Cycle covering of plane triangulations

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Abstract. Bondy conjectures that if G is a 2-edge-connected simple graph with n vertices, then at most $(2n-1)/3$ cycles in G will cover G. In this note, we show that if G is a plane triangulation with $n \geq 6$ vertices, then at most $(2n-3)/3$ cycles in G will cover G.

1. Introduction

We follow the notation of Bondy and Murty [**BM**], except where noted otherwise. An edge e of a graph G is called a *multiple* edge if $G-e$ has an edge f having the same ends as e, and in this case we say that e is an *extra edge* of $G-e$ *parallel* to the edge f. Graphs may have multiple edges but loops are prohibited. Let G be a graph. For $X \subseteq E(G)$, the *contraction* G/X is the graph obtained from G by identifying the ends of each edge in X and then deleting the resulting loops. A collection C of cycles in G is called a *cycle cover* (CC) of G, if every edge of G lies in at least one cycle in C. It is obvious that G has a CC if and only if G is 2-edge-connected. For a graph with $\kappa'(G) \geq 2$, define

$$cc(G) = \min\{|C| : C \text{ is a CC of } G\}.$$

In [**B**], Bondy raised the following conjecture.

Conjecture SCC: If G is a simple 2-edge-connnected graph with n vertices, then

$$cc(G) \leq \frac{2n-1}{3}.$$

If C is a collection of cycles of G and if every edge in G lies in exactly 2 members of C, then C is called a *cycle double cover* (CDC) of G. The eminent cycle double cover conjecture, due to Seymour [S1] and Szekeres [S2], says that every 2-edge-connected graph admits a CDC. The following conjecture is also posted by Bondy in [**B**].

Conjecture SCDC: If G is a simple 2-edge-connected graph with n vertices, then G admits a CDC with at most $n-1$ cycles.

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Theorem 1.1. (Bondy and Seyffarth [B]) *If G is a simple plane triangulation with n vertices, then G has a CDC with at most $n-1$ cycles.*

In this paper, we shall show that if G is a simple plane triangulation with $n \geq 6$ vertices, then

$$cc(G) \leq \frac{2n-3}{3}.$$

We shall prove a multigraph version of Conjecture SCC for plane triangulations. Let G be a graph. Define an equivalence relation on $E(G)$ such that e is related to e' if and only if $e = e'$ or e and e' share the same ends (e and e' are parallel edges). Let $[e]$ denote the equivalence class containing e and $[G]$ the collection of all equivalence classes. Define

$$\mu(G) = \sum_{[e]\in[G]} (|[e]| - 1).$$

Hence a graph G is simple if and only if $\mu(G) = 0$. We define a (multi) graph G to be a *plane triangulation* if G is a plane graph each of whose faces has degree 2 or 3. In Section 2, we develop some reduction techniques, and in Section 3, we shall show Theorem 1.2 below. Some of the routine and repeated arguments in the proofs are omitted. Interested readers may contact the authors for details.

Theorem 1.2. *If G is a plane triangulation of $n \geq 6$ vertices, then*

$$cc(G) \leq \frac{2n-3}{3} + \frac{\mu(G)}{2}. \tag{1}$$

2. Reductions

Let X, Y be two sets. The *symmetric difference* of X and Y, denoted by $X \triangle Y$, is $X \cup Y - X \cap Y$. If G is a graph and H and J are subgraphs of G, then denote

$$H \triangle J = G[E(H) \triangle E(J)].$$

If G has 2 subgraphs G' and G'' such that $G = G' \cup G''$ and such that $G' \cap G''$ is a 2-cycle of G, then G is called a C_2*-sum* of G' and G''.

Lemma 2.1. *Let G be a graph with $\kappa'(G) \geq 2$. If G is a C_2-sum of G_1 and G_2, where $\kappa'(G_i) \geq 2$, then*

$$cc(G) \leq cc(G_1) + cc(G_2) - 1.$$

Proof: Let $\{e_1, e_2\}$ be the edges of the 2-cycle C commonly shared by G_1 and G_2. For $i \in \{1, 2\}$, let C_i be a CC of G_i. Let C_j^i be a cycle in C_i that contains the edge e_j, $(1 \leq i, j \leq 2)$. If $C_1^1 = C_2^1$, then $C = C_1^1 = C_2^1$, and so $(C_1 - \{C\}) \cup C_2$

is a CC of G and Lemma 2.1 follows. Hence we may assume that $C_1^i \neq C_2^i$ and so $E(C_j^i) \cap E(C) = \{e_j\}$. Thus

$$(C_1 - \{C_1^1, C_2^1\}) \cup (C_2 - \{C_1^2, C_2^2\}) \cup \{C_1^1 \triangle C_1^2, C_2^1 \triangle C_2^2\}$$

is a CC of G and so Lemma 2.1 follows again. ∎

Let H be a subgraph of G. The *vertices of attachment* of H in G, denoted by $A_G(H)$, are the vertices in $V(H)$ that are incident with some edges in $E(G) - E(H)$.

For a graph H, H^+ denotes a graph obtained from H by adding an extra edge parallel to some edge of H.

Lemma 2.2. *Suppose that* $H = \Gamma_1$ *or* $H = \Gamma_1^+$ *(see Figure 1) with an extra edge* e *that is parallel to an edge in* $E(\Gamma_1) - \{v_1v_2, v_2v_3, v_3v_1\}$, *such that* H *is a subgraph of* G *with* $A_G(H) \subseteq \{v_1, v_2, v_3\}$. *Let* e_1 *be an extra edge parallel to* v_1v_2, *and* e_2 *be an extra edge parallel to* v_2v_3. *Let* $V_H = V(H) - \{v_1, v_2, v_3\}$.

(i) *If* $H = \Gamma_1$, *then let* $G' = (G - V_H) + e_2$ *and we have*

$$cc(G) \leq cc(G') + 1. \tag{2}$$

(ii) *If* $H = \Gamma_1^+$, *then let* $G'' = (G - V_H) + \{e_1, e_2\}$ *and we have*

$$cc(G) \leq cc(G'') + 1. \tag{3}$$

Proof: We shall show (i) first. Let $\mathcal{C}$ be a CC of G', and let $C \in \mathcal{C}$ be a cycle containing e_2. Let $C' = C - e_2 + \{v_2v_5, v_5v_4, v_4v_6, v_6v_3\}$, and $F = v_2v_4v_1v_6v_5v_3v_2$. Thus $(\mathcal{C} - \{C\}) \cup \{C', F\}$ is a CC of G and so (2) holds.

The proof for (ii) is similar and uses the fact that we can always assume that e_1 and e_2 are in distinct cycles of any CC of G''. ∎

Lemma 2.3. *Suppose that* $H = \Gamma_i$ *or* H *is* Γ_i^+ *(see Figures 1 and 2) with an extra edge* e *that is parallel to an edge of* $E(\Gamma_i) - \{v_1v_2, v_2v_3, v_3v_1\}$, $(2 \leq i \leq 4)$, *such that* H *is a subgraph of* G *with* $A_G(H) \subseteq \{v_1, v_2, v_3\}$. *Let* e_i *be an extra edge parallel to* v_iv_3, $(1 \leq i \leq 2)$, *and let* $V_H = V(H) - \{v_1, v_2, v_3\}$.

(i) *If* $H = \Gamma_i$, *then let* $G' = G - V_H$ *and we have*

$$cc(G) \leq cc(G') + 2. \tag{4}$$

(ii) *Suppose that* $H = \Gamma_i^+$. *If* e *in not incident with* v_1, *then let* $G'' = G - V_H + e_1$, *and if* e *is incident with* v_1, *then let* $G'' = G - V_H + e_2$. *In either case, we have*

$$cc(G) \leq cc(G'') + 2.$$

Proof: We consider the following cases.

Case 1: $i = 2$.

Let $\mathcal{C}$ be a CC of G' and let C be a cycle in $\mathcal{C}$ that contains $v_1 v_3$. Let $C' = C - v_1 v_3 + \{v_1 v_6, v_6 v_4, v_4 v_5, v_5 v_3\}$, let $F_1 = v_1 v_3 v_6 v_5 v_1$ and $F_2 = v_1 v_2 v_5 v_4 v_1$. Then $(\mathcal{C} - \{C\}) \cup \{C', F_1, F_2\}$ is a CC of G and so (4) holds.

The proof for (5) is similar and uses the fact that we can assume that e_1 and $v_1 v_3$ are in distinct cycles of any CC of G''.

Case 2: $i = 3$.

Let $\mathcal{C}$ be a CC of G' and let C_1, C_2 be cycles in $\mathcal{C}$ that contain $v_1 v_3$ and $v_2 v_3$, respectively. (It may happen that $C_1 = C_2$). Let $C_1' = C_1 - \{v_1 v_3\} + \{v_1 v_4, v_4 v_6, v_6 v_3\}$, $C_2' = C_2 - \{v_2 v_3\} + \{v_2 v_5, v_5 v_7, v_7 v_3\}$, (if $C_1 = C_2$, then $C_1' = C_2'$ is obtained by replacing $v_1 v_3$, $v_2 v_3$ by the above two paths, respectively), and let $F_1 = v_1 v_3 v_2 v_7 v_6 v_5 v_1$ and $F_2 = v_1 v_2 v_5 v_4 v_6 v_1$. Thus $(\mathcal{C} - \{C_1, C_2\}) \cup \{C_1', C_2', F_1, F_2\}$ is a CC of G, and so (4) holds.

Suppose that H is Γ_3^+ and e is not incident with v_1. Let $\mathcal{C}'$ be a CC for G'' and let C_1, C_2 be defined as above and let C_e be the cycle in $\mathcal{C}$ containing e_1.

If $E(C_2) \neq \{e_1, v_2 v_3\}$, then $C_e \neq C_2$. Since e is not incident with v_1, there is a (v_2, v_3)-path P in Γ_3 containing e such that the internal vertices of P are in V_H. Thus we can define C_e' to be $C_e - e_1$ plus the (v_2, v_3)-path P, and define C_1', C_2', F_1, F_2 as above. It follows that $(\mathcal{C} - \{C_1, C_2, C_e\}) \cup \{C_1', C_2', C_e', F_1, F_2\}$ is a CC of G and so (5) holds.

Thus we assume that $E(C_2) = E(C_e) = \{e_1, v_2 v_3\}$. Without loss of generality, we assume that e is not parallel to $v_5 v_7$. Let $F_3 = v_1 v_4 v_6 v_3 v_7 v_2 v_5 v_1$, $F_4 = v_1 v_4 v_5 v_6 v_7 v_2 v_3 v_1$, $C_1'' = C_1 - \{v_1 v_3\} + \{v_1 v_6, v_6 v_3\}$, and let F_5 be any cycle containing both $v_5 v_7$ and e. Thus $(\mathcal{C} - \{C_2, C_1\}) \cup \{C_1'', F_3, F_4, F_5\}$ is a CC of G, and so (5) holds.

The case when e is incident with v_1 can be shown similarly.

Case 3: $i = 4$.

Let $\mathcal{C}$ be a CC of G' and let C_1, C_2 be cycles in $\mathcal{C}$ containing $v_1 v_3$ and $v_2 v_3$, respectively. (Possibly $C_1 = C_2$). Let $C_1' = C_1 - \{v_1 v_3\} + \{v_1 v_4, v_4 v_6, v_6 v_3\}$ and $C_2' = C_2 - \{v_2 v_3\} + \{v_2 v_5, v_5 v_6, v_6 v_7, v_7 v_3\}$, and let $F_1 = v_1 v_3 v_2 v_8 v_7 v_5 v_6 v_1$ and $F_2 = v_1 v_2 v_7 v_6 v_4 v_5 v_1$. Then $(\mathcal{C} - \{C_1, C_2\}) \cup \{C_1', C_2', F_1, F_2\}$ is a CC of G and so (4) holds.

The proof when $H = \Gamma_4^+$ is similar to that for the Case of $i = 3$. ∎

Lemma 2.4. *Suppose that $H = \Gamma_5$ or $H = \Gamma_5^+$ (see Figure 3) with an extra edge e that is parallel to an edge of $E(\Gamma_5) - \{v_1 v_2, v_2 v_3, v_3 v_1\}$ such that H is a subgraph of G with $A_G(H) \subseteq \{v_1, v_2, v_3\}$. Let $V_H = V(H) - \{v_1, v_2, v_3\}$ and let $G' = G - V_H$. Then*

$$cc(G) \leq cc(G') + 3. \tag{6}$$

Proof: Let $\mathcal{C}$ be a CC of G' and let C_1, C_2, C_3 be cycles in $\mathcal{C}$ containing v_1v_3, v_2v_3 and v_1v_2, respectively.

Assume first that $H = \Gamma_5$. Then let $C_1' = C_1 - \{v_1v_3\} + \{v_1v_4, v_4v_6, v_6v_3\}$, $C_2' = C_2 - \{v_2v_3\} + \{v_2v_8, v_8v_7, v_7v_9, v_9v_3\}$, and $C_3' = C_3 - \{v_1v_2\} + \{v_1v_5, v_5v_2\}$, and let $F_1 = v_1v_4v_5v_6v_7v_2v_3v_1$, $F_2 = v_1v_6v_9v_7v_5v_8v_2v_1$ and let F_3 be any cycle in G containing v_3v_7. Thus $\mathcal{C} - \{C_1, C_2, C_3\} \cup \{C_1', C_2', C_3', F_1, F_2, F_3\}$ is a CC of G and so (6) holds.

Now we assume that H has one multiple edge e. Without loss of generality, we may assume that e is not parallel to v_3v_7. Thus one can choose F_3 above so that $e, v_3v_7 \in E(F_3)$ and so (6) holds also. ∎

Lemma 2.5. *Suppose that $H = \{\Gamma(6), \Gamma(6)^+\}$ (see Figure 3) such that H is a subgraph of G with $A_G(H) \subseteq \{w_1, w_2, w_3\}$. Let e be an extra edge not in $E(G)$ that is parallel to w_1w_2. If $H = \Gamma(6)$, then let $G' = G - \{w_4, w_5, w_6\}$, and if $H = \Gamma(6)^+$ (without loss of generality, we assume that the multiple edge in H is parallel to one of $\{w_1w_4, w_4w_5, w_5w_6, w_2w_4, w_2w_5, w_2w_6\}$), then let $G' = G - \{w_4, w_5, w_6\} + e$. In any case, we have*

$$cc(G) \leq cc(G') + 2.$$

Proof: Let $F_1 = w_1w_4w_2w_5w_3w_1$ and $F_2 = w_2w_6w_5w_3w_4w_2$. Let $\mathcal{C}$ be a CC of G' and let C_1 be a cycle in $\mathcal{C}$ containing w_1w_3. Assume first that $H = \Gamma(6)$. Define $C_1' = C_1 - w_1w_3 + w_1w_4w_5w_6w_3$. Then $(\mathcal{C} - \{C_1\}) \cup \{C_1', F_1, F_2\}$ is a CC of G, and so (7) must hold.

Then we assume that $H = \Gamma(6)^+$. Since e is parallel to w_1w_2 in G', we may assume that e does not lie in C_1. Let $C_e \in \mathcal{C}$ be a cycle containing e. Let C_e' be obtained from C_e by replacing e by a (w_1, w_2)-path in H that covers the multiple edge. Hence $(\mathcal{C} - \{C_1, C_e\}) \cup \{C_1', C_e', F_1, F_2\}$ is a CC of G, and so (7) holds also. ∎

Lemma 2.6. *Let H be a subgraph of G.*

(i) *Suppose that $H = \Gamma_6$ (see Figure 4) or $H = \Gamma_6^+$ with an extra edge e parallel to an edge of $E(\Gamma_6 - \{x_1x_2, x_2x_3, x_3x_4, x_4x_1\})$ and with $A_G(H) \subseteq \{x_1, x_2, x_3, x_4\}$. If $H = \Gamma_6$, then define $G_1 = G - \{x_5, x_6\}$; and if $H = \Gamma_6^+$, then define $G_1 = G - \{x_5, x_6\} + e'$, where $e' \notin E(G)$ is an extra edge parallel to x_2x_4. We have*

$$cc(G) \leq cc(G_1) + 1. \qquad (8)$$

(ii) *Suppose that $H = L_6$ or $H = L_6^+$ (see Figure 4) with an extra edge e that is parallel to an edge of $E(\Gamma_6 - \{x_1x_2, x_2x_3, x_3x_1\})$ and with $A_G(H) \subseteq \{x_1, x_2, x_3, x_4\}$ and $e' \notin E(G)$ be an extra edge. If $H = L_6$, then let $G_2 = G - \{x_5, x_6\} + x_2x_4$; and if $H = L_6^+$, then let $G_2 = G - \{x_5, x_6\} + x_2x_4 + e'$,* *where e' is parallel to x_4x_5 if e is parallel to x_3x_6 or x_4x_6; or where e' is parallel to x_2x_4 if e is parallel to x_2x_6 or x_2x_5; or where e' is parallel to x_1x_4 if e is parallel to x_1x_5 or x_1x_6. In any case, we have*

$$cc(G) \leq cc(G_2) + 1. \tag{9}$$

(iii) *Let $L \in \{L'_8, L''_8, L'''_8\}$ (see Figure 6) and let $e' \notin E(G)$ be an extra edge parallel to x_2x_4. If $H = L$ or $H = L^+$ with an extra edge e that is parallel to an edge of $E(H - \{x_1x_2, x_2x_3, x_3x_4, x_4x_1\})$ and with $A_G(H) \subseteq \{x_1, x_2, x_3, x_4\}$, then defining $G_3 = G - \{x_5, x_6, x_7, x_8\} + e'$, we have*

$$cc(G) \leq cc(G_3) + 2. \tag{10}$$

(iv) *Let $L''_7 = L''_8 - x_8$. If $H = L''_7$ or $H = L''^+_7$ with an extra edge e parallel to an edge in $E(H) - \{x_1x_2, x_2x_3, x_3x_4, x_4x_1\}$ and with $A_G(H) \subseteq \{x_1, x_2, x_3, x_4\}$, then defining $G_4 = G - \{x_5, x_6, x_7\}$, we have*

$$cc(G) \leq cc(G_4) + 2. \tag{11}$$

(v) *Let $L'_6 = L''_8 - \{x_5, x_6\}$ and let $L''_6 = L''_8 - \{x_7, x_8\}$. Suppose that $H \in \{L'_6, L''_6\}$ or $H \in \{L'^+_6, L''^+_6\}$ with an extra edge e parallel to an edge in $E(H) - \{x_1x_2, x_2x_3, x_3x_4, x_4x_1\}$ and with $A_G(H) \subseteq \{x_1, x_2, x_3, x_4\}$. If $H \in \{L'_6, L''_6\}$, then let $G_5 = (G - \{x_5, x_6, x_1x_4, x_4x_3\})/\{x_2x_4\}$, and if $H \in \{L'^+_6, L''^+_6\}$, then let $G_5 = (G - \{x_5, x_6, x_1x_4, x_4x_3\} + e')/\{x_2x_4\}$, where $e' \notin E(G)$ is an extra edge parallel to x_2x_3. In any case, we have*

$$cc(G) \leq cc(G_5) + 2. \tag{12}$$

Proof: We shall show (i) first. Suppose that $H = \Gamma_6$ and that C is a CC of G_1 and let C_1 and C_2 be cycles in C such that $x_1x_4 \in E(C_1)$ and $x_3x_4 \in E(C_2)$. Let $C'_1 = C_1 - x_1x_4 + x_1x_5x_4$, $C'_2 = C_2 - x_3x_4 + x_3x_6x_4$, and let $F = x_1x_5x_2x_6x_3x_4x_1$. Then $(C - \{C_1, C_2\}) \cup \{C'_1, C'_2, F\}$ is a CC of G and so (8) holds.

Suppose that $H = \Gamma_6^+$. By the case when $H = \Gamma_6$, we may assume that e is parallel to one of $\{x_2x_5, x_5x_4, x_2x_6, x_6x_4\}$ and so we can replace e' by a (x_2, x_4)-path that passes the multiple edge e.

To show (ii) of Lemma 2.6, we assume that $H = L_6$ and let C be a CC of G_1. Let $C_1, C_2, C_3 \in C$ be cycles containing x_1x_2, x_2x_4 and x_2x_3, respectively. Since no cycle can contain x_1x_2, x_2x_4, x_2x_3 simultaneously, we may assume that either $C_1 \neq C_2$ or $C_2 \neq C_3$.

If $C_1 \neq C_2$, then let $C'_1 = C_1 - x_1x_2 + x_1x_6x_2$ and let $C'_2 = C_2 - x_2x_4 + x_2x_5x_6x_4$. Let $F = x_1x_2x_3x_6x_5x_1$. Replace x_2x_4 by $x_2x_6x_4$ in any cycle in

$C - \{C_1, C_2\}$ containing x_2x_4 and still denote the resulting collection by $C - \{C_1, C_2\}$, for convenience. Thus $(C - \{C_1, C_2\}) \cup \{C_1', C_2', F\}$ is a CC of G and so (9) must hold.

If $C_2 \neq C_3$, then let $C_2' = C_2 - x_2x_4 + x_2x_6x_4$ and let $C_3' = C_3 - x_2x_3 + x_2x_5x_6x_3$, and let $F' = x_1x_5x_2x_3x_6x_1$. Thus $(C - \{C_2, C_3\}) \cup \{C_2', C_3'\}$ is a CC of G. Hence (9) must hold again.

When $H = L_6^+$, we can replace e' in the cycle containing e' by a path in H containing the multiple edge and so (9) holds again.

To show (iii), we let C be a CC of G_3 and let $C_e, C_0, C_3, C_4 \in C$ be cycles that contain e', x_2x_4, x_3x_4, and x_4x_1 respectively.

Assume that $H = L_8''$ or $L_8''^{+}$. Since $[e'] = \{e', x_2x_4\}$, we may assume that $C_4 \neq C_0$ and $C_3 \neq C_e$. Let $F_1 = x_1x_7x_8x_2x_5x_6x_3x_4x_1$ and let $C_0' = C_0 - x_2x_4 + x_2x_5x_6x_4$, $C_3' = C_3 - x_3x_4 + x_3x_5x_4$, $C_e' = C_e - e' + x_2x_7x_8x_4$ and $C_4' = C_4 - x_4x_1 + x_4x_7x_1$, and let F_2 be a cycle containing x_2x_4 and the multiple edge (if it exists). Thus $(C - \{C_0, C_3, C_4, C_e\} \cup \{C_0', C_3', C_4', C_e', F_1, F_2\}$ is a CC of G and so (10) holds also.

The proof for the case when $H \in \{L_8', L_8'^{+}, L_8''', L_8'''^{+}\}$ is similar to that for $H \in \{L_8'', L_8''^{+}\}$ and the proof for (iv) is similar to that for (iii). Thus they are omitted.

We shall show (v) for $H \in \{L_6'', L_6''^{+}\}$. The proof for $H \in \{L_6', L_6'^{+}\}$ is similar. Let Let v denote the vertex in G_5 to which x_2x_4 is contracted. Let C be a CC of G' and let C_e, C_3 be cycles in C containing e' and vx_3, respectively. (If $H = L_6''$, then just take C_3).

If $C_e = C_3$, then let $F' = x_1x_2x_3x_6x_5x_4x_1$, $F'' = x_2x_5x_3x_6x_4x_2$ and let F''' be a cycle that contains the multiple edge e. Thus $(C - \{C_3\}) \cup \{F', F'', F'''\}$ is a CC of G and so (12) holds.

Thus we assume that $C_e \neq C_3$. Let $C_3' = G[E(C_3) - vx_3]$. Thus either $C_3' + x_2x_3$ or $C_3' + x_3x_4$ is a cycle in G. Note that any cycle in $C - \{C_e, C_3\}$ can easily be adjusted to cycles in G (still denoted by $C - \{C_e, C_3\}$, for convenience).

Let $F_1 = x_1x_2x_5x_6x_3x_4x_1$, $F_2 = x_2x_3x_6x_4x_2$, and let C_e' be obtained from C_e by replacing e' by an (x_2x_3)-path containing e. If $C_3' + x_2x_3$ is a cycle in G, then let $C_3'' = C_3' + x_2x_6x_5x_4x_3$; and if $C_3' + x_3x_4$ is a cycle in G, then let $C_3'' = C_3' + x_4x_2x_6x_5x_3$. Thus in any case, $(C - \{C_3, C_e\}) \cup \{C_3'', C_e', F_1, F_2\}$ is a CC of G, and so (12) holds. ∎

Let C be a cycle of a plane graph G. Define $IntC$ to be the vertices of G inside (exclusively) C. Define $ExtC$ similarly. The cycle C is *trivial* if $IntC = \emptyset$; and is *acyclic* if the underlying simple graph of $G[IntC]$ is acyclic.

A *k-face* of a plane graph G is a face of degree k. Define $L(n)$ as the graph in Figure 5.

Lemma 2.7. *Let G be a plane triangulation with $\mu(G) = 1$ and with $n = |V(G)| \geq 3$. If the exterior face of G is a 2-cycle C, and if C is acyclic, then*

$G \cong L(n)$.

Proof: Let v_1, v_2 be the two vertices in $V(C)$ and let e_1, e_2 be the two edges in $E(C)$. Since $\mu(G) \leq 1$, and since G is a plane triangulation, e_1 must lie in a 3-face C_1 inside C. Let v_3 be the vertex in $V(C_1) - \{v_1, v_2\}$. If v_3 has degree at least 4, then since G is a triangulation, v_3 and two of its neighbors other than v_1, v_2 would form a 3-cycle inside C, contrary to the assumption that C is acyclic. If v_3 has degree 2, then we have $n = 3$ and $G = L(3)$. Hence v_3 has degree 3, and so $G - v_3$ is also a plane triangulation with C as a acyclic exterior face. Thus by induction, $G - v_3 = L(n-1)$ and so $G = L(n)$. ∎

Lemma 2.8. *Let G be a simple plane triangulation. If the exterior face of G is a 3-cycle C and if C is acyclic, then either G contains a subgraph $H \in \{L_6, \Gamma_6\}$ (using the notation in Figure 4) with $A_G(H) \subseteq \{x_1, x_2, x_3, x_4\}$ or $G = \Gamma(n)$, where $n = |V(G)|$.*

Proof: Let $C = v_1 v_2 v_3 v_1$. Since G is a plane triangulation, $v_2 v_3$ lies in a 3-face $C_1 = v_2 v_3 v v_2$ with $v \in IntC$. Let $v_2 = u_1, u_2, \ldots, u_m = v_3$ be the neighbors of v in G such that they are ordered clockwise by the planar imbedding of G.

Since G is a simple plane triangulation, $v v_2 u_2 v, v u_2 u_3 v, \ldots,$ must be 3-faces. Since C is acyclic, either $m = 3$, or $4 \leq m \leq 5$ and $u_3 = v_1$.

If $m = 5$ and $u_3 = v_1$, then G contains a subgraph $H = \Gamma_6$ with $x_1 = v_1, x_2 = v_2, x_3 = v, x_4 = v_3$. If $m = 4$ and $u_3 = v_1$, then, since $v v_1 v_3 v$ and $v_2 v v_3 v_2$ are now 3-faces, $G = \Gamma(5)$. Hence we may asssume that $m = 3$. If $x_2 = v_1$, then $G = \Gamma(4)$. Thus we assume that $x_2 \neq v_1$, and so $G - v$ is also a plane triangulation with C as the exterior face. By induction, Lemma 2.8 holds. ∎

Lemma 2.9. *Suppose that G is a plane graph, and that G has a nontrivial 2-cycle C with $V(C) = \{v_1, v_2\}$ and $E(C) = \{e_1, e_2\}$. Let $H = G - ExtC$.*

(i) *If $H = L(3)^+$ such that the extra edge e is parallel to $v_1 v_3$, then letting $G' = G/v_3 v_2$, we have $cc(G) \leq cc(G')$.*

(ii) *If $H = L(4)$, then letting $G' = G - IntC$, we have $cc(G) \leq cc(G') + 1$.*

(iii) *If $H = L(4)^+$ such that the extra edge e is not parallel to any of $\{e_1, e_2\}$, then letting e' be an extra edge parallel to e_1 and let $G' = G - IntC + e'$, we have $cc(G) \leq cc(G') + 1$.*

(iv) *If H is isomorphic to $\Gamma(5)^+$, such that the exterior face of H is C, then letting $G' = G - IntC$, we have $cc(G) \leq cc(G') + 2$.*

Proof: (i) of Lemma 2.9 is trivial. We now show (ii). Let $\mathcal{C}$ be a CC of G' and let C_1 be a cycle in $\mathcal{C}$ containing e_1. Define $C_1' = C_1 - e_1 + \{v_1 v_3, v_3 v_4, v_4 v_2\}$ and $F = G[\{e_1, v_1 v_4, v_4 v_3, v_3 v_2\}]$. Thus $\mathcal{C} - \{C_1\} \cup \{C_1', F\}$ is a CC of G and so (ii) of Lemma 2.9 holds.

Now we show (iii). Let $\mathcal{C}$ be a CC of G'. Note that $[e_1] = \{e_1, e_2, e'\}$ in G' this time. We may assume that e_1 and e' are in distinct cycles C_1 and C_e, respectively. Define C_1' and F as above. Since e is not parallel to e_1, there is a (v_1, v_2)-path P in $H - \{e_1, e_2\}$ containing e. Define $C_e' = C_e - e' + P$. Thus $C - \{C_1, C_e\} \cup \{C_1', C_e', F\}$ is a CC of G.

Now we show (iv). Let C be a CC of G' and let C_i be a cycle in C containing e_i, $(1 \leq i \leq 2)$. Note that no matter where e_1, e_2 lie in H, $H - \{e_1, e_2\}$ always has a spanning cycle and so $H - \{e_1, e_2\}$ has two internally disjoint (v_1, v_2)-paths P_1 and P_2. Let $C_i = C_i - e_i + P_i$, $(1 \leq i \leq 2)$. (When $C_1 = C_2 = C$, let $C_1' = P_1 \cup P_2$.) Thus it is easy to see that the edges in $H - E(P_1) \cup E(P_2)$ can be covered by two cycles in H and so (iv) follows. ∎

Define plane graphs $\Gamma^4, \Gamma^5, \Gamma^6$ as the graphs in Figure 6.

Lemma 2.10. *Let G be a plane triangulation with $\mu(G) \leq 1$ and with $4 \leq |V(G)| \leq 5$. If the exterior face of G is a 3-face, then G is isomorphic to one graph in* $\{\Gamma(4), \Gamma(5), \Gamma(4)^+, \Gamma(5)^+, \Gamma^4, \Gamma^5\}$.

Proof: The proof is straightforward. ∎

For each i, $(1 \leq i \leq 5)$, define Γ_i' to be the simple plane triangulation obtained from Γ_i by adding a new vertex v_0 in the exterior face of Γ_i and by joining v_0 to each of v_1, v_2, v_3 with a new edge, respectively.

Lemma 2.11. *: If G is isomorphic to one of the graphs below,*

$$\{\Gamma_i, \Gamma_i', \Gamma_i^+, (\Gamma_i')^+,\ (1 \leq i \leq 5), \Gamma(6), \Gamma(6)^+, L(6), L(6)^+, \Gamma_6, \Gamma_6^+\},$$

then

$$cc(G) \leq \frac{2|V(G)| - 3}{3} + \frac{\mu(G)}{2}.$$

Proof: The proof is routine and so is omitted. ∎

3. The Proof of Theorem 1.2

We argue by contradiction and assume that

$$G \text{ is a counterexample to Theorem 1.2} \tag{13}$$

such that

$$|V(G)| + \mu(G) \text{ is as small as possible,} \tag{14}$$

and subject to (14),

$$|E(G)| \text{ is minimized.} \tag{15}$$

If G has two 2-faces, then we pick two distinct edges, e, e' (say), from each of these 2-faces. Thus $\mu(G) = \mu(G - \{e, e'\}) + 2$ and so by (14) and (15), G is not a counterexample, contrary to (13). Hence we assume that

$$G \text{ has at most one 2-face.} \tag{16}$$

Since G is a plane triangulation, and by (14), we have

$$\kappa(G) \geq 2 \text{ and } \delta(G) \geq 3. \tag{17}$$

Lemma 3.1. *If C is a nontrivial 2-face of G, then C is the exterior face of G.*

Proof: Suppose that G has a nontrivial 2-cycle C that is not the exterior face of G. Suppose also that C is so chosen that there is no nontrivial 2-cycle properly contained in the interior of C.

Case 1: $|IntC| = 1$.

By (16) and (17), $G[IntC \cup V(C)] = L(3)^+$. Define G' as in (i) of Lemma 2.9. If $|V(G')| \geq 6$, then by (14) we have

$$cc(G) \leq cc(G') \leq \frac{2(n-1)-3}{3} + \frac{\mu(G)+1}{2},$$

and so G is not a counterexample, a contradiction.

Hence by $|V(G)| \geq 6$, $|V(G')| = 5$. It follows by Lemma 2.10 and by (17) that G' is either spanned by $\Gamma(5)$ with $\mu(G') = 3$ or spanned by Γ^5 with $\mu(G') = 4$ Thus $cc(G) \leq cc(G') \leq 4$, by (i) of Lemma 2.9, a contradiction.

Case 2: $|IntC| = 2$.

By (16), $G[IntC \cup V(C)] = L(4)$ or $L(4)^+$. Define G' as in (ii) or (iii) of Lemma 2.9. If $|V(G')| \geq 6$, then by arguing as above, one can derive a contradiction. Hence we assume that $|V(G')| \leq 5$, and so by Lemma 2.10 and by the fact that G' must have a 2-face, G' is isomorphic to one of $\{\Gamma(4)^+, \Gamma(5)^+, \Gamma^{4+}, \Gamma^{5+}\}$. Since $cc(\Gamma(4)^+) = cc(\Gamma^{4+}) = 2$ and $cc(\Gamma(5)^+) = cc(\Gamma^{5+}) = 3$, and since $\mu(G') \leq 1$, it follows by (ii) and (iii) of Lemma 2.9 that G satisfies (1), contrary to (13).

Case 3: $|IntC| \geq 3$.

We may similarly assume that $|ExtC| \geq 3$. If both $|IntC| = |ExtC| = 3$, then $G[IntC \cup V(C)] \cong \Gamma(5)^+$ and so by Lemma 2.10 and by (iv) of Lemma 2.9, $cc(G) \leq 3+2 = 5$, contrary to (13). Thus we assume that $|ExtC| \geq 4$. Let $H = G[IntC \cup V(C)]$. If $|V(H)| \geq 6$, then by Lemma 2.1, by (14), and noting that

$$\mu(H) + \mu(G - IntC) = \mu(G) + 1,$$

we have

$$cc(G) \leq cc(G - IntC) + cc(H) - 1 \leq \frac{2n-3}{3} + \frac{\mu(G)}{2}. \tag{18}$$

Hence we assume that $|V(H)| \leq 5$. Since H has an exterior 2-face, it is then easy to see by Lemma 2.9 that G satisfies (1), contrary to (13). ∎

Lemma 3.2. $\mu(G) \leq 1$.

Proof: Suppose that $\mu(G) \geq 2$. Then by (16) and Lemma 3.1, G must have parallel edges e_1, e_1' and parallel edges e_2, e_2' with $[e_1] \neq [e_2]$ such that, for each i, $G[\{e_i, e_i'\}]$ is a trivial 2-cycle of G. Note that $G' = G - \{e_1', e_2'\}$ is a plane triangulation. By (17) and since $[e_1] \neq [e_2]$, G has a cycle containing both e_1' and e_2'. Note that $\mu(G') = \mu(G) - 2$. By (15),

$$cc(G) \leq cc(G') + 1 \leq \frac{2n-3}{3} + \frac{\mu(G)-2}{2} + 1,$$

contrary to (13). ∎

Lemma 3.3. *Each of the following subgraphs is forbidden in G:*

(i) $H \in \{L(3)^+, L(4), L(4)^+\}$ *with* $A_G(H) \subseteq \{v_1, v_2\}$.
(ii) $H = \Gamma_i$ *or* Γ_i^+ *with* $A_G(H) \subseteq \{v_1, v_2, v_3\}, (1 \leq i \leq 5)$.
(iii) $H \in \{L, L^+\}$ *where* $L \in \{\Gamma_6, L_6, L_7'', L_8', L_8'', L_8'''\}$ *with* $A_G(H) \subseteq \{x_1, x_2, x_3, x_4\}$.
(iv) $H = \Gamma(6)$ *or* $\Gamma(6)^+$ *with* $A_G(H) \subseteq \{v_1, v_2, v_3\}$.

Proof: Assume that $H = \Gamma_1$ and let G' be defined as in Lemma 2.2. If $|V(G')| = 3$, then $G = \Gamma_1$ or Γ_1^+ and so (1) holds for G. If $|V(G')| \geq 6$, then by (14) and by Lemma 2.2,

$$cc(G) \leq cc(G') + 1 \leq \frac{2(n-3)-3}{3} + \frac{\mu(G)}{2} + 1,$$

and so G is not a counterexample, a contradiction. Thus by $|V(G)| \geq 6$, we have $4 \leq |V(G')| \leq 5$. Thus by Lemma 2.10, one can easily check that G is not a counterexample, contrary to (13).

The proofs for the other cases are similar, by using reduction lemmas in section 2. ∎

Proof of Theorem 1.2: Since G is a plane triangulation, the exterior face of G is either a 2-face or a 3-face. Since $|V(G)| \geq 6$, G must have a nontrivial 3-cycle. If every nontrivial 3-cycle of G is acyclic, then in particular, the exterior 3-face or the 3-face obtained by deleting an edge from the exterior 2-face is also acyclic. Thus by Lemma 2.7 and Lemma 2.8, G must contain either Γ_6, L_6 or $\Gamma(6)$, contrary to Lemma 3.3. Hence G has a cyclic 3-cycle. Let C_0 be a cyclic 3-cycle of G such that

$$|IntC_0| \text{ is minimized.} \quad (19)$$

By (19), any 3-cycle contained in $G[IntC_0]$ is either trivial or acyclic. By Lemmas 2.8 and 3.3, if Z is a nontrivial 3-cycle in $G[IntC_0]$, then

$$|IntZ| \leq 2. \quad (20)$$

Let $M = G[IntC_0 \cup V(C_0)]$. Let $C = u_1u_2u_3u_1$ be a trivial 3-cycle in M such that $|V(C) \cap V(C_0)| = 0$. Since G is a plane triangulation with $\mu(G) \leq 1$, M has a 3-cycle containing u_iu_{i+1}, ($i \equiv 1, 2, 3 \pmod 3$). Let C_i be a 3-cycle in M containing u_iu_{i+1} such that $E(C) \cap E(G[IntC_i \cup V(C_i)]) = \{u_iu_{i+1}\}$ and such that

$$|IntC_i| \text{ is maximized.} \tag{21}$$

Case 1: For $i \neq j$, $E(C_i) \cap E(C_j) = \emptyset$.

Let $C_1 = u_1u_2u_4u_1$, $C_2 = u_2u_3u_5u_2$, and $C_3 = u_1u_3u_6u_1$. Thus u_4, u_5, u_6 are distinct.

Suppose first that $|IntC_i| = 0$, $(1 \leq i \leq 3)$. Define

$$G_a = (G - \{u_1u_6, u_3u_5, u_2u_4\})/E(C), \tag{22}$$

and let u denote the vertex in G_a to which C is contracted. By (21), no new multiple edge will be produced by the contraction, and so

$$\mu(G_a) \leq \mu(G). \tag{23}$$

Since G is a plane triangulation and since the boundaries of other faces not incident with $V(C)$ are unchanged, G_a is also a plane triangulation. We shall show

$$cc(G) \leq cc(G_a) + 1. \tag{24}$$

Let $\mathcal{C}$ be a CC of G_a, and let $C_j \in \mathcal{C}$ such that $uu_j \in V(C_j)$, $(4 \leq j \leq 6)$. For any cycle L in G_a, L can be extended to a cycle L' in G, by using edges in $E(C)$, if neccessary.

It is easy to see that we can extend C_4, C_5, C_6 to C_4', C_5', C_6' so that any specified edge in $E(C)$ can be covered twice by C_4', C_5', C_6'.

In fact, without loss of generality, we may assume that $|[u_1u_2]| = 2$. Define $L_j = G[E(C_j) - \{uu_4, uu_5, uu_6\}]$. When C_4, C_5, C_6 are distinct, L_j is a path in G joining u_j to a vertex $u_j' \in V(C)$, $(4 \leq j \leq 6)$. If $C_4 = C_5$, then define C_4', C_6' as follows:

if $u_6' = u_1$, then $C_4' = L_4 + u_4u_2u_1u_3u_5$ and $C_6' = L_6 + u_6u_3u_2u_1$;
if $u_6' = u_2$, then $C_4' = L_4 + u_4u_1u_2u_3u_5$ and $C_6' = L_6 + u_6u_3u_1u_2$;
if $u_6' = u_3$, then $C_4' = L_4 + u_4u_2u_1u_3u_5$ and $C_6' = L_6 + u_6u_1u_2u_3$.

When C_4, C_5, C_6 are all distinct, define C_4', C_5', C_6' as follows:

if $u_5' = u_1$, then $C_5' = L_5 + u_5u_3u_2u_1$;
if $u_5' = u_2$, then $C_5' = L_5 + u_5u_3u_1u_2$;
if $u_5' = u_3$, then $C_5' = L_5 + u_5u_2u_1u_3$;
if $u_6' = u_1$, then $C_6' = L_6 + u_6u_3u_2u_1$;
if $u_6' = u_2$, then $C_6' = L_6 + u_6u_1u_2u_3$;
if $u_6' = u_3$, then $C_6' = L_6 + u_6u_1u_2u_3$;

and choose C_4' so that the remaining edge in $E(C)$, if there is any, is covered by C_4'.

Let $F_1 = u_1u_4u_2u_5u_3u_6u_1$. Then $\{L' : L \in \mathcal{C}\} \cup \{F\}$ is a CC of G and so (23) holds.

If $|V(G_a)| \geq 6$, then by (14), (22) and (23),

$$cc(G) \leq cc(G_a) + 1 \leq \frac{2(n-2)-3}{3} + \frac{\mu(G)}{2} + 1,$$

contrary to (13).

If $|V(G_a)| \leq 5$, then since $|V(C) \cap V(C_0)| = 0$ and since u is a vertex of degree at least 3 in G_a, it follows by Lemma 2.10 that $G \in \{\Gamma_1, \Gamma_1^+, \Gamma_1', (\Gamma_1')^+\}$ and so by Lemma 2.11, G is not a counterexample, either.

By (20), we need to consider the cases when exactly k of the C_i's are nontrivial, where $1 \leq k \leq 3$. The proofs for these subcases are similar to that when $k = 0$ and so are omitted.

Case 2: For some $i \neq j$, $E(C_i) \cap E(C_j) \neq \emptyset$.

If $E(C_i) \cap E(C_j) \neq \emptyset$, for every $i \neq j$, then $u_4 = u_5 = u_6$, contrary to Lemma 3.1 or to the assumption that $n \geq 6$. Hence we assume that

$$E(C_3) \cap (E(C_1) \cup E(C_2)) = \emptyset \text{ and } u_4 = u_5. \tag{25}$$

(2A) $IntC_1 \neq \emptyset$ and $IntC_2 \neq \emptyset$ or $|IntC_3| \geq 1$ and $IntC_1 = IntC_2 = \emptyset$. Then $G[IntC_1 \cup IntC_2 \cup \{u_1, u_2, u_3, u_4\}]$ contains a subgraph isomorphic to one of $\{\Gamma_6, \Gamma_6^+, L_7'', L_7''^+\}$, contrary to Lemma 3.3. Thus we assume that

$$IntC_1 = \emptyset. \tag{26}$$

(2B) $|IntC_1| = 0$ and $|IntC_2| > 0$. Then G has a forbidden subgraph H isomorphic to one of $\{L_6', L_6''\}$. This case can be excluded by applying (v) of Lemma 2.6.

(2C) $IntC_1 = IntC_2 = IntC_3 = \emptyset$ and $u_4 \in IntC_0$.

Since G is a triangulation, there are u_7 and u_8 in $V(G)$, such that $C_4 = u_1u_7u_4u_1$ and $C_5 = u_3u_8u_4u_3$ are 3-cycles satisfying (21). Applying the previous argument to the 3-cycles C_4 and C_5, we conclude that $IntC_4 = IntC_5 = \emptyset$. Let $H = G[\{u_1, u_2, u_3, u_4\}]$ and let $G_b = (G - \{u_1u_6, u_7u_4, u_8u_3\})/E(H)$. Imitating the proof for (24), we can similarly show first that

$$\mu(G_b) \leq \mu(G) \text{ and } cc(G) \leq cc(G_b) + 2, \tag{27}$$

and then that G is not a counterexample, contrary to (13).

(2D) $IntC_1 = IntC_2 = IntC_3 = \emptyset$ and $u_4 \notin IntC_0$.

Thus $u_4 \in V(C_0)$. It follows from Case 1 and Cases (2A) - (2C) that for any trivial 3-cycle $C' = z_1 z_2 z_3 z_1$ in $G[IntC_0]$, there must be $z_4, z_5 \in V(C_0)$ such that $z_4 z_2 z_1 z_4$, $z_4 z_2 z_3 z_4$ and $z_1 z_3 z_5 z_1$ are trivial 3-faces in M, with $z_i = u_i$ $(1 \leq i \leq 4)$ and $z_5 = u_6$. Note that $C'' = z_1 z_5 z_4 z_1$ must be a trivial 3-face since otherwise G contains a L_6, contrary to Lemma 3.3. Call $G[\{z_1, z_2, z_3, z_4, z_5\}]$ an associated $\Gamma(5)$ with edge $z_4 z_5 \in E(C_0)$. For each edge in $E(C_0)$, there is at most one associated $\Gamma(5)$ with the given edge. Delete z_1, z_2 from the associated $\Gamma(5)$ with $z_4 z_5$, and do the same for other associated $\Gamma(5)$'s with other edges in $E(C_0)$, (if there are any). Then the resulting graph is again a triangulation in which C_0 is an acyclic 3-cycle, and so by Lemmas 7 and 8, either M contains a trivial 3-cycle that satisfies Case 1 or one of Cases (2A) - (2C), or G contains L_6 or $\Gamma(6)$, or $M - E(C_0)$ is isomorphic to the graph L_{11} in Figure 8.

Thus we may assume that $M - E(C_0) \cong L_{11}$. Let $G_c = G - \{z_8, z_9, z_{10}\}$. Then it is easy to see that

$$cc(G) \leq cc(G_c) + 2. \qquad (28)$$

Thus by (14) and since $|V(G_c)| \geq 7$, G must satisfy (1), contrary to (13).

Since every case leads to a contradiction, Theorem 1.2 is proved. ∎

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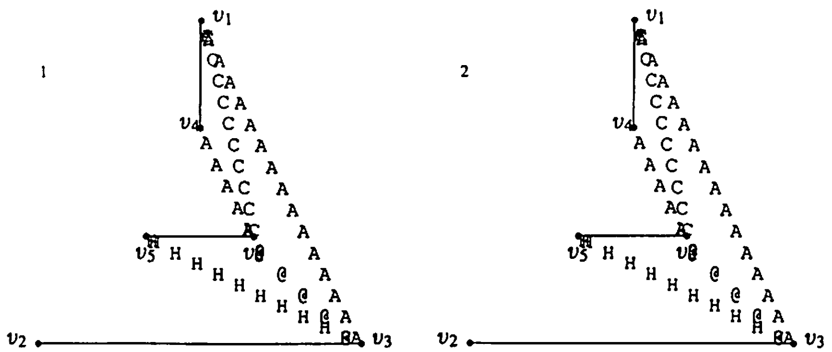

Figure 1 : The graphs $_1$ and $_2$

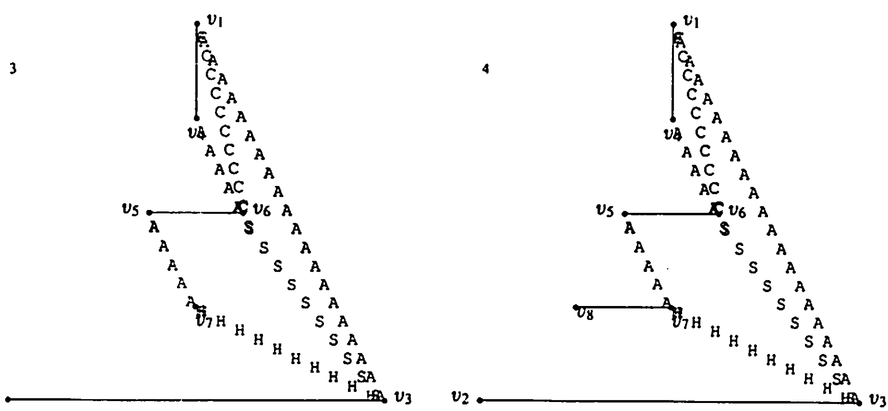

Figure 2 : The graphs $_3$ and $_4$

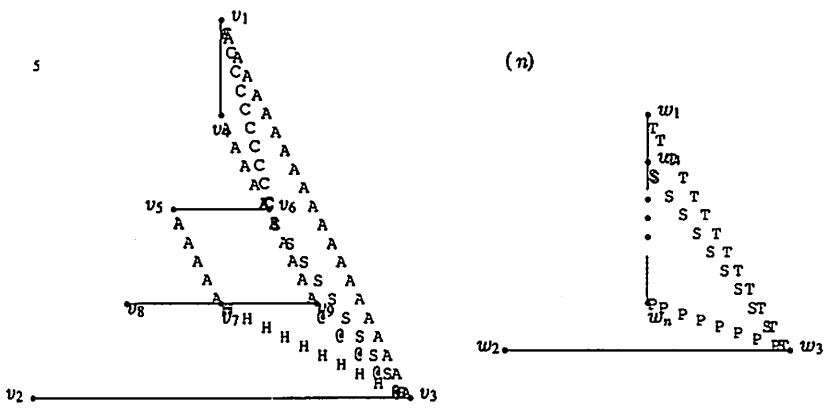

Figure 3 : The graphs $_5$ and (n)

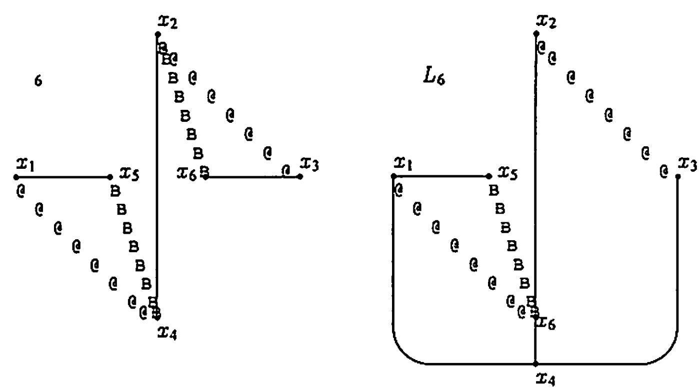

Figure 4 : The graphs $_6$ and L_6

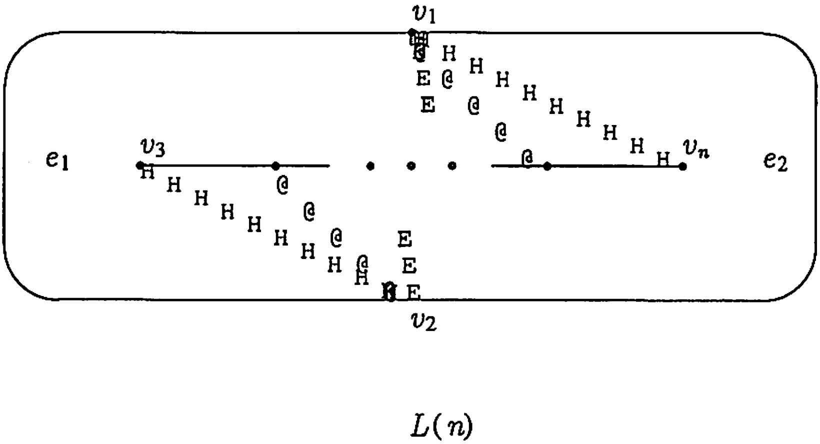

Figure 5 : The graph $L(n)$

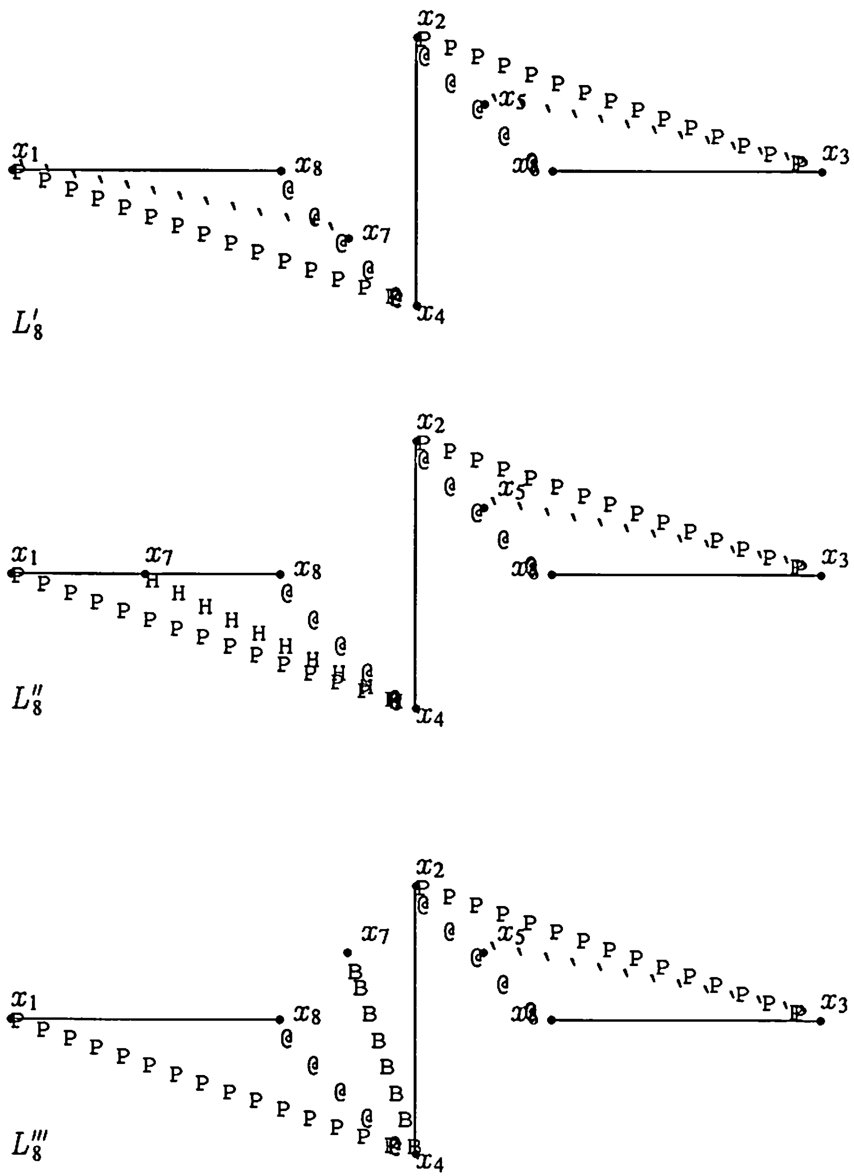

Figure 6 : Graphs L'_8, L''_8 and L'''_8

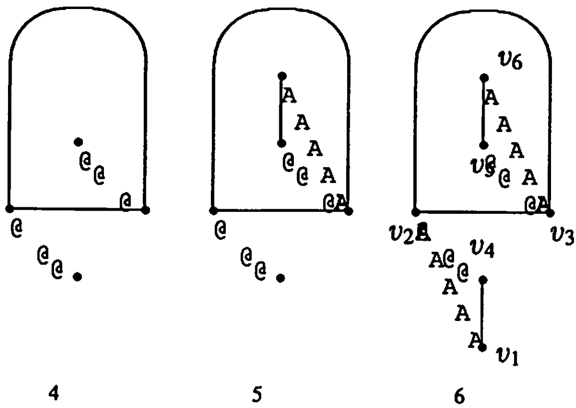

Figure 7 : Graphs $^{4}, ^{5}, ^{6}$

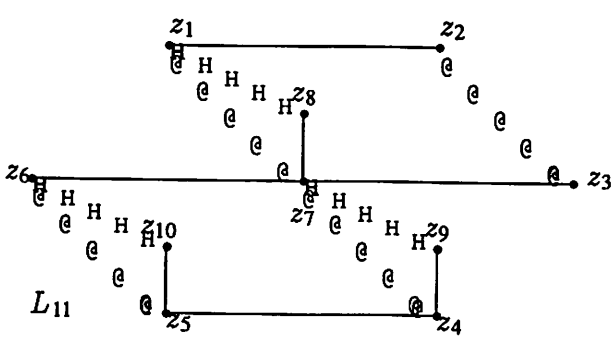

Figure 8 : Graph L_{11}