Graphs of maximum degree 3 and order at most 16 which are critical with respect to the total chromatic number.

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Abstract. We give a list of all graphs of maximum degree three and order at most sixteen which are critical with respect to the total chromatic number.

Introduction

A *total colouring* of a graph G is a map $\phi : E(G) \cup V(G) \rightarrow C$, where C is a set of colours, such that no two incident or adjacent elements of $E(G) \cup V(G)$ receive the same colour. The *total chromatic number* $\chi_T(G)$ of G is the least value of $|C|$ for which G has a total colouring. The maximum and minimum degrees of G are denoted by $\Delta(G)$ and $\delta(G)$ respectively.

In 1965 Behzad [1] conjectured that, for a simple graph G

$$\Delta(G) + 1 \leq \chi_T(G) \leq \Delta(G) + 2.$$

The lower bound is trivial, but the upper bound has so far been intractable. If $\chi_T(G) = \Delta(G) + 1$, then G is *Type 1*, and if $\chi_T(G) = \Delta(G) + 2$, then G is *Type 2*. So far no simple graphs have been found which are not Type 1 or Type 2. The total chromatic number conjecture of Behzad [1] is known to be true for graphs G satisfying $\Delta(G) \leq 4$ or $\Delta(G) \geq \frac{3}{4}|V(G)|$ ([9]). It has also been proved for quite a large number of other classes of graphs. McDiarmid and Sanchez-Arroyo [10] have recently shown that the problem of determining the total chromatic number of cubic bipartite graphs is NP-complete.

A graph G is *critical* with respect to the total chromatic number if it is connected and

$$\chi_T(G\backslash e) < \chi_T(G)$$

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for each edge e of G. In this paper we give a list of all critical graphs of maximum degree three and order at most sixteen. The graphs are listed overall according to the number of vertices they have (i.e. their order). Subject to that, they are listed in eigenvalue order, with the graphs with largest eigenvalue preceding graphs of lower eigenvalue; when the largest eigenvalues are the same, the second largest eigenvalue is used to determine the order. The critical graphs of maximum degree three and order at most ten were found earlier by Chetwynd [3].

This list suggests various constructions for obtaining critical graphs of maximum degree three, and we hope to describe some of these constructions in a later paper. For the most part, the graphs have been drawn so as to suggest the most obvious construction.

There is little in the literature concerning critical graphs. However it is shown in [4] that the circuits C_n are critical if and only if $n \not\equiv 0 \pmod 3$, $n \geq 3$. Also in [4] it is shown that the Möbius ladders M_{2n} are critical if $n \geq 2$, $n \neq 3$ (see 8.1, 10.7, 12.9, 14.5 and 16.11 in the attached list), and that 6.1 and 10.6 are critical. It is well-known that, for $n \geq 1$, K_{2n} is Type 2, and it follows from the result in [7] that if J is a subgraph of K_{2n} with $j + |E(J)| = n - 1$, where j is the maximum size (i.e. number of edges) of a matching in J, then $K_{2n} \backslash E(J)$ is critical. It similarly follows from the result in [8] that if H is a subgraph of $K_{n,n}$ with $h + |E(H)| = n - 1$, where h is the maximum size of a matching in H, then $K_{n,n} \backslash E(H)$ is critical. Bor Liang Chen and Hung-Lin Fu [2] have recently proved a result which implies that if G is a graph of order $2n$ and maximum degree $2n-2$ then G is critical if and only if $\bar{G}$ consists of the star $K_{1,2n-3}$ and a disjoint K_2.

2. Production of our list

Our raw material was the list of all cubic graphs of order $N \leq 14$ produced by Halberstam and Quintas [6]. Initially a computer program was written to test whether a graph was critical. Applied directly to Halberstam and Quintas' list, we found that the number of cubic graphs of order $N \leq 14$ compared with the number of critical cubic graphs of order N was as given in Chart 1. By applying the method described in this paper, we also found the number of critical graphs of order 16.

We used the list to generate all cubic graphs of order 16 by taking in turn each pair of edges from each cubic graph of order 14, inserting a vertex into each edge of each such pair, and joining the two inserted vertices. We need the following lemma.

Lemma 1. *The only simple cubic graph of order 16 which cannot be obtained from a simple cubic graph of order 14 by the process described is the graph A of Figure 1.*

N	Cubic	Critical Cubic
4	1	1
6	2	0
8	5	1
10	19	8
12	85	14
14	509	6
16	–	14

Chart 1

A:

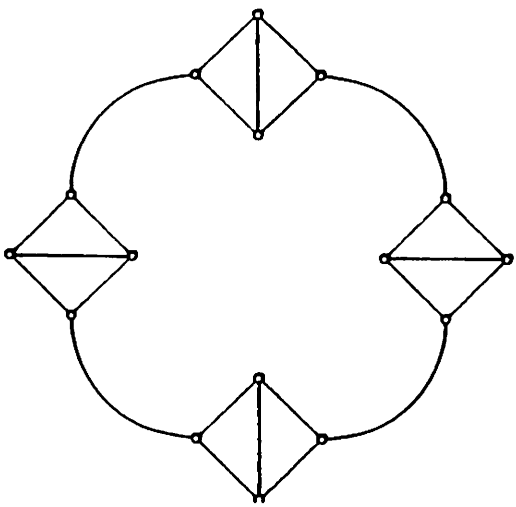

Figure 1. The graph A.

Proof of Lemma 1: Let B be a cubic graph of order 16 which cannot be produced by our process. Then the removal of any edge and the replacement of each of the pair of edges joined to each of the resulting vertices of degree two by single edge yields at leat one double edge. This removal and replacement process is illustrated in Figure 2.

If the two resulting edges, say e_1 and e_2, meet in two vertices, so that e_1 and e_2 together form a double edge, then originally we had the graph I of Figure 3.

If the two resulting edges e_1 and e_2 meet in one vertex, so that, for example, $c = d$, then originally B contained as a subgraph the graph of Figure 4. But since at least one double edge is produced, c must be joined to at least one of f and g, say f; then we may suppose that originally we had the graph II of Figure 3.

If the two resulting edges do not meet, but we now have a double edge joining, say, c and f, then originally B contained as a subgraph the graph of Figure 5, with c, d, f and g all distinct.

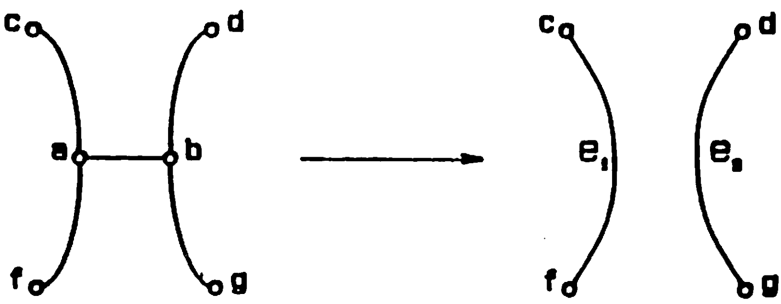

Figure 2.

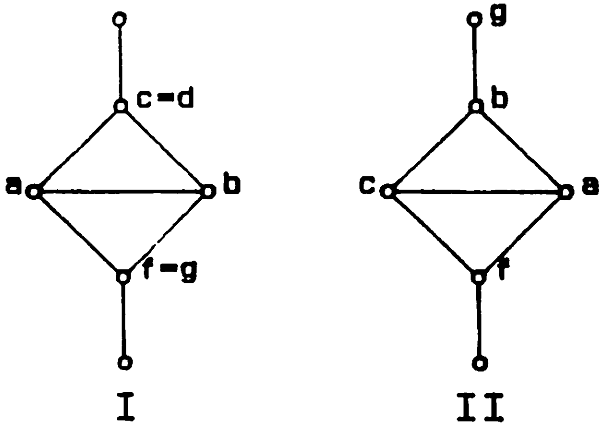

Figure 3.

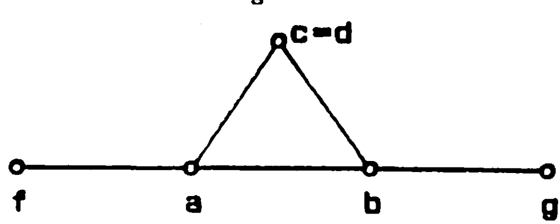

Figure 4.

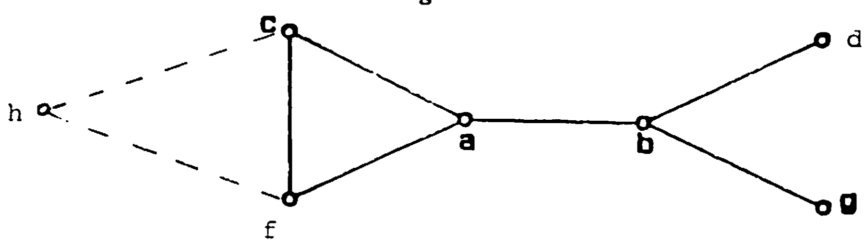

Figure 5.

By applying the earlier argument to different edges we see that c and f are both joined to the same further vertex (h), so that we again have one of the diamond shapes of Figure 3 with cf as the central edge; we also see that either d and g are joined, and that dg is a central edge of a diamond, or that d and g are not joined, but there is a diamond on d and on g.

In the latter case there is no way of completing the subgraph to form a cubic graph of order 16 which cannot be produced by our process (although it can be completed to form a cubic graph of order 14, for example, which cannot be produced by our process from a graph of order 12).

Thus we must have a second diamond with dg as the central edge. It now follows easily that $B = A$. ∎

Our procedure normally produces several copies of the same graph. We tested each copy for criticality, and then removed duplications. By this means we found the 14 critical cubic graphs of order 16.

It is obvious that critical graphs with maximum degree three are connected and have minimum degree at least two. If we take a connected graph of maximum degree three and minimum degree two, and then 'forget' the vertices of degree two [i.e. replace a pair of edges e_1 and e_2 incident with vertices a, b and b, c respectively and replace them by an edge e incident with a and c, and then repeat this operation as necessary], we may produce some multiple edges. But since our data base is only a list of simple cubic graphs, we need to examine carefully how to produce all critical simple graphs of maximum degree three and order at most sixteen from this list.

We shall call vertices of degree two diodes. If G is a graph of maximum degree three and minimum degree at least two, let G_f be the cubic (multi-)graph obtained from G by forgetting the diodes. We first notice that if we can produce a double edge by 'forgetting' one diode, then the graph cannot be critical. We state this formally in Lemma 2.

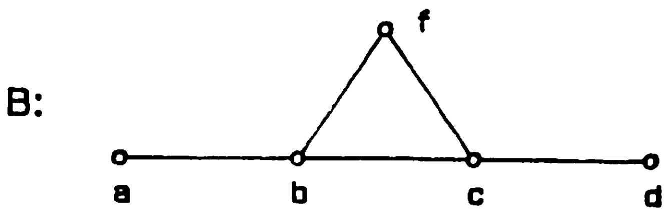

Figure 6. The graph B.

Lemma 2. *If a graph G of maximum degree three contains the graph B with f as a diode of G, then G is not critical.*

Proof: Suppose that G is a critical graph with maximum degree three which contains B as a subgraph, with f as a diode of G. Then $G\backslash\{f\}$ can be coloured with

4 colours, say 1, 2, 3 and 4. If a and d have the same colour, then we may suppose that a, ab, cd, d are coloured 1, 2, 2, 1 or 1, 2, 3, 1 respectively. But in both cases, the colouring can be extended to a colouring of G. If a and d have different colours, then we may suppose that a, ab, cd, d are coloured 1, 2, 1, 2, or 1, 2, 3, 2, or 1, 3, 3, 2 or 1, 3, 4, 2, respectively. However the first possibility can be excluded, for there is no way of extending this to a colouring of the path a, ab, b, bc, c, cd, d; in the other three cases, the colouring can be extended to a total colouring of G. This contradiction proves Lemma 2. ∎

By contrast, if there are no single diodes which can be 'forgotten' to produce a double edge, but there are two adjacent diodes which can be forgotten to produce a double edge, then the graph may well be critical. Thus we have to allow for the replacement of an edge $e = \ell m$ in a cubic simple graph by the graph C of Figure 7.

C:

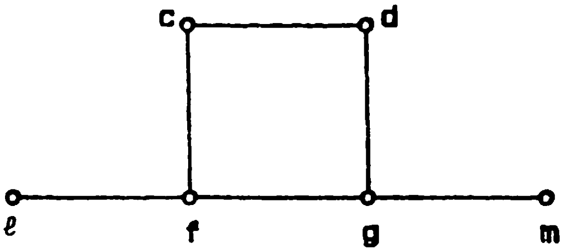

Figure 7. The graph C.

As it turns out, we only need to consider graphs obtained from simple cubic graphs by inserting one or two vertices in some edges, and by replacing some other edges by the graph C. But this is not obvious.

To show this, we first observe that a critical graph cannot contain three diodes a, b, c with b adjacent to both a and c.

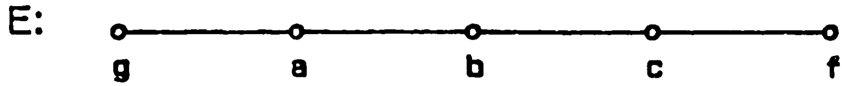

Figure 8. The graph E.

Lemma 3. *If a G of maximum degree three contains the graph E as a subgraph, with a, b, c as diodes of G, then G is not critical.*

Proof: Suppose that G is critical. Then $G\backslash\{bc\}$ can be totally coloured with four colours. But it is easy to verify that, whatever the colours of f and g and the edge incident with them, the colouring can be extended to a total colouring of G. ∎

Next we show that if a critical graph contains C, then ℓ and m must have degree three.

Lemma 4. *If a graph G with maximum degree three contains the graph C as a subgraph, with c and d as diodes of G, and if* ℓ *or* m *is a diode, then G is not critical.*

Proof: Suppose that G is a critical graph and that ℓ is a diode of G. Then $G\backslash\{cd\}$ can be totally coloured with four colours. It is a simple matter to check that no matter what colours are used on m and the edges incident with m, and whatever colours are used on ℓ and the edges of G incident with m other than ℓ, then the total colouring can be extended to a total colouring of all of G. This is a contradiction. Therefore G is not critical. ∎

We next observe that if a graph G contains any of the following subgraphs, then it cannot be critical.

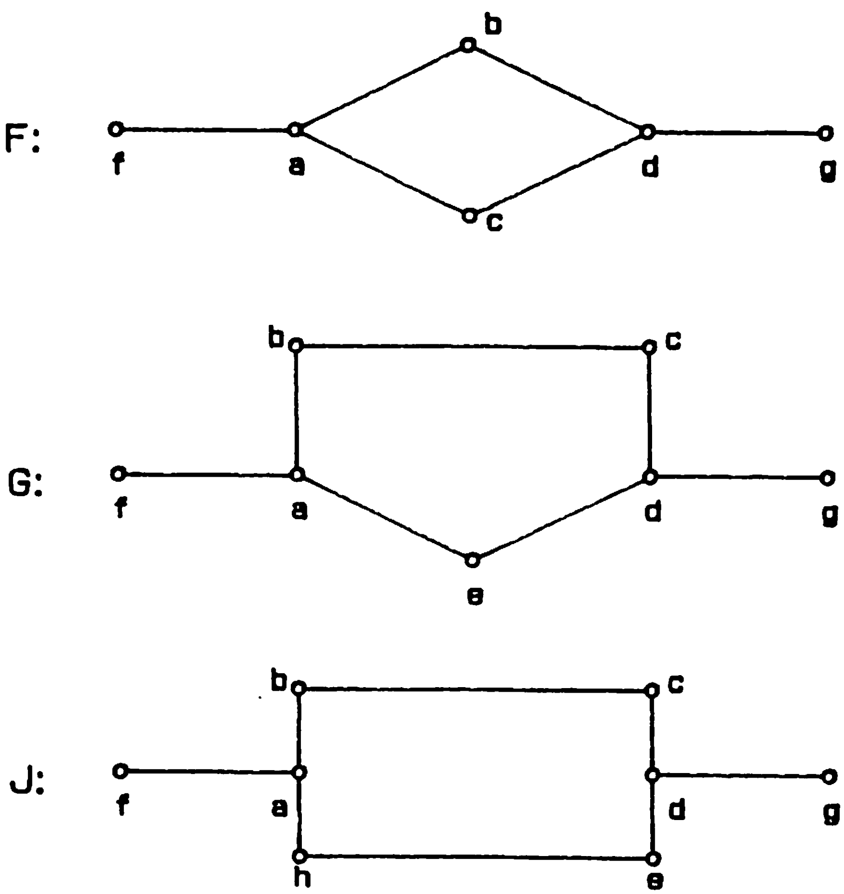

Figure 9.

Lemma 5. *If a graph G of maximum degree three contains any of the graphs F, H, J as subgraphs, with b, c, e, h (as appropriate) as diodes, then G cannot be critical.*

Proof: This is similar to the proof of Lemma 3. ∎

It follows from Lemmas 2, 3, 4, and 5 that, if G is critical and G_f contains a double edge joining two vertices f and g, then, in G, f and g are joined as in the graph C, and it also follows that ℓ and m are not diodes. If in G_f ℓ and m are non-adjacent then the subgraph C in G can be obtained by replacing a simple edge joining ℓ and m in an appropriate cubic graph. This is not possible, though, if ℓ and m are adjacent in G_f (the case $\ell = m$ clearly does not occur if G is critical).

The next lemma shows that this exceptional case does not occur.

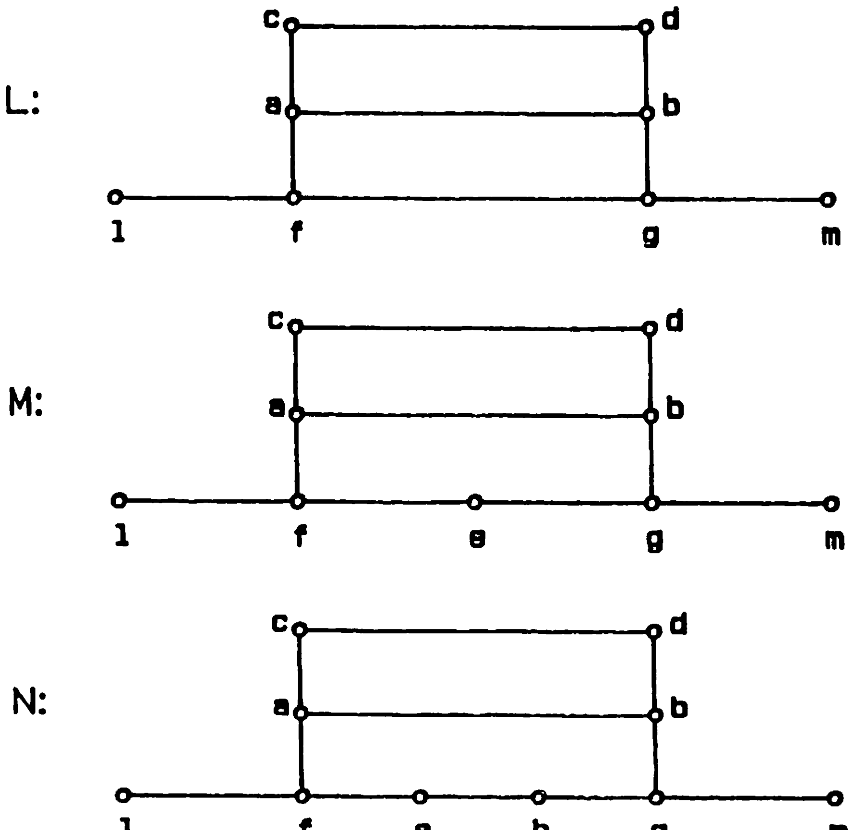

Figure 10.

Lemma 6. *If a graph G of maximum degree three contains any of the graphs L,* *M and N as subgraphs, with c, d, e, h (as appropriate) as diodes, then G is not critical.*

Proof: This is similar to the proof of Lemma 2. ∎

We conclude that if G has maximum degree three and is critical, and if G_f contains a double edge joining, say, vertices f and g, and if the other vertices adjacent to f and g are ℓ and m respectively, then there is a cubic graph with ℓ and m joined by a simple edge e from which G can be formed by replacing e by the graph C.

The following lemma shows that in fact we can assume that the double edges in G_f are not linked together by an edge. It follows that if G_f contains double edges then G can be formed from some simple cubic graph G^* by replacing some simple edges of G^* by the graph C and by inserting one or two vertices in some other edges of G^*.

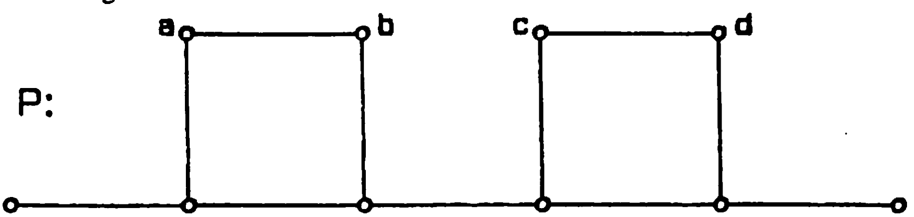

Figure 11.

Lemma 7. *If a graph G of maximum degree three contains the graph P as a subgraph, with a, b, c, d as diodes, then G is not critical.*

Proof: This is similar to the proof of Lemma 3. ∎

If G is critical, then G_f is not the graph consisting of two vertices joined by three edges. For it is easy to check that all simple graphs obtained from such a graph by inserting one or two vertices into at least two of the edges are Type 1. This final remark proves the following lemma.

Lemma 8. *Each critical graph of maximum degree three can be obtained from a simple cubic graph by replacing some of the edges by the graph C, and by inserting one or two vertices in some other edges.*

Lemma 8 is of course the key to our method. From the list of cubic simple graphs of order up to 14, we first generated all the critical cubic graphs of order 16, as described earlier. We then generated the critical graphs with one diode, then those with two diodes, then those with three diodes, etc.. The method was simply to insert into different edges of each simple cubic graph of the appropriate orders either one vertex, or two vertices, or to replace the edge by the graph C, in such a way that the graph obtained has the required number of diodes. The graph was then tested for criticality.

3. The list of critical graphs

The graphs are listed according to the number of their vertices, and, subject to that, in the order of their largest eigenvalues; when the number of vertices and the largest eigenvalue is the same, the second largest eigenvalue is used to decide the place in the list. Each graph is given with an indication of which graph in Halberstam and Quintas' list it is derived from (by inserting vertices and by replacing edges by the graph C). For example graph 12.18 has, in brackets, $N_8 = 4$. This indicates that 12.18 is derived from the fourth graph of order eight in Halberstam and Quintas' list.

We first collect together some facts which can be gleaned from the list. The number of critical graphs with a given order and number of diodes is given in Figure 12; note that the number of diodes necessarily has the same parity as the order.

	Number of diodes				Total
Order	0 or 1	2 or 3	4 or 5	≥ 5	
4	1	0	0	0	1
5	0	0	0	0	0
6	0	1	0	0	1
7	1	0	0	0	1
8	1	0	0	0	1
9	3	1	0	0	4
10	8	0	0	0	8
11	0	1	0	0	1
12	14	10	0	0	24
13	3	0	0	0	3
14	6	12	1	0	19
15	71	2	0	0	73
16	14	14	1	0	29

Figure 12. The number of critical graphs with given order and number of diodes.

Despite the irregularities at the start, it seems highly likely that the number of critical graphs of maximum degree three with a given number d of diodes increases monotonically with the order (of the same parity as d) from some point onwards.

We now give the list of critical graphs (note that there are critical graphs with maximum degree less than three, namely K_2 and C_{3m+1}, C_{3m+2} $(m = 1, 2, 3, \ldots)$; these are not included in the list below):

References

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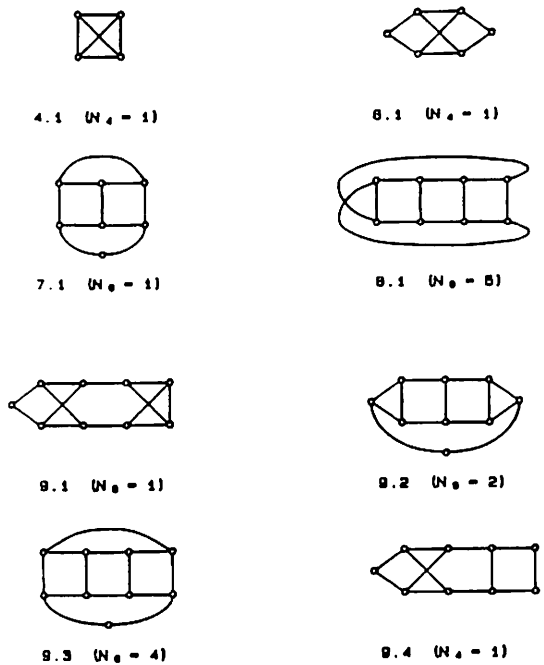
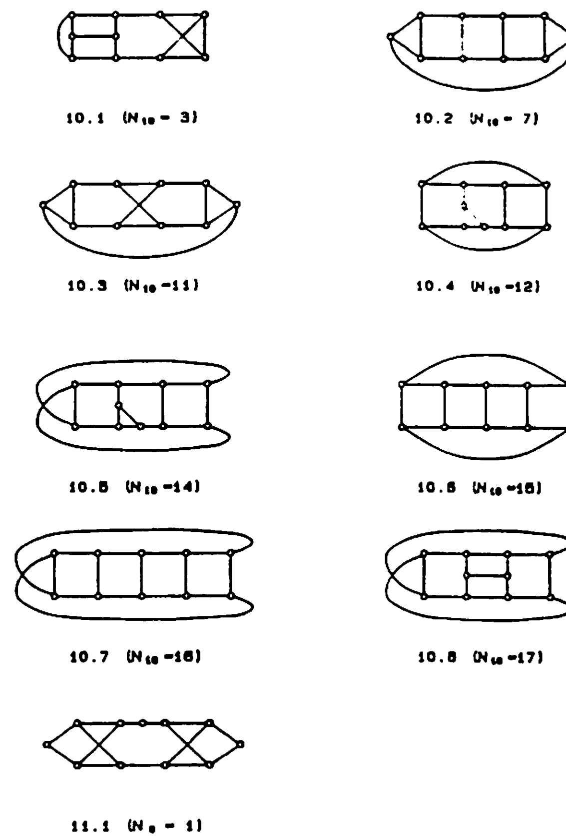
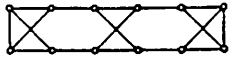


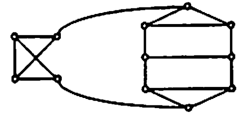
EARLY MEMBERS OF SEQUENCE



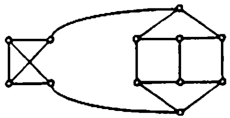
CRITICAL GRAPHS OF ORDER TEN AND ELEVEN



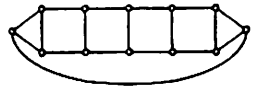
12.1 $(N_{12} = 9)$



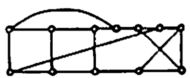
12.2 $(N_{12} = 23)$



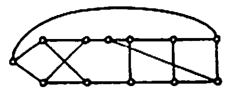
12.3 $(N_{12} = 25)$



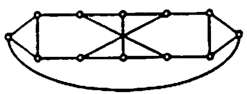
12.4 $(N_{12} = 33)$



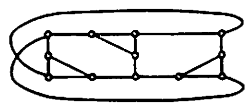
12.5 $(N_{12} = 38)$



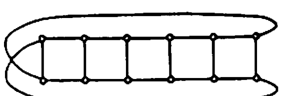
12.6 $(N_{12} = 49)$



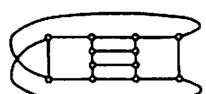
12.7 $(N_{12} = 64)$



12.8 $(N_{12} = 65)$

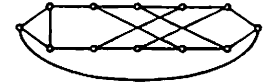


12.9 $(N_{12} = 67)$

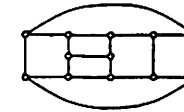


12.10 $(N_{12} = 68)$

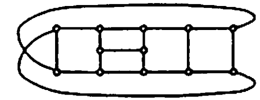
CRITICAL GRAPHS OF ORDER TWELVE



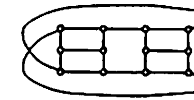
12.11 $(N_{12} = 69)$



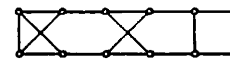
12.12 $(N_{12} = 70)$



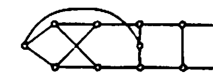
12.13 $(N_{12} = 71)$



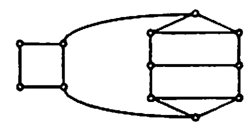
12.14 $(N_{12} = 73)$



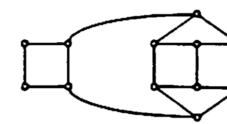
12.15 $(N_{8} = 1)$



12.16 $(N_{8} = 3)$

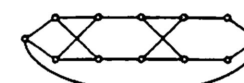


12.17 $(N_{8} = 2)$



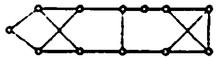
12.18 $(N_{8} = 4)$

12.19 $(N_{10} = 8)$

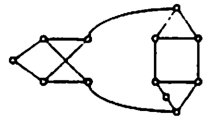


12.20 $(N_{10} = 10)$

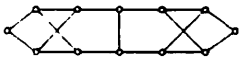
CRITICAL GRAPHS OF ORDER TWELVE (cont)



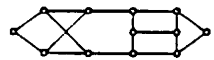
12.21 (N_{10} = 2)



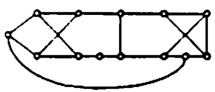
12.22 (N_{10} = 4)



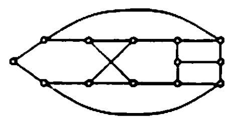
12.23 (N_{10} = 2)



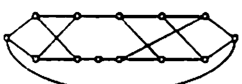
12.24 (N_{10} = 3)



13.1 (N_{12} =32)

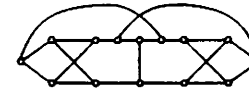


13.2 (N_{12} =75)

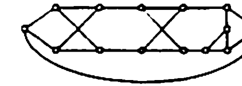


13.3 (N_{12} =49)

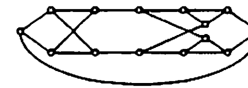
RITICAL GRAPHS OF ORDER TWELVE (cont) AND THIRTEEN



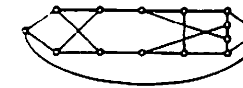
14.1 (N_{14} =220)



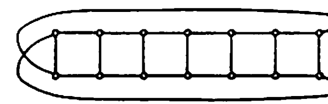
14.2 (N_{14} =223)



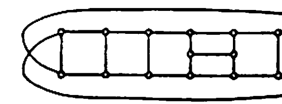
14.3 (N_{14} =317)



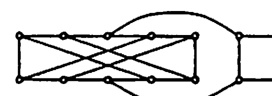
14.4 (N_{14} =319)



14.5 (N_{14} =355)

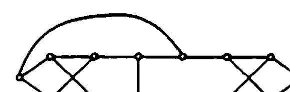


14.6 (N_{14} =361)

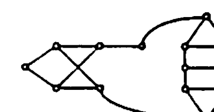


14.7 (N_{12} =16)

14.8 (N_{12} = 9)

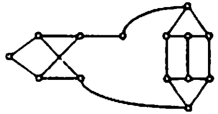


14.9 (N_{12} =19)

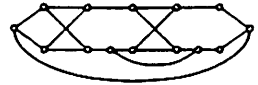


14.10 (N_{12} =23)

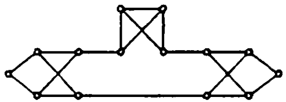
CRITICAL GRAPHS OF ORDER FOURTEEN



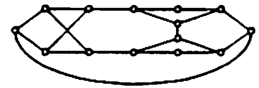
14.11 (N_{13}=25)



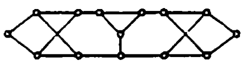
14.12 (N_{13}=49)



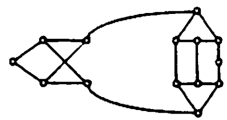
14.13 (N_{13}=22)



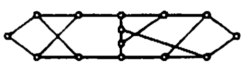
14.14 (N_{13}=48)



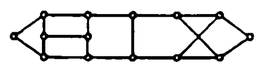
14.15 (N_{13}=10)



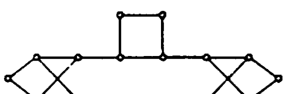
14.16 (N_{13}=25)



14.17 (N_{13}=21)

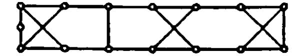


14.18 (N_{13}= 6)

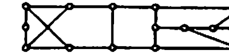


14.19 (N_{13}=10)

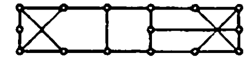
CRITICAL GRAPHS OF ORDER FOURTEEN (cont)



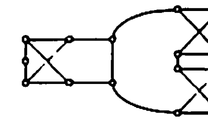
15.1 (N_{14}=34)



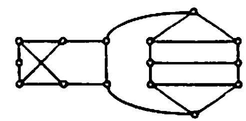
15.2 (N_{14}=44)



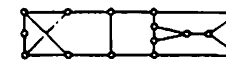
15.3 (N_{14}=45)



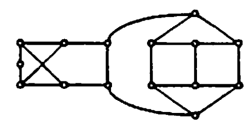
15.4 (N_{14}=50)



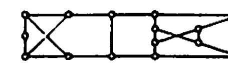
15.5 (N_{14}=51)



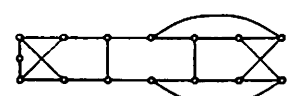
15.6 (N_{14}=52)



15.7 (N_{14}=53)



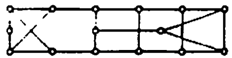
15.8 (N_{14}=54)



15.9 (N_{14}=55)

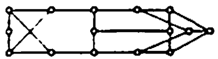
15.10 (N_{14}=51)

CRITICAL GRAPHS OF ORDER FIFTEEN



15.11 (N_{14} =89)

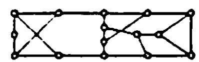
15.12 (N_{14} =112)



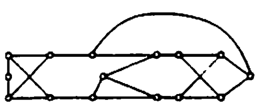
15.13 (N_{14} =113)

15.14 (N_{14} =76)

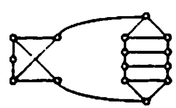
15.15 (N_{14} =120)



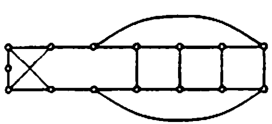
15.16 (N_{14} =122)



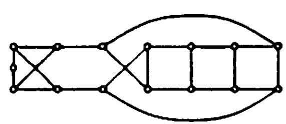
15.17 (N_{14} =123)



15.18 (N_{14} =131)

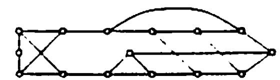


15.19 (N_{14} =133)

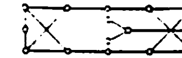


15.20 (N_{14} =134)

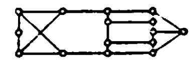
CRITICAL GRAPHS OF ORDER FIFTEEN (cont)



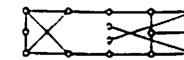
15.21 (N_{14} =139)



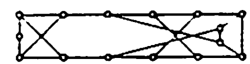
15.22 (N_{14} =143)



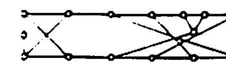
15.23 (N_{14} =146)



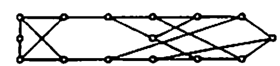
15.24 (N_{14} =147)



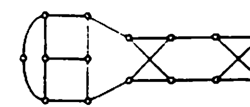
15.25 (N_{14} =157)



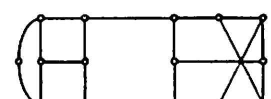
15.26 (N_{14} =159)



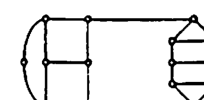
15.27 (N_{14} =161)



15.28 (N_{14} =42)

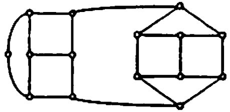


15.29 (N_{14} =65)

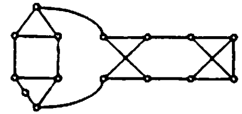


15.30 (N_{14} =78)

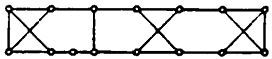
CRITICAL GRAPHS OF ORDER FIFTEEN (cont)



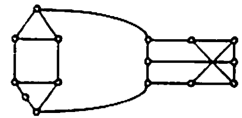
15.31 $(N_{14} = 60)$



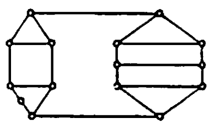
15.32 $(N_{14} = 48)$



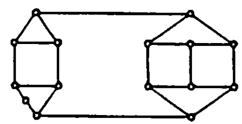
15.33 $(N_{14} = 34)$



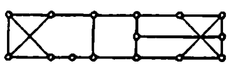
15.34 $(N_{14} = 72)$



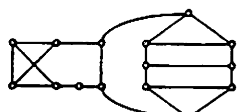
15.35 $(N_{14} = 93)$



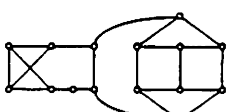
15.36 $(N_{14} = 95)$



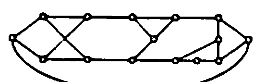
15.37 $(N_{14} = 45)$



15.38 $(N_{14} = 61)$

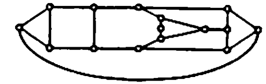


15.39 $(N_{14} = 53)$

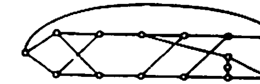


15.40 $(N_{14} = 244)$

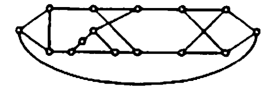
CRITICAL GRAPHS OF ORDER FIFTEEN (cont)



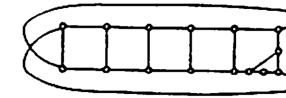
15.41 $(N_{14} = 235)$



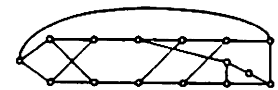
15.42 $(N_{14} = 292)$



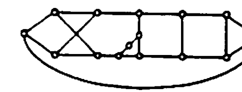
15.43 $(N_{14} = 293)$



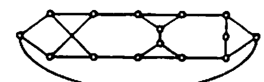
15.44 $(N_{14} = 327)$



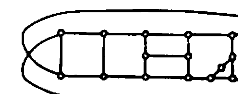
15.45 $(N_{14} = 292)$



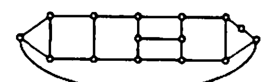
15.46 $(N_{14} = 270)$



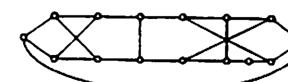
15.47 $(N_{14} = 316)$



15.48 $(N_{14} = 375)$

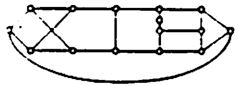


15.49 $(N_{14} = 288)$

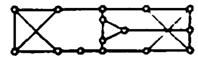


15.50 $(N_{14} = 212)$

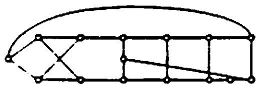
CRITICAL GRAPHS OF ORDER FIFTEEN (cont)



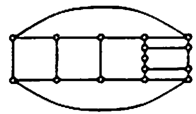
15.51 (N_{14} =208)



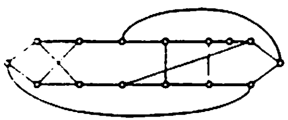
15.52 (N_{14} =143)



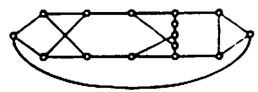
15.53 (N_{14} =297)



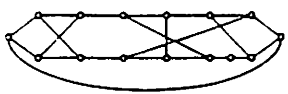
15.54 (N_{14} =360)



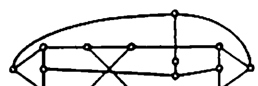
15.55 (N_{14} =304)



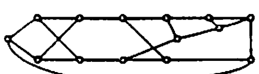
15.56 (N_{14} =316)



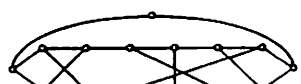
15.57 (N_{14} =308)



15.58 (N_{14} =454)

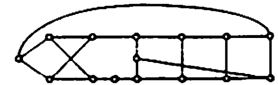


15.59 (N_{14} =207)

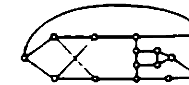


15.60 (N_{14} =308)

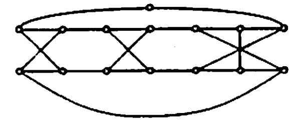
CRITICAL GRAPHS OF ORDER FIFTEEN (cont)



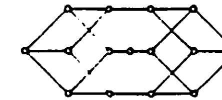
15.61 (N_{14} =297)



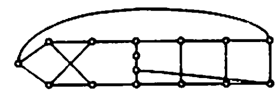
15.62 (N_{14} =173)



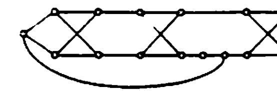
15.63 (N_{14} =466)



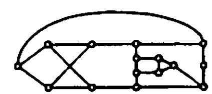
15.64 (N_{14} =241)



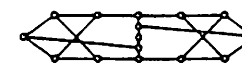
15.65 (N_{14} =297)



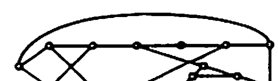
15.66 (N_{14} =125)



15.67 (N_{14} =173)



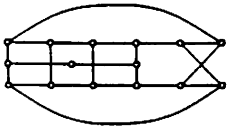
15.68 (N_{14} =180)



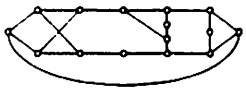
15.69 (N_{14} =207)

15.70 (N_{14} =220)

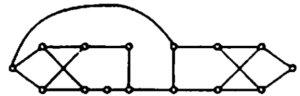
CRITICAL GRAPHS OF ORDER FIFTEEN (cont)



15.71 (N_{14} = 488)

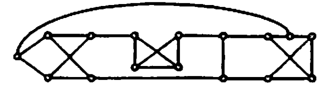


15.72 (N_{12} = 48)

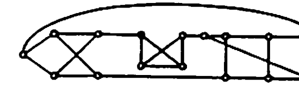


15.73 (N_{12} = 8)

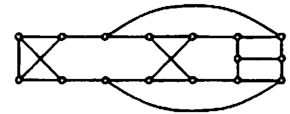
CRITICAL GRAPHS OF ORDER FIFTEEN (cont)



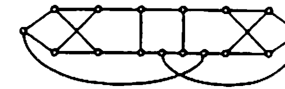
16.1



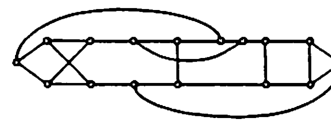
16.2



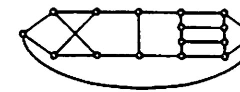
16.3



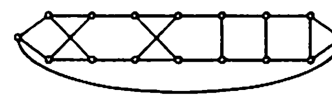
16.4



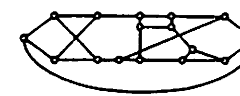
16.5



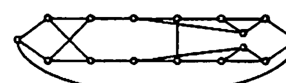
16.6



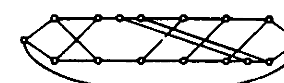
16.7



16.8

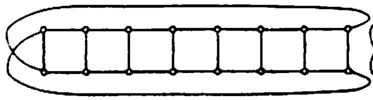


16.9

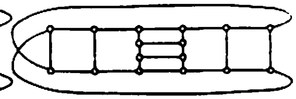


16.10

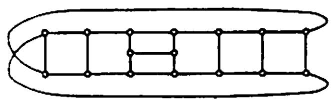
CRITICAL GRAPHS OF ORDER SIXTEEN



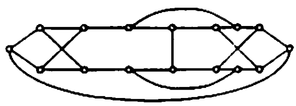
16.11



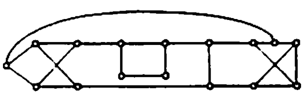
16.12



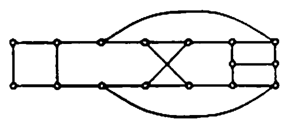
16.13



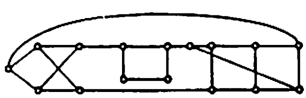
16.14



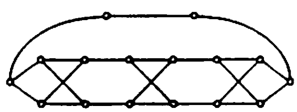
16.15 (N_{12} =32)



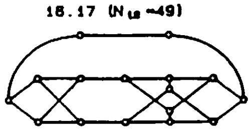
16.16 (N_{12} =75)



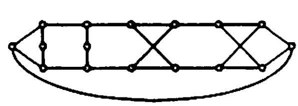
16.17 (N_{12} =49)



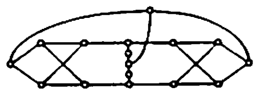
16.18 (N_{14} =239)



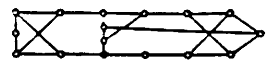
16.19 (N_{14} =316)



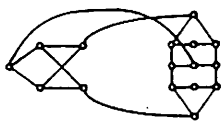
16.20 (N_{14} =237)



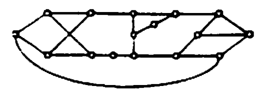
16.21 (N_{14} -220)



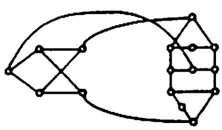
16.22 (N_{14} -125)



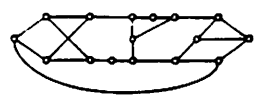
16.23 (N_{14} -314)



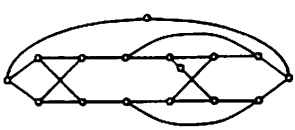
16.24 (N_{14} -233)



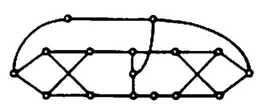
16.25 (N_{14} -314)



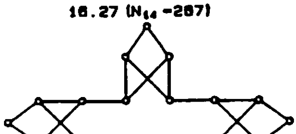
16.26 (N_{14} -233)



16.27 (N_{14} -287)



16.28 (N_{14} -220)



16.29 (N_{16} -22)

CRITICAL GRAPHS OF ORDER SIXTEEN (cont)