ARC-MINIMAL DIGRAPHS OF SPECIFIED DIAMETER

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ABSTRACT: In this paper we consider the problem of characterizing directed graphs of specified diameter. We are especially interested in the minimal number of arcs $a(d,n)$ required to construct a directed graph on n vertices with diameter d. Classes of graphs considered include general digraphs, digraphs without cycles of length 2, and digraphs with regular indegree or regular outdegree. Upper bounds are developed in cases where the exact solutions are not known.

1: Introduction

Let D be a directed graph. We define $d(u,v)$, the distance from vertex u to vertex v, to be the number of arcs in a shortest uv-path in D in which no arc is traversed against its orientation. If no such path exists, $d(u,v) = \infty$

The diameter of a directed graph $d(D)$ is defined as follows:

$$d(D) = \max_{u,v}(d(u,v))$$

The use of directed graphs rather than undirected graphs to model communications networks is suggested by a practical consideration: the implementation details are simpler. The network will require less physical space and less message-management overhead.

The problem of orienting a given undirected graph in such a way that the resulting digraph has the least possible diameter seems to be very difficult. In fact, Chvátal and Thomassen [1] show that the problem of determining if a given graph has an orientation with diameter 2 is NP-hard.

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We address the related problem of determining the minimum number of arcs required in a digraph on n vertices with diameter d. More precisely, we define $a(d,n)$ to be the least integer such that there exists a directed graph with diameter d on n vertices with $a(d,n)$ arcs. A restricted form of this question was posed by Erdös, Rényi, and Sós [2]. We discuss this variant in Section 6.

Let $\{0,1,\ldots,n-1\}$ represent a set of n vertices. Let $S_x = \{a,b,c,\ldots\}$ represent arcs from vertex x to each of a, b, c, etc. Then finding a directed graph which realizes $a(d,n)$ is equivalent to the following:

Find $S = \{S_0,S_1,\ldots,S_{n-1}\}$ such that both

a) for $0 \le i,j \le n-1$, $\exists\ r_1,r_2,\ldots,r_k$, $k \le d$

$$\text{where } r_1 \in S_i,\quad r_2 \in S_{r_1},\quad \vdots \quad r_k = j$$

AND b) $\sum_i |S_i|$ is minimized.

Then $a(d,n) = \sum_i |S_i|$, and a directed graph realizing this value may be constructed in the obvious way.

In the following three sections, we present a number of results and bounds related to $a(d,n)$. In the subsequent sections, we present results concerning variants of this problem.

2: The restricted case d = 2

Theorem 1: $a(2,n) = 2(n-1)$, $n \ge 4$

Proof: Consider the following sets:

$$S_0 = \{1,2,\ldots,n-1\}$$

$$S_i = \{0\},\ 1 \le i \le n-1$$

Clearly these satisfy requirement a) above. It remains to show that $\sum_{i} |S_i|$ is minimal.

Suppose $a(2,n) < 2(n-1)$. Let D be a directed graph of diameter 2 with n vertices and $a(2,n)$ arcs. A trivial counting argument shows that at least one vertex of D must have outdegree 1. (In fact, at least three vertices must have outdegree 1, but we require only one for this proof.) Let x be such a vertex, and let (xy) be the only arc originating at x. Consider the breadth-first spanning tree of D rooted at x. Because the graph has diameter 2, this tree must have the form illustrated in Figure 1. This partial subgraph contains n-1 arcs. Each of the n-2 leaves at the bottom in Figure 1 must have outdegree at least 1. This brings the number of arcs to at least 2(n-1)-1. If any arc originating at a leaf z of the rooted tree has x as its

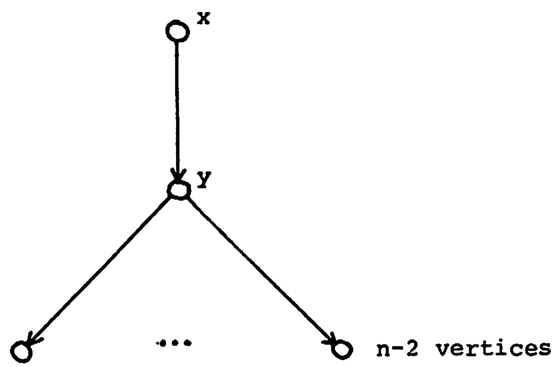

Figure 1

other end-point, then z must also be the origin of another arc (else there cannot exist paths of length ≤ 2 from z to all other leaves of the tree). Thus no arc from the vertices at the bottom enters x (else the number of arcs is at least 2(n-1)). But x must have indegree ≥ 1. Thus there must be an arc from y to x, which again brings the number of arcs to at least 2(n-1).

3: The restricted case d = 3

Theorem 2: $a(3,n) = 2(n-2) + 1$, $n \geq 8$.

Proof: Consider the sets $S_0 = \{1,2,\ldots,n-2\}$

$$S_i = \{n-1\}, \quad 1 \leq i \leq n-2$$

$$S_{n-1} = \{0\}$$

Again it is clear that these sets satisfy the first requirement, and we need only show the minimality. The argument is similar to the previous theorem. If $a(3,n) \leq 2(n-2)$, then there exists at least one vertex of outdegree 1. Let x be such a vertex, and let $(xy) \in A(D)$, the set of arcs of the digraph D. Let the outdegree of y be k. Thus if we construct a breadth-first spanning tree rooted at x, it will resemble the graph in Figure 2. If $n-(k+2) \leq 1$ the proof is

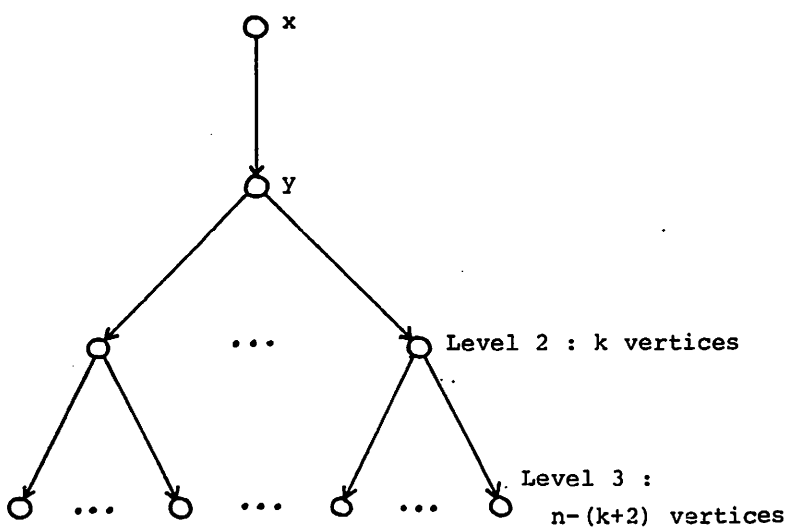

Figure 2

trivial. We therefore assume that $n-(k+2) > 1$. The number of arcs from vertices of level 2 to vertices of level 3 must be at least $n-(k+2)$. Let it be $n-(k+2) + a$, $a \geq 0$. Each vertex on level 3 must have outdegree ≥ 1. Let b be the number of arcs from vertices on level 3 to x $(b \geq 0)$. Using these arcs, at most a + b vertices on level 2 are connected by paths of length 3 to vertex y. Thus k−(a+b) vertices on level 2 each require at least one more arc to be appropriately connected to vertex y. Finally, we observe that each of the b vertices in level 3 that has an arc to vertex x must also be the origin of at least one other arc, to allow paths of length ≤ 3 to the other vertices in level 3. Summing, we find the number of arcs to be at least

$$1 + k + (n-(k+2)+a) + (n-(k+2)) + (k-(a+b)) + b = 2n - 3 ,$$

which yields a contradiction.

4: The general case: d > 3

Here we have no formula for a(d,n), but give an upper bound linear in n.

Theorem 3: $a(d,n) \le \dfrac{n-1}{\lfloor \frac{d}{2} \rfloor} + (n-1) + O(1) , \quad d \ge 4$

Proof: If d is even and (n−1) is a multiple of $\frac{d}{2}$, we construct the digraph illustrated in Figure 3, formed by identifying one vertex of each of $\dfrac{n-1}{\frac{d}{2}}$ directed cycles, each of length $(1 + \frac{d}{2})$. Clearly this digraph has diameter d and uses no more than the specified number of arcs. For odd values of d, and other values of n, the construction proceeds in a similar manner.

We conjecture that this is best possible.

5: Regularity of outdegree

A feature of the digraphs constructed above is the presence of one vertex at which the outdegree is a function of d and n, while the outdegree of all other vertices is 1. In short, one vertex forms a bottleneck, which renders these graphs undesirable as prototype network designs. We therefore now

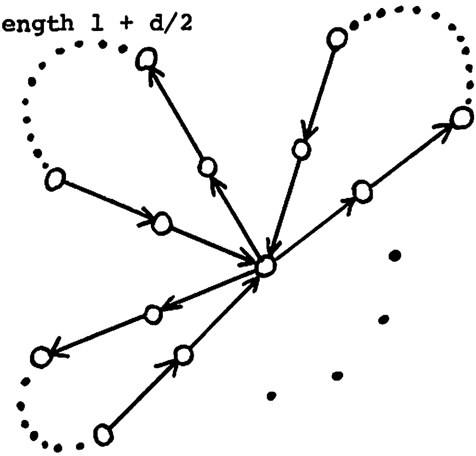

Figure 3

address the problem of determining a'(d,n), the least number of arcs required to construct a digraph with diameter d on n vertices, such that all vertices have the same outdegree.

Theorem 4: $a'(d,n) \geq \dfrac{n^{\frac{d+1}{d}}}{2^{\frac{1}{d}}}$

Proof: Let D be an outdegree regular digraph of diameter d on n vertices with a'(d,n) arcs. Let the outdegree of each vertex be k. Let v be any vertex of D, and consider the breadth-first spanning tree T_v rooted at v. Since D has diameter d, we know that the depth of T_v is at most d, and that at depth i from v, T_v has at most k^i vertices. Since the number of vertices in T_v is n, we observe that

$$1 + k + k^2 + k^3 + \ldots + k^d \geq n$$

$$\Rightarrow \quad 2*k^d \geq n$$

$$\Rightarrow \quad k \geq \left(\frac{n}{2}\right)^{\frac{1}{d}}$$

Thus $a'(d,n) = n*k \geq \dfrac{n^{\frac{d+1}{d}}}{2^{\frac{1}{d}}}$.

Theorem 5: $a'(d,n) \le d*(n^{\frac{d+1}{d}} - n) + n$

Proof: We give a constructive proof for values of n that are d-powers (i.e squares for d = 2, cubes for d = 3, etc.). The modification of the construction for other values of n is straight-forward, and is treated briefly at the end of the proof. Consider first the case d = 2. Assume as before that the vertices are {0,1,...,n-1}. Let S be the set

$$\{1,2,\ldots,n^{\frac{1}{2}}-1, n^{\frac{1}{2}}, 2*n^{\frac{1}{2}}, 3*n^{\frac{1}{2}}, \ldots, (n^{\frac{1}{2}}-1)*n^{\frac{1}{2}}\}.$$

For each vertex i, let $S_i = \{k \mid k = (i + j) \bmod n,\ j \in S\}$. The corresponding digraph is partially illustrated in Figure 4. We see immediately that this digraph has diameter = 2, and by the definitions of S and S_i, we see that the outdegree of each vertex is $2*(n^{\frac{1}{2}} - 1)$. Thus the number of arcs is $2*n*(n^{\frac{1}{2}} - 1)$.

For arbitrary d ≥ 3, the construction is similar. We let

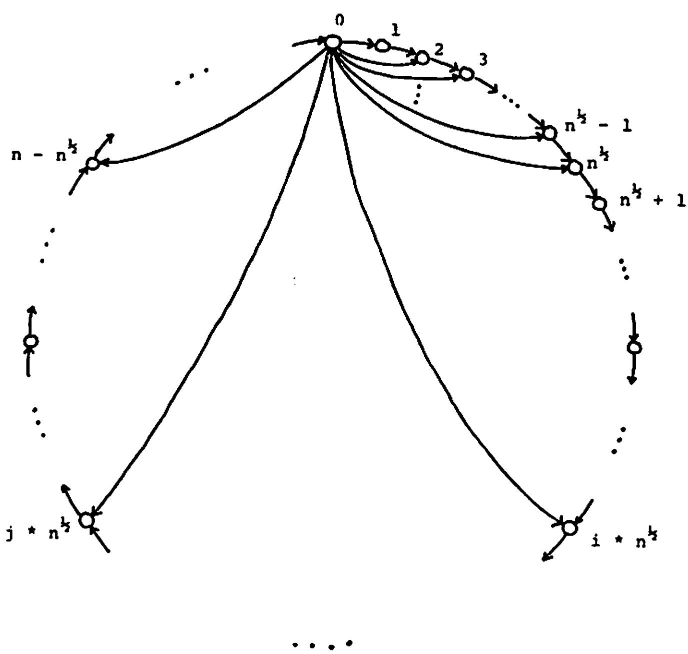

Figure 4

S be the set $\{1,2,3,\ldots,n^{\frac{1}{d}}-1,$

$$n^{\frac{1}{d}},2*n^{\frac{1}{d}},\ldots,(n^{\frac{1}{d}}-1)*n^{\frac{1}{d}},$$
$$n^{\frac{2}{d}},2*n^{\frac{2}{d}},\ldots,(n^{\frac{1}{d}}-1)*n^{\frac{2}{d}},$$
$$\vdots$$
$$n^{\frac{(d-1)}{d}},2*n^{\frac{(d-1)}{d}},\ldots,(n^{\frac{1}{d}}-1)*n^{\frac{(d-1)}{d}}\}$$

Again, letting $S_i = \{k \mid k = (i+j) \bmod n,\ j \in S\}$ for $0 \le i \le n-1$ constructs a digraph with diameter d, and as each vertex has outdegree $d*(n^{\frac{1}{d}}-1)$, the number of arcs is

$$d*n*(n^{\frac{1}{d}}-1) = d*(n^{\frac{d+1}{d}} - n)$$

It can easily be shown by a similar construction that for other values of n, $a'(d,n) \le d*(n*\lfloor n^{\frac{1}{d}} \rfloor - n) + n$, from which the theorem follows.

6: Regarding cycles of length 2

One of the significant reasons for using digraphs rather than undirected graphs as network models is that two-way communications between nodes may be prohibitively expensive or impractical. Thus it is appropriate to consider digraphs without cycles of length 2. The constructions given in the last section do not satisfy this requirement. For example, the arcs $(0,(n-n^{\frac{d-1}{d}}))$ and $((n-n^{\frac{d-1}{d}}),0)$ are both found in all the digraphs constructed in Theorem 5.

Another variant of the problem is to relax the regularity constraint, while prohibiting cycles of length 2. Letting $a''(n)$ be the least number of arcs required to construct a digraph on n vertices with diameter = 2 and no cycles of length 2, Katona and Szemerédi [3] provide a lower bound on $a''(n)$.

Theorem 6 [3]: $a''(n) \geq \frac{n}{2} * \log \frac{n}{2}$ (log base 2).

The proof of this theorem is analytic and yields no immediate evidence that $a''(n) = O(\frac{n}{2} * \log \frac{n}{2})$.

We define $a'''(d,n)$ to be the least number of arcs required to construct an outdegree-regular digraph of diameter d on n vertices, with no cycles of length 2.

It is clear that Theorem 4 gives a valid lower bound on $a'''(d,n)$. We restate this result only for completeness.

Theorem 7: $$a'''(d,n) \geq \frac{n^{\frac{d+1}{d}}}{2^{\frac{1}{d}}}$$

We now show that the upper bound derived in a previous section is also valid for $a'''(d,n)$ for some values of d and n. We will modify the construction given in the proof of Theorem 5, and thereby eliminate the cycles of length 2.

Theorem 8: For n a perfect d-power,

$$a'''(d,n) \leq d*(n^{\frac{d+1}{d}} - n)$$

Proof: We examine the case d = 2 in detail, and give only an outline for other values of d.

As before, we will construct a set S, and connect each vertex i to all vertices k such that $k = (i+j) \bmod n$ for some $j \in S$. A cycle of length 2 will result from this construction if and only if there exist $p, q \in S$ such that $0 = (p + q) \bmod n$. Thus we require that

$$0 = (p + q) \bmod n \Rightarrow p \notin S \text{ or } q \notin S \qquad \text{(I)}$$

We modify the set S used in Theorem 8, so as to avoid this condition, while still ensuring that the constructed digraph has diameter 2. This is equivalent to requiring that

$$\{t \mid t = (\sum_{i=1}^{k} y_i) \bmod n,\ k \le 2,\ y_i \in S\} = \{1,2,\dots,n-1\} \qquad \text{(II)}$$

That conditions (I) and (II) are sufficient can be seen as follows: due to the symmetry of the digraph, we need only show that $d(0,v) \le 2$ for all other vertices v, and that $d(0,0) \ge 3$. Thus for each $v \in \{1,2,\dots,n-1\}$ there must be either an arc $(0,v)$ or a pair of arcs $\{(0,x),(x,v)\}$ for some $x \in \{1,2,\dots,n-1\}$. Thus either $v \in S$, or $x \in S$ and $(v-x) \bmod n \in S$, which is equivalent to (II) above. Clearly, $d(0,0) \ge 3$ is ensured by condition (I).

Let $S = \{1,2,\dots,n^{\frac{1}{2}}-2,$

$n^{\frac{1}{2}}-1, 2*n^{\frac{1}{2}}-1, \dots, (n^{\frac{1}{2}}-1)*n^{\frac{1}{2}}-1,$

$(n^{\frac{1}{2}}-1)*n^{\frac{1}{2}}\}$

and define S_i as before. It can easily be verified that the set S satisfies (I) and (II), so the constructed digraph will have diameter 2. It is also trivial to show that there do not exist $p,q \in S$ such that $(p + q) \bmod n = 0$.

For $d \ge 3$, condition (II) generalizes to

$$\{t \mid t = (\sum_{i=1}^{k} y_i) \bmod n,\ k \le d,\ y_i \in S\} \supseteq \{1,2,\dots,n-1\} \qquad \text{(III)}$$

We define S as shown below, and use S to construct the digraph as before. Again, little effort is required to show that S satisfies conditions (I) and (III).

$$
\begin{aligned}
S = \{ & 1, 2, \ldots, n^{\frac{1}{d}}-2, \\
& n^{\frac{1}{d}}-1, 2*n^{\frac{1}{d}}-1, \ldots, (n^{\frac{1}{d}}-1)*n^{\frac{1}{d}}-1, \\
& n^{\frac{2}{d}}-1, 2*n^{\frac{2}{d}}-1, \ldots, (n^{\frac{1}{d}}-1)*n^{\frac{2}{d}}-1, \\
& \vdots \\
& n^{\frac{d-1}{d}}-1, 2*n^{\frac{d-1}{d}}-1, \ldots, (n^{\frac{1}{d}}-1)*n^{\frac{d-1}{d}}-1, \\
& (n^{\frac{1}{d}}-1)*n^{\frac{d-1}{d}} \}.
\end{aligned}
$$

7: Remarks

Theorem 1 may be viewed as a special case of Theorem 3, in which equality is achieved. It appears that Theorem 2 may represent an exceptional case.

Inspection of a'''(d,n) for small values of d and n suggests that the lower bound of Theorem 7 is tighter than the upper bound of Theorem 8.

To construct a regular digraph with diameter d it suffices to find any set S that satisfies properties (I) and (III) as developed in the proof of Theorem 8. For values of n that are not perfect powers of d, we can construct such a set S with the following algorithm:

```
Not_Used ← {1,2,...,n-1}
if n is even, then Not_Used ← Not_Used - { n/2 }
Reached ← Ø
S ← Ø
repeat
    randomly choose i ∈ Not_Used                    (*)
    Not_Used ← Not_Used - {i,n-i}
```

$S \leftarrow S \cup \{i\}$

Reached $\leftarrow$ Reached $\cup$

$$\{t \mid t = (\sum_{i=1}^{k} y_i) \bmod n,\ k \le d,\ y_i \in S\}$$

until $\{1,2,\ldots,n-1\} \subseteq$ Reached.

It can be shown that the algorithm may be expected to terminate after approximately $c*(n*\log n)^{\frac{1}{d}}$ iterations, for some constant c.

Obviously this algorithm can easily be improved. Selecting random elements from Not_Used is not very effective once Reached is a large set. A better algorithm (that is, one which may be expected to construct a smaller set S) will result if we replace line (*) by

pick $i \in$ Not_Used such that the size of the set

$$\{t \mid t = (\sum_{i=1}^{k} y_i) \bmod n,\ k \le d,\ y_i \in S \cup \{i\}\}$$

is maximized

Whether or not this leads to a set S of size $O(n^{\frac{1}{d}})$ is not yet known.

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