Figure 74A

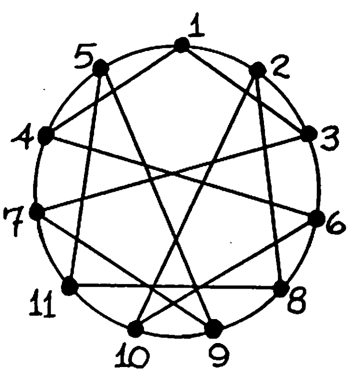

Figure 74B

Suppose we now join 6 to both 3 and 4, and 7 to both 3 and 5. Since two of 8, 9, 10, 11, must be joined to 2, take (8,2) arbitrarily. Then (2,10) is not possible (11 would not reach 4 in two steps). By symmetry, we take (9,2). The distance property requires (10,4); the final edge is (11,5). The graph (Figure 75) is isomorphic to the one in Figure 47.

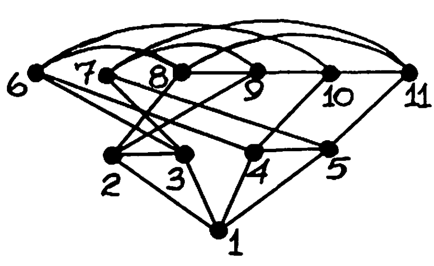

Figure 75

Finally, join 6 to 2 and 4, 7 to 3 and 5. If (8,2) is an edge, symmetry of (9,5) and (11,5) in satisfying the distance criterion for 8 allows us to take (9,5).

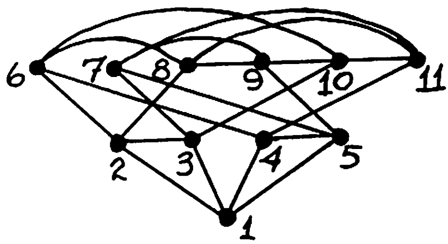

Figure 76A

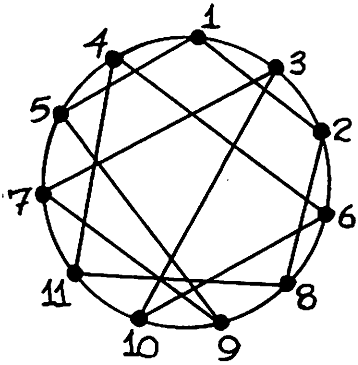

Figure 76B

Figures 76A and 76B present the graph when (3,10) and (4,11) are edges; Figures 77A and 77B when (3,11) and (4,10) are edges.

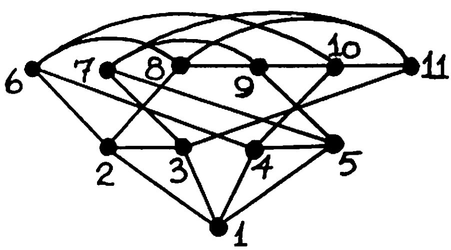

Figure 77A

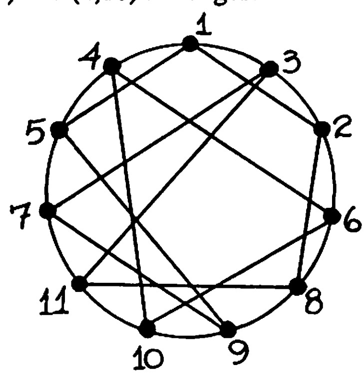

Figure 77B

Instead of joining 2 to 8, we may join 2 to 9. So that 2 may reach 11, take (3,11). Without loss of generality, we now take (8,4) and (10,5). Figure 78 illustrates the completed graph, which is isomorphic to that in Figure 76.

We summarize the results of this section in

Theorem 2. If (a,b,c) = (8,8,2), there are 33 non-isomorphic tetravalent GM graphs that are not subsumed in the earlier (9,6,3) case.

Note that these graphs, unlike the graphs in Theorem 1, do not have three triangles at any vertex.

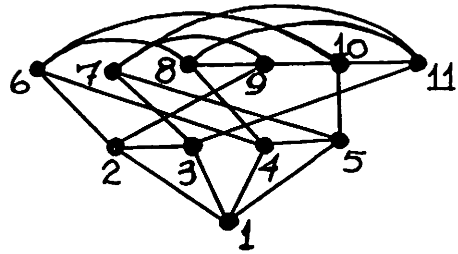

Figure 78

4. The (7,10,1) Case.

If (a,b,c) = (7,10,1), there are only seven subcases.

Case 4.1. We start with the situation of Figure 79. To avoid forming two triangles (or more) through any vertex, 6 must be joined to 2, 3, and 5.

To avoid two triangles through a vertex, 7 can not be joined to 3. If (7,2) is a join, the distance criterion requires 8 or 9 to be joined to 2. This creates two triangles through 2; hence 7 must be joined to 4.

So that 10 may reach 6 in two steps, take (10,3); similarly, take (11,5). In order for 5 to reach 10, and for 2 to reach 11, in at most two steps, take (5,9) and (2,8), without loss of generality. The graph (Figures 80A and 80B) is completed by edges (10,2) and (11,4).

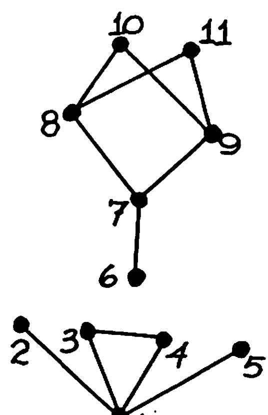

Figure 79

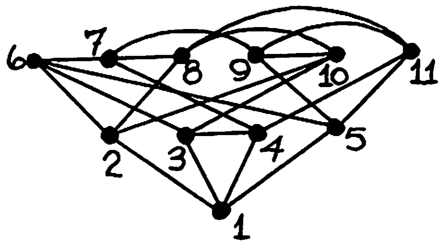

Figure 80A

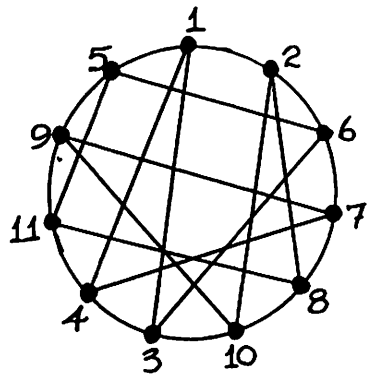

Figure 80B

If we join 11 to both 2 and 5, and join 10 to 2, there is no loss of generality in taking (8,3) so that 10 and 11 can reach 3 in two steps. The distance criterion for 9 requires (9,4). The edge (10,5) completes the graph (Figure 81), which is isomorphic to that in Figure 74.

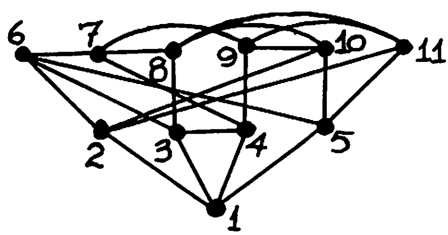

Figure 81

Case 7.2. Figure 82 presents a different level-2 configuration.

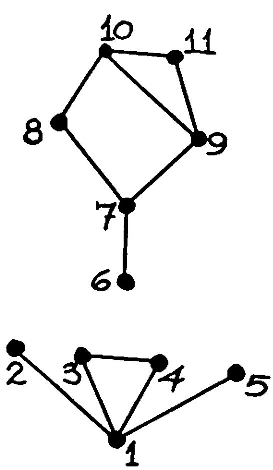

Figure 82

We must join 6 to 2, 3, and 5 to avoid creating two triangles through a vertex. Since 11 can not be joined to both 3 and 4, it must be joined to 2 or 5; choose (11,5).

If we choose (10,2), so that 6 may reach 10 in two steps, we take (7,2) to avoid two triangles through 10. Now 8 can not be joined to both 3 and 4, nor to 2; so we take (8,5). To avoid two triangles through 11, we need (8,4). Joins (9,3) and (11,4) give the graph in Figures 83A and 83B. Joins (9,4) and (11,3) give a graph (Figure 84) isomorphic to that in Figure 80.

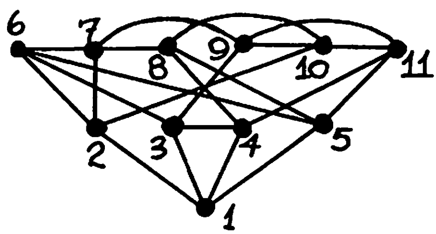

Figure 83A

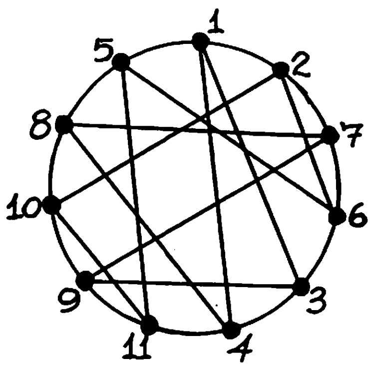

Figure 83B

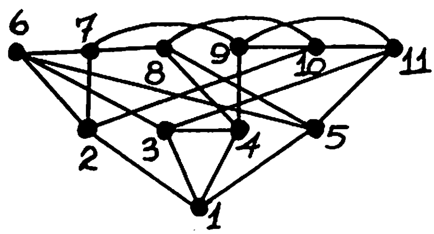

Figure 84

If 10 is joined to 3 so that 6 can reach 10 in two steps, and if 5 is joined to 7 to reach 8 through at most two edges, then 8 must be joined to both 2 and 4. The joins (4,9) and (2,11) give a graph (Figure 85) isomorphic to that in Figure 84. The joins (4,11) and (2,9) give a graph (Figure 86) isomorphic to that in Figure 83.

Vertex 5 can also be joined to 8. To avoid obtaining a previous graph,we must take (7,4). This forces the edge (11,4); then 8 and 9 must be joined to 2. The graph is displayed in Figures 87A and 87B.

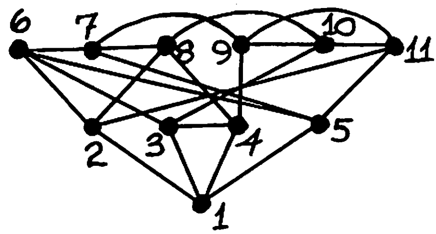

Figure 85

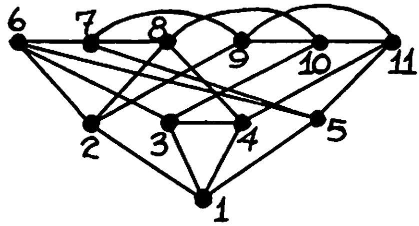

Figure 86

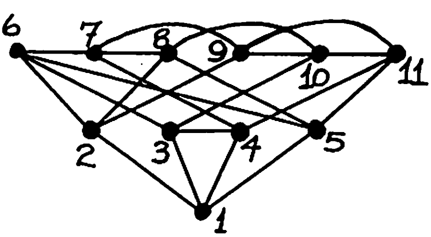

Figure 87A

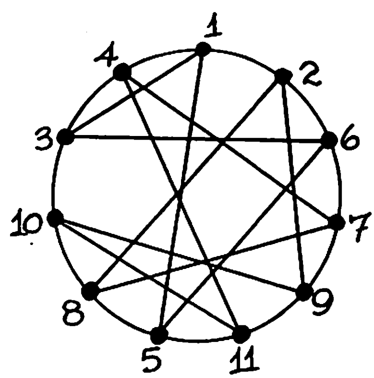

Figure 87B

Case 7.3. Figure 88 presents the next possible level-2 configuration.

Vertex 6 must be joined to 2, 3, and 5 (to avoid two triangles at a vertex). Similarly, 7 must be joined to 4. Since 3 can not be joined to both 10 and 11 (this would result in two triangles through 4), 3 may be joined to either 8 or 9; we choose (3,8), forcing the join (4,11). Vertex 10 can only be joined to 2 and 5; since 2 and 5 are symmetric as regards 11, take (11,2), and hence (9,5). The graph (Figure 89) is isomorphic to that in Figure 83.

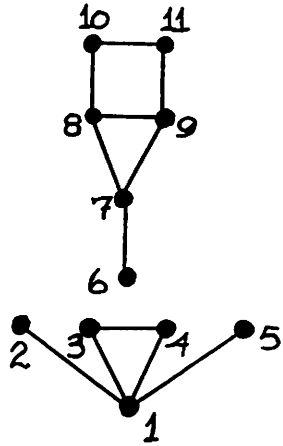

Figure 88

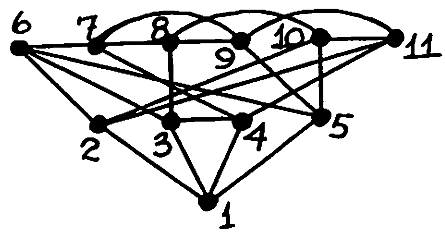

Figure 89

Case 7.4. Another level-2 configuration is displayed in Figure 90.

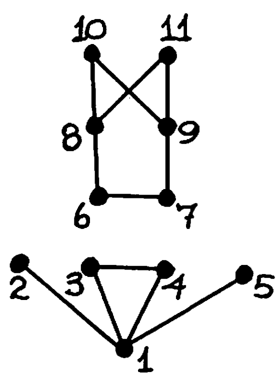

Figure 90

If 6 is joined to 2 and 3, and 7 to 2 and 4, then the distance property for 6 and 7 forces 8 and 9 to join 5. Now, joining either 10 or 11 to 5 forces two triangles through the same vertex.

If 6 is joined to 2 and 3, and 7 to 2 and 5,we require (9,4). To avoid two triangles through 4, we require (8,4). Without loss of generality, we take (3,10). Then 11 must be joined to 2 and 5, and 10 to 5. But then 10 can not reach 2 through two edges.

Thus, 6 must be joined to 2 and 3, and 7 to 4 and 5. The distance criterion for 6 and 7 requires (8,5) and (9,2). Since 10 and 11 are symmetric, form (10,3) and (11,4); finally, choose (10,2) and (11,5). The graph (Figure 91) is isomorphic that in Figure 80.

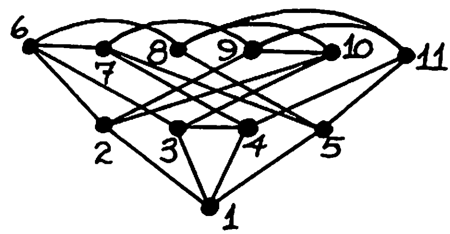

Figure 91

Case 7.5. Figure 92 displays another possible level-2 configuration.

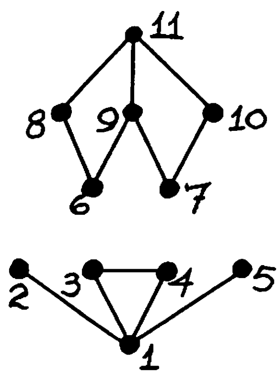

Figure 92

Only one new case results from this configuration. This graph occurs when 6 is joined to 2 and 3, and 8 to 4 and 5. If 7 is joined to 4, then 3 must be able to reach both 10 and 11. To avoid previous graphs, join 3 to 10, and 7 to 2. There is only one set of joins that avoids two triangles through a vertex, namely, (11,2), (9,5), and (10,5). Figures 93A and 93B present the graph; any other graph is isomorphic to either Figure 83 or Figure 87.

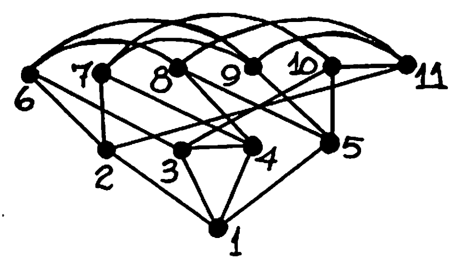

Figure 93A

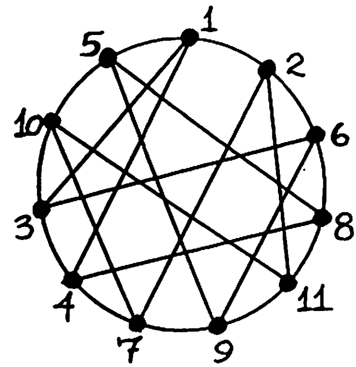

Figure 93B

Case 7.6. Figure 94 displays the penultimate level-2 configuration.

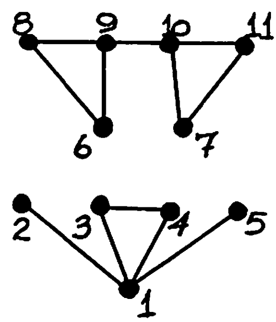

Figure 94

Both 6 and 8 must be joined to both 7 and 11 without forming any more triangles; the pigeon-hole principle shows this to be impossible.

Case 7.7. The last (7,10,1) case is illustrated in Figure 95.

First,we join 6 to 2 and 3. If 8 is joined to 2 and 4, the distance property for 8 requires (9,5). In order for 9 to reach 3 in two steps, take(7,3). Joining 7 to 5 produces two triangles through a vertex; so we take (7,2). But then we have (10,5) and (11,5), and this gives two triangles through 5.

If 8 is joined to 2 and 5, take (9,4) so that 8 may reach 4 in two steps. Since 7 and 10 may not be joined to 4, take (11,4). If 3 is joined to 7, then 7 can not reach 2 and 5 in two steps; if 3 is joined to 11, we need (7,2), (7,5), and (10,5). This gives two triangles through 3 and 4.

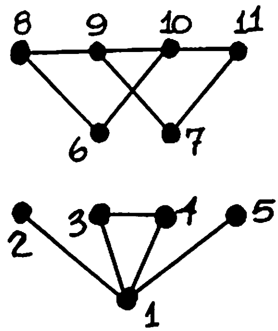

Figure 95

If 8 is joined to 4 and 5, then 3 can not be joined to 11 (this forces (4,9) and thus two triangles through 4). If 3 is joined to 9, the distance criterion for 4 requires (4,11). Then (7,2) and (7,5) are forced. The joins (11,2) and (5,10) give a graph (Figure 96) isomorphic to that in Figure 87; the joins (5,11) and (10,2) give a graph (Figure 97) isomorphic to that in Figure 83.

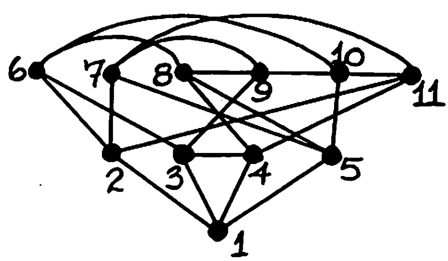

Figure 96

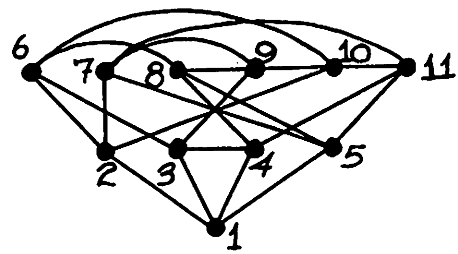

Figure 97

If 3 is joined to 7, then 11 may be joined to 2. If 7 is joined to 2, we need (9,5). To avoid two triangles at a vertex, we require (10,4) and then (11,5). The graph (Figure 98) is isomorphic to that in Figure 83.

If we take (7,5), we need (10,5) and (11,4) to avoid two triangles at a vertex. Take (9,2), and the graph (Figure 99) is isomorphic to that in Figure 93 .

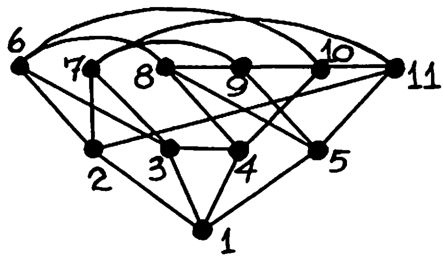

Figure 98

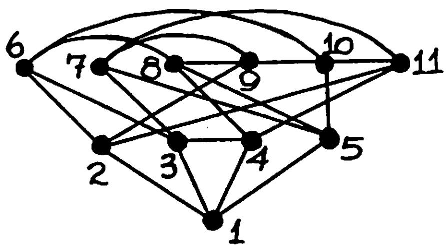

Figure 99

We now summarize the results of Section 4 in

Theorem 3. If (a,b,c) = (7,10,1), there are four tetravalent GM graphs that are not subsumed in the earlier (9,6,3) and (8,8,2) cases. No vertex has more than one triangle through it.

5. The (6,12,0) Case.

In the (6,12,0) case, there are only five subcases.

Case 5.1. The first subcase is shown in Figure 100; since there can be no triangles in the figure, only one interconnection is possible.

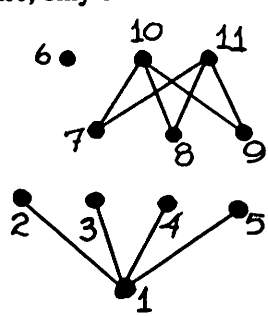

Figure 100

Vertex 6 must be joined to all the level-1 vertices. To avoid triangles, 10 and 11 must be joined to the same level-1 vertex, say 5. We may arbitrarily join 7 to 2 and 3; then 8 must be joined to 2 and 4, and 9 to 3 and 4. The graph is shown in Figures 101A and 101B.

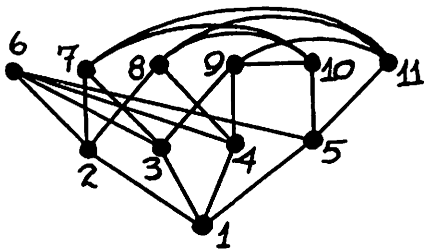

Figure 101A

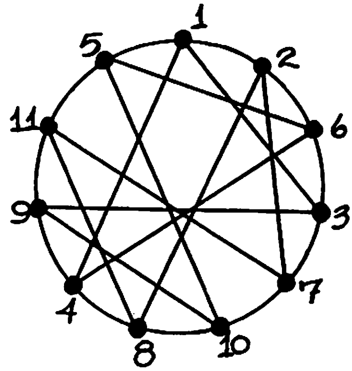

Figure 101B

Case 5.2. Figure 102 shows the second level-2 configuration.

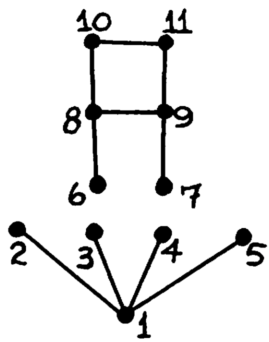

Figure 102

Vertices 6 and 7 can not be joined to the same three vertices (if they do, 8 and 9 are joined to the same vertex and a triangle is formed). Hence, join 6 to 2, 3, 4, and join 7 to 3, 4, 5. The distance criterion requires (8,5) and (9,2). Vertices 6 and 11 must be joined to the same level-1 vertex. Since they may not be joined to 2 (a triangle is formed), and since 3 and 4 are symmetric, take (11,3); similarly, take (10,4). To avoid triangles, join 10 to 2; then we need (11,5) and obtain the graph shown in Figures 103A and 103B.

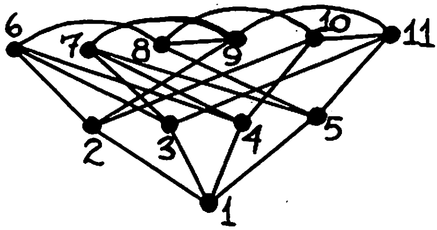

Figure 103A

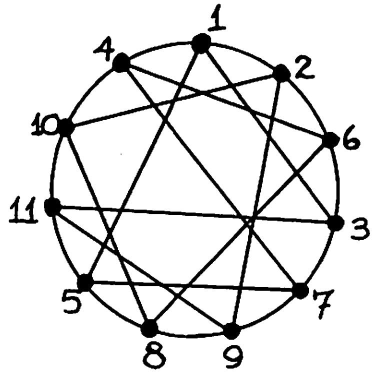

Figure 103B

Case 5.3. Figure 104 shows the third possible level-2 configuration.

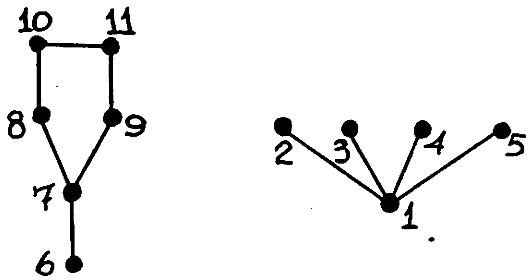

Figure 104

Take (6,2), (6,3), and (6,4),with no loss of generality; this forces (7,5). Now 8 and 9 may not be joined to 5; so 10 and 11 must be (this forms a triangle).

Case 5.4. The fourth level-2 interconnection is shown in Figure 105.

With no loss of generality, join 6 to 2, 3, 4; this forces (7,5). We are unable to join 7 to another level-1 vertex without forming a triangle.

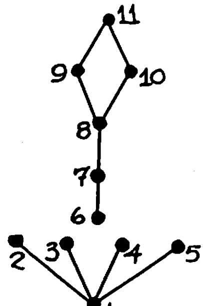

Figure 105

Case 7.5. The fifth and last (6,12,0) case is shown in Figure 106.

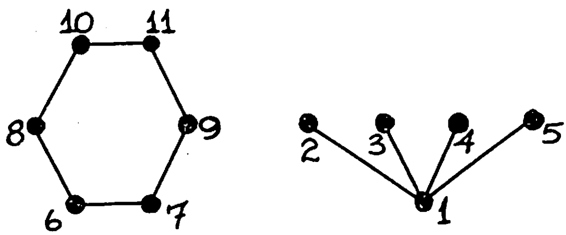

Figure 106

With no loss of generality, join 6 to 2 and 3. To avoid triangles, 7 and 8 must be joined to 4 and 5; then 9 and 10 are joined to 2 and 3, and11 is joined to 4 and 5. The graph (Figure 107) is isomorphic to that in Figure 101.

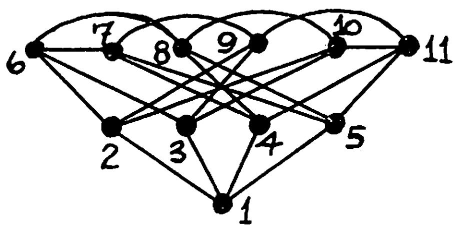

Figure 107

We summarize the discussion of this section in

Theorem 4. If (a,b,c) = (6,12,0), and there are no triangles in the graph (that is, the graph is not subsumed in an earlier case), then there are two tetravalent GM graphs.

We now combine the results of Theorems 1,2,3,4.

Theorem 5. There are 47 non-isomorphic tetravalent GM graphs.

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