SOME PROPERTIES OF A GENERALIZED SPERNER LABELING

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Abstract. A generalization of Sperner's labeling for simplices is considered. It allows us to give any label not only to points from the interior of the simplex but also to points from the relative interior of each facet, while the Sperner labeling rule is preserved for all points on the boundary of each facet. Some properties of this labeling and its behavior on the facets of the simplex are discussed. Also necessary and sufficient conditions for existence of an odd number of completely labelled simplices in any triangulation of the simplex are given.

1. Preliminaries

Let $S = (e^0,\dots,e^n)$ be an n-simplex with vertices $e^0,\dots,e^n$ and let $N = \{0,\dots,n\}$.

Any function $\ell\colon S \to N$ we will call a *labeling function* on S. We say that a function $\ell\colon S \to N$ is a *Sperner labeling* on S if

1) $\ell(e^i) = i$ for $i \in N$,

2) $\ell(x) \in \{\ell(e^{i_0}),\dots,\ell(e^{i_k})\}$ whenever $(e^{i_0},\dots,e^{i_k})$ is the face of the smallest dimension of S, on which x lies; $k = 0,\dots,n$.

For any subsimplex $\sigma = (a^{i_0},\dots,a^{i_k})$ of S we define

$$\ell(\sigma) = \bigcup_{j=0}^{k} \ell(a^{i_j}) \ : \ k = 0,\dots,n.$$

We say that an n-simplex σ is *complete* if $\ell(\sigma) = N$. An (n-1)-simplex σ is *i-complete* if $\ell(\sigma) = N\setminus\{i\}$.

Theorem 1.1 (Sperner Lemma [4]). If T is any triangulation of an n-simplex $S = (e^0,\dots,e^n)$ and if ℓ is a Sperner labeling on S (or on the set of the vertices of T), then the number of complete simplices in T is odd.

To simplify the formulation of the Lemma for pseudomanifolds [1] we will introduce the following notation:

Let P by an n-pseudomanifold and let T be any triangulation of P. By ∂P we denote the boundary of the pseudomanifold P with its triangulation generated by T. Denote by

C the set of all complete simplices in T,

C_i the set of all i-complete (n-1)-simplices in T,

$C_{i_{/P'}}$ the set of all i-complete simplices in $T' \subseteq T$, where T' is a triangulation of $P' \subseteq P$.

Our version of the Sperner Lemma for pseudomanifolds can be formulated in the following way (see also [2] or [3]):

<u>Theorem 1.2</u>. If T is any triangulation of an n-pseudomanifold P and ℓ is any labeling function ℓ: P $\rightarrow$ {0, 1, ..., n} = N, then $\#C = \#C_{i_{/\partial P}} \pmod 2$, for every $i \in N$.

Note that when the n-pseudomanifold is the simplex $S = (e^0, \ldots, e^n)$ and ℓ is a Sperner labeling function we have that $\#C_{i_{/\partial S}} = \#C_{i_{/\sigma_i}}$, where σ_i is the facet of S opposite to the vertex e^i (with respect to the triangulation generated by T), and $\#C_{i_{/\sigma_i}} = 1 \pmod 2$, by inductive hypothesis (see [1]).

From the above it is clear that for any labeling function defined on the simplex S and for any triangulation of the simplex the number of complete simplices in the triangulation has the same parity as the number of facets on which there is an odd number of i-complete (n-1)-simplices.

Define

$$K_i = \{j \in N \mid \#C_{i_{/\sigma_j}} = 1 \pmod 2\}.$$

Note that any Sperner labeling function on a simplex has the property that $i \in K_i$ for all $i \in N$ (we will refer to this as Sperner property).

We will show that some of the labelings we consider here also have this property and that this implies that $K_i = \{i\}$ for all $i \in N$ (see Theorem 2.1). Furthermore, this is equivalent to the property that there exists $i \in N$ such that $K_i = \{i\}$ (see Proposition 2.3).

We will give a full characterization of the sets K_i, $i \in N$, in the next section.

2. A generalized Sperner labeling and its properties on the facets of the simplex.

Let $S = (e^0,\ldots,e^n)$ be an n-simplex and let ℓ be a labeling function, which is a Sperner labeling for all points of the boundary of each facet, i.e. on $\partial(\partial S)$, and arbitrary for all points in the relative interior of each facet.

We will describe the properties of the labeling in terms of the sets K_i, $i \in N$, defined in Section 1.

The case when there exists an $i \in N$ such that $i \in K_i$ is covered by the following Theorems 2.1 and 2.2.

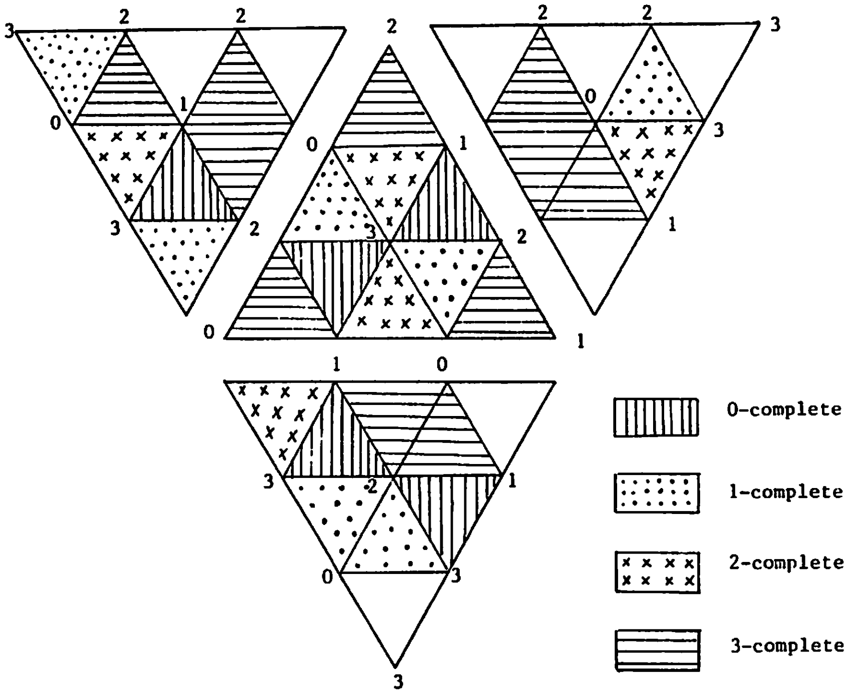
Theorem 2.1. If $i \in K_i$ for every $i \in N$, then $K_i = \{i\}$ for all $i \in N$.

Theorem 2.2. If there exists an $i \in N$ such that $K_i = \{i\}$ then $K_j = \{j\}$ for all $j \in N$.

These two theorems give a characterization of labelings which have the Sperner property. Moreover, one can show

Proposition 2.3. Theorem 2.1 and Theorem 2.2 are equivalent.

Other interesting properties of the sets K_i in cases where not for all $i \in N$ do we have $i \in K_i$ are given in the following Theorems 2.4 and 2.5.



We have here

$K_0 = \{1\}, \quad K_1 = \{0\}, \quad K_2 = \{0,1,2\}, \quad K_3 = \{0,1,3\}$

$K_2 = K_3 \setminus \{3\} \cup \{2\}$

$K_1 = K_3 \setminus \{3,1\} = K_2 \setminus \{2,1\}$

$K_0 = K_3 \setminus \{3,0\} = K_2 \setminus \{2,0\}$

Figure 2.1

Theorem 2.4. There exists at most one $j \in N$ such that either $K_j = \emptyset$ or $K_j = N$.

$K_j = \emptyset$ if and only if $K_i = \{i,j\}$ for all $i \in N\setminus\{j\}$.

$K_j = N$ if and only if $K_i = N\setminus\{i,j\}$ for all $i \in N\setminus\{j\}$.

Theorem 2.5. If there exist $i_0, i_1 \in N$, $i_0 \neq i_1$, such that $i_0, i_1 \in K_{i_0}$ and $K_{i_1} \neq \emptyset$, then for each $j \in N$ either

$$K_j = K_{i_0}\setminus\{i_0, j\}$$

or

$$K_j = (K_{i_0}\setminus\{i_0\}) \cup \{j\}.$$

Moreover

$$\#K_{i_0} = n + 2 - \#\{i \mid i \in K_i\}.$$

Theorem 2.5 is illustrated by the example in Figure 2.1, when $n = 3$. Note that in the example we have two sets $K_2 = \{0,1,2\}$ and $K_3 = \{0,1,3\}$ with the property described in Theorem 2.5.

The case $i \notin K_i$ for all $i \in N$ is described by the following theorem.

Theorem 2.6. If $i \notin K_i$ for all $i \in N$, then $K_i = N\setminus\{i\}$ for all $i \in N$.

3. Proofs of Theorems 2.1-2.6.

From the definition of the sets K_i and from the Theorem 1.2 we get that for any labeling function we have $\#C = \#C_{i/\partial S} \equiv \#K_i \pmod 2$ for each $i \in N$.

If a labeling function ℓ is as described in Section 2, we can prove

Theorem 3.1. For any triangulation of a facet σ_i of the simplex S we have one of the following two possibilities:

1) $\#C_{i/\sigma_i} = 1 \pmod 2$ and then $\#C_{j/\sigma_i} = 0 \pmod 2$ for all $j \in N\setminus\{i\}$,

2) $\#C_{i_{/\sigma_i}} = 0 \pmod 2$ and then $\#C_{j_{/\sigma_i}} = 1 \pmod 2$ for all $j \in N\backslash\{i\}$.

<u>Proof</u>. Let ℓ' be a labeling function on S such that $\ell' = \ell$ on σ_i and ℓ' is a Sperner labeling for all $x \notin \sigma_i$. Then applying Sperner's Lemma (Theorem 1.1) to any facet σ_j for $j \in N\backslash\{i\}$ with the labeling ℓ' we get that

$$\#C_{j_{/\sigma_j}} = 1 \pmod 2,$$

$$\#C_{j_{/\sigma_k}} = 0 \pmod 2 \text{ for any } k \in N\backslash\{i,j\} \text{ and}$$

$$\#C_{i_{/\sigma_k}} = 0 \pmod 2 \text{ for any } k \in N\backslash\{i\}.$$

From this we obtain

$$\#C_{j_{/\bigcup_{k\in N\backslash\{i\}}\sigma_k}} = 1 \pmod 2 \text{ for all } j \neq i$$

and

$$\#C_{i_{/\bigcup_{k\in N\backslash\{i\}}\sigma_k}} = 0 \pmod 2.$$

Hence, by Theorem 1.2 we get either

1) $\#C_{i_{/\sigma_i}} = 1 \pmod 2$ and $\#C_{j_{/\sigma_i}} = 0 \pmod 2$ for $j \neq i$ or

2) $\#C_{i_{/\sigma_i}} = 0 \pmod 2$ and $\#C_{j_{/\sigma_i}} = 1 \pmod 2$ for $j \neq i$

for the labeling ℓ'. The fact that $\ell' = \ell$ on σ_i completes the proof of Theorem 3.1.

From this theorem we get immediately a number of corollaries on the sets K_i, which will be used to prove the theorems of the previous section.

<u>Corollary 3.2</u>. If $i \in K_i$ for some $i \in N$, then $i \notin K_j$ for all $j \in N\backslash\{i\}$.

<u>Corollary 3.3</u>. If $i \notin K_i$ for some $i \in N$, then $i \in K_j$ for all $j \in N\backslash\{i\}$.

Corollary 3.4. If $j \in K_{i_0}$ for some $i_0, j \in N$, $i_0 \neq j$, then $j \in K_i$ for all $i \neq j$ and $j \notin K_j$.

Corollary 3.5. If $j \notin K_{i_0}$ for some $i_0, j \in N$, $i_0 \neq j$ then $j \notin K_i$ for all $i \neq j$ and $j \in K_j$.

From the above corollaries we have immediately

Proposition 3.6. For every $i \in N$ we have exactly one of the following possibilities: either $i \in K_i$ or $i \in K_j$ for all $j \in N \setminus \{i\}$.

We now come to the proofs.

1) Theorem 2.1 can be proved using Corollaries 3.2 and 3.3.

2) Using Corollaries 3.2, 3.5 and Proposition 3.6 the proof of Theorem 2.2 is immediate.

From 1) and 2) it is easy to see that

3) Proposition 2.3 holds.

4) Theorem 2.4 is a consequence of Corollaries 3.3-3.5.

5) Theorem 2.5 can be proved using Corollaries 3.2 and 3.4.

6) Theorem 2.6 can be proved using Corollaries 3.2-3.5.

4. Conclusion.

The characterization of the sets K_i given in Section 2 provides an answer to the question of the existence of an odd number of complete simplices in a triangulation of an n-simplex S with a labeling function as defined in Section 2. It depends on the properties of the sets K_i and also on the dimension n of the simplex.

First let us recall that from Theorem 1.2 and the definition of the sets K_i we get $\#C = \#C_{i/\partial S} = \#K_i \pmod 2$ for all $i \in N$, which together with the results of Section 2 simplifies the formulation of the following

Proposition 4.1. $\#C = n + 2 - \#\{i \in N \mid i \in K_i\} \pmod 2$.

This can be proved easily using Theorems 2.1, 2.2, 2.5 and 2.6.

References

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