ON A CORRESPONDENCE BETWEEN GRAPHS AND GROUPS

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ABSTRACT

A group satisfies PP3 (the permutation property of degree 3) if any product of 3 elements remains invariant under some nontrivial permutation of its factors, or equivalently, if G has at most one nontrivial commutator of order 2. A PP3 group is elementary if it is a finite group of exponent at most 4. There is an algorithm that associates an elementary PP3 group to an arbitrary graph. It follows, for instance, that almost every nontrivial graph automorphism has order a power of 2 and that the first-order theory of (elementary) PP3 groups is hereditarily undecidable.

1. INTRODUCTION

It is a well-known fact that the first-order theory of the class of all abelian groups is decidable [16], even in triple exponential Turing time [8]. On the other hand, A. Malcev[10] established that the first order theory of metabelian groups of exponent a prime p > 2 is essentially undecidable. A. Tarski raised the question whether every variety of groups properly containing the variety of all abelian groups has an undecidable elementary theory. J. Ershov [2] observed that if such a variety contains a nonabelian finite group then it has an undecidable theory. This result led him to conjecture that every nonabelian variety of groups is undecidable (This result is established in [18]). Ershov's proof consists in proving that (a) the varieties of 2-nilpotent groups of exponent p > 2, the variety of 2-nilpotent groups of exponent 4 with

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commutator subgroup of exponent 2, and the variety generated by the Frobenius groups $Z_p \ltimes Z_q$ (p,q > 2 are distinct primes) have hereditarily undecidable theories; and (b) every nonabelian variety of groups must contain one of them.

The purpose of this note is to establish an effective correspondence between the class of graphs and a certain class of almost abelian groups. Among other things, this correspondence strengthens Ershov's result and establishes a Galois-type connection between well-known problem combinatorial problems in group and graph theory. This note can be regarded as a preliminary report on this connection. Full proofs and further results will be contained in an upcoming paper [5].

2. ELEMENTARY PP3-GROUPS

Permutation properties (here denoted PP) of groups and semigroups has been studied in [1], [13] and [14]. They were apparently first introduced by Restivo and Reutenauer [13], who show that the strong Burnside Problem for semigroups (viz., is every finitely generated torsion semigroup finite?) has a positive solution for semigroups with the permutation property (for a survey, see [14]).

Definition 2.1. Let $n \geq 2$ be a positive integer and S a semigroup. An n-tuple $(x_1, x_2, \ldots, x_n)$ of factors of S satisfies PPn -the permutation property of degree n- if there exists a nontrivial permutation $\sigma \in S_n$ of its factors such that

(1) $\quad x_1 x_2 \ldots x_n = x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)}$.

S <u>satisfies PPn</u> if every n-tuple of elements of S satisfies PPn. □

The objects of interest here are groups satisfying PP3. Recall that the commutator subgroup G' is the subgroup of G generated by all commutators $[x,y]=xyx^{-1}y^{-1}$. For undefined algebraic terminology and notation see [7] or [15].

The next 3 results have been established in [4] or [5].

<u>Theorem 2.2</u>. A group G satisfies PP3 if and only if $|G'|\leq 2$.□

<u>Lemma 2.3</u>. There are exactly three isomorphism types of two generator nonabelian PP3 2-groups which are presented by

$$G_1 := \langle x,y \mid u^2, x^{2^r}, y^{2^s}, [u,x], [u,y]\rangle,$$

(*)
$$G_2 := G_1/\langle x^y x^{2^{r-1}-1}\rangle$$
$$= \langle x,y \mid u^2, x^{2^r}, y^{2^s}, x^y = x^{2^{r-1}+1}, [u,x], [u,y]\rangle,$$

and

$$G_3 := G_2/\langle y^x y^{2^{s-1}-1}\rangle$$
$$= \langle x,y \mid u^2, x^{2^r}, y^{2^s}, x^y = x^{2^{r-1}+1}, y^x = y^{2^{s-1}+1}, [u,x], [u,y]\rangle.$$

where u abbreviates the commutator [x,y]. □

The latter two possibilities in the lemma actually give rise to nonisomorphic groups since the quaternion Q and dihedral group D_4 are their homomorphic images.

The class of PP3 groups is closed under subgroups and homomorphic images but not under direct products. However it is closed under a slight variation of the direct product operation, which is in fact a generalization of the direct product of abelian groups. As it turns out, this is the basic construction needed to build arbitrary PP3 groups.

Definition 2.4. Let u (respectively v) be a central involution in a group G (H, respectively, of the same order as u). The amalgamated direct product of G and H (with u and v amalgamated) is the quotient of GxH by the normal subgroup {(1,1),(u,v)}. □

The definition can be obviously extended to any finite number of factors. Note that if the involutions are both trivial then the amalgamated product is (isomorphic to) the ordinary direct product. Informally, the amalgamated product is obtained from the ordinary product by identifying the two involutions u and v.

Now it is possible to give a general structure theorem for PP3 2-groups. Note that an arbitrary finitely generated PP3 group is nilpotent, and thus is a direct sum of finitely generated free abelian and finite PP3 2-groups and odd order abelian groups.

Theorem 2.5. The following conditions are equivalent on a finitely generated group G:

(1) G is a PP3 2-group.

(2) G is an amalgamated products of cyclic 2-groups and groups of type (*).

(3) G is (isomorphic to) a group presented by

(**) $\langle u, x_1, \ldots, x_n \mid u^2, x_i^{2^{t_i}} = u^{a_{o,i}}, [x_i, x_j] = u^{a_{ij}}, [x_i, u] \rangle$

for some (unique) positive integers t_i ($1 \leq i \leq n$) and a_{ij} equal to 1 or 0 (according as whether u is a power of x_j, if i=0, or whether x_i and x_j

commute, if $i>0$).

The set $\{t_1,...,t_n,[a_{ij}]\}$ forms (up to a permutation of the indices) a complete set of invariants of the group G.□

Although PP3 groups are almost abelian they can be quite complex. In fact, there is no upper bound on the degrees of irreducible representations of PP3 groups. The degree of a nonlinear absolutely irreducible representation of a finite PP3-group equals the square root of $[G:Z(G)]$ [6, lemma 2.3]) which is a perfect square and can be made arbitrarily large (see groups G_1 and G_2 above).

<u>DEFINITION 2.6</u>. An <u>elementary PP3 group</u> is a finitely generated PP3 2-group of exponent 4. □

An elementary PP3 group is finite and is completely determined up to isomorphism by its matrix of invariants $[a_{ij}]$. This matrix is essentially the adjacency matrix of a graph. The graph, however, is a <u>pointed</u> graph, i.e., it has a distinguished vertex preserved by all graph isomorphisms. (It is possible to consider instead arbitrary PP3 groups and vertex-colored (vc-) graphs with colors from the set $\mathbb{N}$ of positive integers but the elementary class will suffice here). In the remainder of this section π denotes a group presentation or its associated group and Γ denotes a (finite) graph.

The proof of the next theorem is contained in the following correspondence.

CONSTRUCTION. Let π be the presentation of an elementary PP3 group (**) with invariant matrix $[a_{ij}]$ $(t_1=\ldots=t_n=1)$. Let $\Gamma(\pi)$ be the (unlabeled) pointed graph (V,E,u) with vertex set the generators of π, distinguished vertex u and with adjacency matrix $[a_{ij}]$.

Conversely, given a pointed graph Γ = (V,E,u) with adjacency matrix $[a_{ij}]$ let $\pi(\Gamma)$ be the group presented by (**), with generating set V and invariant matrix $[a_{ij}]$. □

THEOREM 2.7. Two elementary PP3 groups are isomorphic if and only if the associated pointed graphs are isomorphic. □

In particular, to every isomorphism of the graph π there corresponds an isomorphism of the group $\Gamma(\pi)$ by Von Dyck's theorem. Clearly this correspondence is one to one. Note, however, that this correspondence is very rarely onto because there are asymmetric graphs (i.e., having only trivial automorphisms) but, on the other hand, every group of order at least 3 has a nontrivial automorphism.

The CONSTRUCTION and most of its consequences carry over to arbitrary finitely generated PP3 groups if one allows vc-graphs, where the colors are positive integers corresponding to the orders of the generators of a canonical presentation similar to (**).

3. THE ORDER OF GRAPH AUTOMORPHISMS

It is well-known that not all groups occur as automorphism groups of a group. Thus the following corollary is of interest in its own right. Related results have been established by [17] in the case of torsion-free class 2 nilpotent groups.

COROLLARY 3.1. Every finite group is (isomorphic to) a subgroup of the automorphism group of an elementary PP3 group. □

On the other hand, in a very precise sense[12] almost all graphs are asymmetric (i.e., have no nontrivial automorphisms). However, this result ignores the question of the nature of nontrivial symmetries of graphs. The following corollary follows easily from [11, Theorem 1], which, in particular, states that almost all d-generator groups of Frattini class 2 have no automorphism of odd order. The corollary provides an asymptotic result on the above question. Of course, the odd order of a graph automorphism on n vertices divides $|GL_n(2)|$, the number of nonsingular nxn matrices over the field of 2 elements, but there is little hope for a precise result since every group occurs as the automorphism group of some suitable graph. □

COROLLARY 3.2 If a_n (respectively, e_n) is the number of nontrivial automorphism groups of (unlabeled) graphs on n vertices (which are 2-groups, respectively), then

$$e_n/a_n \to 1 \text{ as } n \to \infty. \quad \square$$

Sketch of proof. Every automorphism of a graph π induces an automorphism of group $\Gamma(\pi)$ and every group automorphism is realized by a graph automorphism of some graph π. □

Thus almost all nontrivial automorphisms of a graph have order a power of 2.

4. UNSOLVABLE PROBLEMS

It follows from theorem 2.2 that many decision problems of infinite PP3 groups are solvable in polynomial time; for instance, properties that follow from properties of the abelianized quotient (finiteness, triviality, word and conjugacy problems). Global properties in the class of PP3 groups, however, are not as immediate since it is not a variety (although it is properly contained in the variety defined by the law $x^2y=yx^2$) and the above theorem of Ershov [2] does not apply.

Nonetheless, structure theorem 2.5 makes it possible to save the proof of the key argument in [2]. In fact, Theorem 2.5 can be regarded as a relatively elementary embedding (in the sense of Ershov [2]) of the first-order theory of graphs (or a reflexive, symmetric binary relation) in the first-order theory of elementary PP3 groups. Therefore, most of his results can be strengthened to classes of groups containing the class of PP3 groups. For instance,

<u>COROLLARY 4.1</u> The first-order theory of elementary (and hence arbitrary) PP3 groups is hereditarily undecidable.

In way similar to [2] one can establish the undecidablity of the first-order theory of <u>finite</u> (elementary) PP3 groups. Since PP3 groups are PPn groups for all n $\geq$ 3 similar results follow for all layers in the PP hierarchy. In fact, it follows that any class of groups containing the class of PP3 groups has a hereditarily undecidable first-order theory.

A different problem is whether or not the first-order theory of a fixed PP3 group is decidable. Ershov [3] has shown that the first-order theory of a finitely generated <u>nilpotent</u> group is decidable if and only if the group is center-by-finite. It has been shown in [1] that the class of finitely generated PP groups coincides with the class of center-by-finite groups. Obvioulsy a finite extension of a group with first-order decidable theory has a decidable theory. In particular, each finitely generated PP group has a decidable first-order theory. It would be interesting to characterize the class of groups each of which has decidable first-order theory, or in particular, its relation to the class of PP groups.

R E F E R E N C E S

[1] M. Curzio, P. Longobardi, M. Maj, and D.J.S. Robinson, A permutational property of groups, Arch. Math. 44(1985),385-389.

[2] J.L. Ershov, Theories of nonabelian varieties of groups, in: <u>Proc. Tarski Sym</u>. (Proc. Sym. Pure Math., vol. 25, AMS, Providence, R.I.,1974), 255-264.

[3] J.L. Ershov, Elementary group theories. Dokl. Akad. Nauk 203:6 (1972); Sov. Math 13:2(1972) 528-532.

[4] M. Garzon and Y. Zalctsein, On permutation properties in groups and semigroups, submitted.

[5] M. Garzon, A Galois-theory of Group theory and combinatorial problems, in preparation.

[6] Isaacs, M. and D. Passman, A characterization of groups in terms of the degrees of their characters, Pacific J. of Math 15, 877-903.

[7] M.I. Kargapolov, and J.I. Merzljakov, <u>Fundamentals of the Theory of Groups</u> (Springer-Verlag, New York, 1979).

[8] L. Lo, On the computational complexity of the theory of abelian groups, Ph.D. thesis, U. of Michigan, Ann Arbor, 1984.

[9] A.I. Mal'cev, On a Correspondence between rings and groups, Uspehi Mat. Nauh 14:5(1959), 208-209.

[10] A.I. Mal'cev, On a Correspondence between rings and groups, Mat. Sbornik (New series) 50(92) (1960), 257-266.

[11] U. Martin, Almost all p-groups have automorphism groups a p-group, Bulletin of the AMS (New Series) 15:1(1986), 79-82.
[12] E.M. Palmer, Graphical Evolution (John Wiley & Sons, 1985).
[13] A. Restivo and C. Reutenauer, On the Burnside Problem for semigroups J. of Algebra 89(1984), 102-104.
[14] A. Restivo and C. Reutenauer, Rational languages and the Burnside Problem, Theor. Comp. Sci. 40:1(1985), 13-30.
[15] D.J.S. Robinson, A course in the Theory of Groups, (Springer-Verlag, GTM #80, 1982).
[16] W. Szmielev, Arithmetical properties of abelian groups Doctorial dissertation, Univ. of California, Berkeley, 1950.
[17] U.H.M. Webb, Using graphs to investigate the automorphism groups of nilpotent groups, in Graphs and Applications. Proc. 1st. Colorado Simp. on Graph Theory (John Wiley & Sons, New York, 1985), 333-347.
[18] A.P. Zamyatin, A non-abelian variety of groups has an undecidable elementary theory, Algebra and Logic 17:1(1978), 13-20.

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