RESOLVABLE GROUP DIVISIBLE DESIGNS WITH BLOCK SIZE 4

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Abstract. In this paper, various constructions for resolvable group divisible designs with block size 4 are given.

1. Introduction

A group divisible design GD(k,m;v) is a triple $(V,\mathcal{G},\mathcal{B})$ where B is a set containing v points, G is a collection of m-subsets (called groups) of V and B is a collection of k-subsets (called blocks) such that

(i) G is a partition of V;

(ii) Each pair of elements of V from distinct groups occurs in a unique block.

A parallel class in a GD(k,m;v) is a set of blocks which partitions V. A group divisible design is called resolvable if the blocks can be partitioned into parallel classes. A resolvable GD(k,m;v) will be denoted by RGD(k,m;v).

It is not difficult to show that the following are necessary conditions for the existence of an RGD(k,m;v):

$$v \equiv 0 \pmod{k}, \qquad v \equiv 0 \pmod{m}$$
$$v-m \equiv 0 \pmod{k-1}. \qquad (1)$$

A natural problem is: Are the necessary conditions (1) for the existence of an RGD(k,m;v) also sufficient?

For k = 3, m = 1 or 3, this is the famous Kirkman's schoolgirl problem. It was completely solved by Ray-Chaudhuri and R. M. Wilson in 1971 ([9]). For k = 3, m = 2, it is called a nearly Kirkman triple system and it was proved ([1],[3],[6],[7]) that, there exists a nearly Kirkman triple system RGD(3,2;v) if and only if v ≡ 0 (mod 6), v > 12.

Recently, R, Rees and D. R. Stinson ([10]) considered the general case for RGD(3,m;v) and proved that the necessary conditions (1) are also sufficient for the existence of an RGD(3,m;v), with three exceptions (m = 2, v = 6 or 12, and m = 6, v = 18) and some unsolved cases. E. Mendelsohn and Shen Hao ([8]) considered the problem independently and constructed some new designs by different methods.

For k = 4, m = 1 or 4, it was proved by H. Hanani, D. K. Ray-Chaudhuri and R. M. Wilson ([5]) that the necessary and sufficient condition is

$$v \equiv 4 \pmod{12}. \tag{2}$$

In this paper, we are going to give several recursive and direct constructions for resolvable group divisible designs with block size 4.

2. General Constructions

A GD(k,m;km) is called a transversal design and denoted TD[k,m]. A resolvable TD[k,m] is denoted $TD^*[k,m]$. It is well known ([2],[4]) that the following statements are equivalent:

(i) There exists a TD[k,m],

(ii) There exists a $TD^*[k-1,m]$.

(iii) There exists a set of k-2 pairwise orthogonal Latin squares of order m.

For given k and m, let RG(k,m) be the set of positive integers v for which an RGD(k,m;v) exists. Similarly, for a given k, the set of integers m for which a TD[k,m] (or a $TD^*[k,m]$) exists will be denoted T(k) (or $T^*(k)$).

Lemma 1. If $4v \in RG(4,m)$, $v \geq 4$, $v \neq 2,6,10$. Then for any integer $s \geq 0$, we have

$$4(3s+1)v \in RG(4,m).$$

Proof. It is well known that for any $s \geq 0$, $4(3s+1) \in RG(4,4)$. Let $(X,\mathcal{G},\mathcal{B})$ be an RGD(4,4;4(3s+1)), where the elements of X are sets having v points each. For each group $G \in \mathcal{G}$, form an RGD(4,m;4v) on the union of the four v-sets of G. As we know ([11],[12]), for any integer $n \geq 4$, $n \neq 6,10$, $n \in T^*(4)$. Thus, for each block $B \in \mathcal{B}$, we can form the blocks of a $TD^*[4,v]$. Now it is easy to verify that this gives an RGD(4,m;4(3s+1)v).

Lemma 2. If $m \geq 4$ and $m \neq 6,10$, then $4m \in RGD(4,m)$.

Proof. The existence of an RGD(4,m;4m) is equivalent to the existence of a resolvable transversal design $TD^*[4,m]$. From the fact that there exist 3 pairwise orthogonal Latin squares of order m for each $m \geq 4$, $m \neq 6,10$, the lemma is now obvious.

Lemma 3. If $v \in RG(4,m)$ and $t \geq 4$, $t \neq 6,10$, then $tv \in RG(4,tm)$.

Proof. From Lemma 2, there exists an RGD(4,t;4t) for every $t \geq 4$, $t \neq 6,10$. For a given RGD(4,m;v), we replace each point a by a t-set $\{a_1,a_2,\ldots,a_t\}$, and replace each block $\{a,b,c,d\}$ by the blocks of an RGD(4,t;4t) with $\{a_1,a_2,\ldots,a_t\}$, $\{b_1,b_2,\ldots,b_t\}$, $\{c_1,c_2,\ldots,c_t\}$ and $\{d_1,d_2,\ldots,d_t\}$ as its groups. The obtained design is an RGD(4,tm;tv).

Let $m = 1$. We have the following corollary.

Corollary. If $t \geq 4$, $t \neq 6,10$, then for any integer $s \geq 0$, we have $4(3s+1)t \in RG(4,t)$.

From Lemmas 1-3, we have the following result.

Theorem 1. Let $m \geq 5$, $(m,6) = 1$, then $v \in RG(4,m)$ if and only if

$$v \equiv 0 \pmod{4m},\quad v-m \equiv 0 \pmod{3}. \tag{3}$$

3. Constructions of RGD(4,3;v)

For the existence of an RGD(4,3;v), the necessary condition is

$$v \equiv 0 \pmod{12}. \tag{4}$$

This condition is not always sufficient. In fact, as $3 \notin T^*(4)$, there does not exist an RGD(4,3; 12). But we have the following construction for RGD(4,3;v).

Theorem 2. If there exist an RGD(4,4;u) and an RGD(4r,4;v) with $v \geq 16$, then $u(v-1) \in RG(4,3)$. In other words, if $r \equiv 1 \pmod 4$ and $s \equiv 1 \pmod 3$ with $r \geq 5$, then $12rs \in RG(4,3)$.

Proof. Let $R_k = \{(1,k),(2,k),\ldots,(3r,k)\}$, $k = 1,2,\ldots,4s$. As $r \equiv 1 \pmod 4$, there exists an RGD(4,4;3r+1) on the set $R_k \cup \{x\}$, where $x \notin R_k$ for every k. Let

$$F_{ik} = F'_{ik} \cup \{x,a_{ik},b_{ik},c_{ik}\}, \quad i = 1,2,\ldots,r$$

be the r parallel classes. Omit the point x from each parallel class, we obtain a combinatorial design on R_k, which can be decomposed into r parallel classes:

$$F'_{ik} \cup \{a_{ik},b_{ik},c_{ik}\}, \quad i = 1,2,\ldots,r.$$

Obviously, $\{a_{1k},b_{1k},c_{1k}\}$, $\{a_{2k},b_{2k},c_{2k}\},\ldots,\{a_{rk},b_{rk},c_{rk}\}$ form a parallel class of R_k.

Let $s = 3q+1$, $K_1,K_2,\ldots,K_{4q+1}$ be the 4q+1 parallel classes of an RGD(4,4;4s) on the set $\{1,2,\ldots,4s\}$, and let $\{x_{ij},y_{ij},z_{ij},w_{ij}\}$, $j = 1,2,\ldots,s$ be the blocks of K_i. As $3r > 4$, $3r \neq 6,10$, there exists an RGD(4,3r;12r) on the set $R_{x_{ij}} \cup R_{y_{ij}} \cup R_{z_{ij}} \cup R_{w_{ij}}$. Let $G_{ij1},G_{ij2},\ldots,kG_{2i,3r}$ be the 3r parallel classes of this design. Denote

$$G_{ik} = \bigcup_{j=1}^{s} G_{ijk}, \quad k = 1,2,\ldots,3r.$$

Then

$$\{G_{ik} \mid i = 1,2,\ldots,4q+1;\ k = 1,2,\ldots,3r\}$$

is a set of 3r(4q+1) parallel classes of $\bigcup_{k=1}^{4s} R_k$. Let G_0 be a fixed parallel class in $\{G_{ik}\}$, without loss of generality, we may suppose.

$$G_0 = \{\{a_{ik}, a_{i,k+1}, a_{i,k+2}, a_{i,k+3}\},$$
$$\{b_{ik}, b_{i,k+1}, b_{i,k+2}, b_{i,k+3}\}, \{c_{ik}, c_{k,k+1}, c_{i,k+2}, c_{i,k+3}\} \mid$$
$$i = 1,2,\ldots,r;\ k = 1,5,9,\ldots,4s-3\}.$$

Denote

$$H_i = \bigcup_{k=1}^{4s} F'_{ik} \cup \{\{a_{ik}, a_{i,k+1}, a_{i,k+2}, a_{i,k+3}\},$$
$$\{b_{ik}, b_{i,k+1}, b_{i,k+2}, b_{i,k+3}\}, \{c_{ik}, c_{k,k+1}, c_{i,k+2}, c_{i,k+3}\} \mid$$
$$k = 1,5,9,\ldots,4s-3\}.$$

Then $\{H_i \mid i = 1,2,\ldots,r\}$ is a set of r parallel classes of $\bigcup_{k=1}^{4s} R_k$. Now it is not difficult to verify that

$$\{H_i \mid i = 1,2,\ldots,r\} \cup \{G_{ik} \mid i = 1,2,\ldots,4q+1,\ k=1,2,\ldots,3r\} \backslash G_0$$

is a set of $r+3r(4q+1)-1 = 4rs-1$ parallel classes of $\bigcup_{k=1}^{4s} R_k$ and form an RGD(4,3;12rs).

In the theorem, if we let $s = 4$, then we have

Corollary. There exists an RGD(4,3;v) for any $v \equiv 12 \pmod{48}$, $v \geq 60$.

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