K_n-domination sequences of graphs

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Abstract. The domination number $\gamma(G)$ and the total domination number $\gamma_t(G)$ of a graph are generalized to the K_n-domination number $\gamma_{K_n}(G)$ and the total K_n-domination number $\gamma^t_{K_n}(G)$ for $n \geq 2$, where $\gamma(G) = \gamma_{K_2}(G)$ and $\gamma_t(G) = \gamma^t_{K_2}(G)$. A nondecreasing sequence $a_2, a_3, \ldots, a_m$ of positive integers is said to be a K_n-domination (total K_n-domination, respectively) sequence if it can be realized as the sequence of generalized domination (total domination, respectively) numbers $\gamma_{K_2}(G)$, $\gamma_{K_3}(G), \ldots, \gamma_{K_m}(G)(\gamma^t_{K_2}(G), \gamma^t_{K_3}(G), \ldots \gamma^t_{K_m}(G)$, respectively) of some graph G. It is shown that every nondecreasing sequence $a_2, a_3, \ldots, a_m$ of positive integers is a K_n-domination sequence (total K_n-domination sequence, if $a_2 \geq 2$, respectively). Further, the minimum order of a graph G with $a_2, a_3, \ldots, a_m$ as K_n-domination sequence is determined. K_n-connectivity is defined and for $a_2 \geq 2$ we establish the existence of a K_m-connected graph with $a_2, a_3, \ldots, a_m$ as K_n-domination sequence for every nondecreasing sequence $a_2, a_3, \ldots, a_m$ of positive integers.

The terminology and notation of [3] will be used throughout. In particular, G will denote a graph with vertex set V, edge set E, order p and size q.

In [2], the K_n-degree of a vertex v in a graph G, denoted by $K_n\deg_G v$ is defined as the number of subgraphs of G, isomorphic to K_n that contain v. Observe that the K_2-degree of a vertex is simply the degree of v in G. This generalization of the concept of degree is related to the following generalization of the concept of adjacency in a graph. If n is an integer, $n \geq 2$ and u and v are distinct vertices of a graph G, then u and v are said to be K_n-*adjacent* vertices of G if there is a subgraph of G, isomorphic to K_n, containing u and v. Therefore, u and v are K_2-adjacent vertices of G if and only if u and v are adjacent vertices of G. The set of all vertices K_n-adjacent to a vertex v in G is denoted by $N^{K_n}(v)$ and $|N^{K_n}(v)|$, the K_n-neighbourhood degree of v, by $\deg(v; K_n)$. In general, $K_n\deg v \neq \deg(v; K_n)$. (For example, if v is a vertex of degree n in the graph $G \cong K_{n+1} - e$, then $K_n\deg v = 2$, while $\deg(v; K_n) = n$.) A vertex that is contained in no subgraph of G, isomorphic to K_n is called a K_n-isolated vertex of G.

This definition of K_n-adjacency suggests a generalization of the concept of connectedness in graphs. Let u and v be vertices of a graph G. A $u - v$ K_n-path of G is a finite, alternating sequence of vertices and subgraphs of G, isomorphic to K_n beginning with u and ending with v, such that the vertices of the sequence are distinct, the subgraphs of the sequence are distinct and every subgraph of the sequence is immediately preceded and succeeded by a vertex that is contained in that subgraph. The vertex u is said to be K_n-connected to the vertex v in G if there exists a $u - v$ K_n-path in G. In Figure 1, a graph G is shown where

$u = u_0, F_1, u_1, F_2, u_2, F_3, u_3, F_4, u_4, F_5, u_5 = v$ is a $u - v$ K_3-path in G. A graph is K_n-connected if every two of its vertices are K_n-connected. A graph that is not K_n-connected is K_n-disconnected. A K_n-component of a graph G is a K_n-connected subgraph of G that is not properly contained in a K_n-connected subgraph of G.

Next we generalize the concepts of domination and total domination in graphs as discussed, for example, in [4] and [5]. Our definition of K_n-adjacency suggests a generalization of domination and total domination in graphs. In [7] ([8], respectively), for $n \geq 2$, a *K_n-dominating* (*total K_n-dominating*, respectively) set of a graph G is defined as a set D of vertices such that every vertex in $V(G) - D(V(G)$, respectively) is K_n-adjacent to a vertex of D. The *K_n-domination* number $\gamma_{K_n}(G)$ (*total K_n-domination number* $\gamma^t_{K_n}(G)$, respectively) of G is the minimum cardinality among the K_n-dominating (total K_n-dominating, respectively) sets of vertices of G. We note that the parameter $\gamma^t_{K_n}(G)$ is defined only for graphs with no K_n-isolated vertex. Observe that $\gamma(G) = \gamma_{K_2}(G)$ and $\gamma_t(G) = \gamma^t_{K_2}(G)$.

In Figure 2, a graph G is shown together with a K_i-dominating set D_i of G for which $\gamma_{K_i}(G) = |D_i|$ for $i = 2, 3, 4, 5$. Further, for each $i = 2, 3, 4$ a spanning subgraph H_i of G is also shown in Figure 2 ((a), (b) and (c), respectively) where the vertices of D_i have been darkened. The graph H shown in Figure 3 is such that $\gamma^t_{K_2}(H) = 3(\{v_1, v_2, v_5\}$ is a total dominating set of H) and $\gamma^t_{K_3}(H) = 4(\{v_1, v_2, v_3, v_4\}$ is a total K_3-dominating set of H).

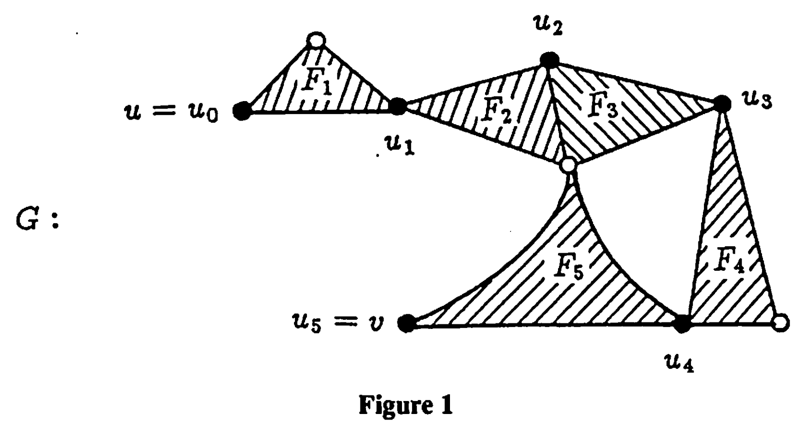
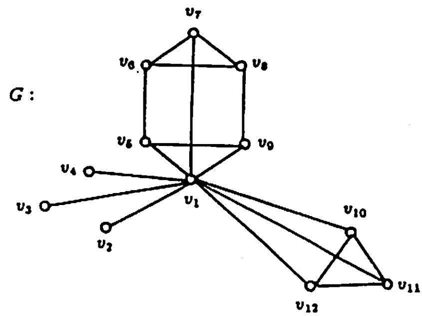
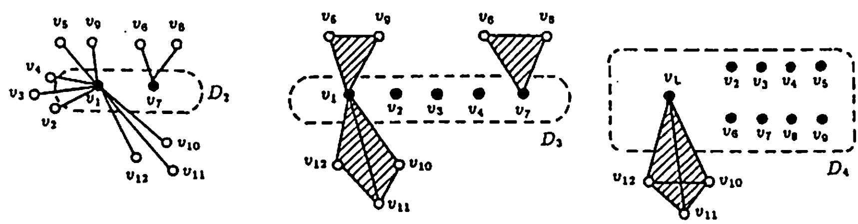


Figure 1

Observe that, for $m \geq 2$ an integer, our definition of K_n-domination (total K_n-domination, respectively) associates a sequence $\gamma_{K_2}(G), \gamma_{K_3}(G), \ldots, \gamma_{K_m}(G)$ $(\gamma^t_{K_2}(H), \gamma^t_{K_3}(H), \ldots, \gamma^t_{K_m}(H)$, respectively) of generalized domination (total domination, respectively) numbers with a graph G (H, respectively, where H has no K_m-isolated vertex). The graph G shown in Figure 2, for example, has



i	$\gamma_{K_i}(G)$	D_i
2	2	$\{v_1, v_7\}$
3	5	$\{v_1, v_2, v_3, v_4, v_7\}$
4	9	$\cup_{i=1}^{9}\{v_i\}$
5	12	$V(G)$



(a) H_2 (b) H_3 (c) H_4

Figure 2

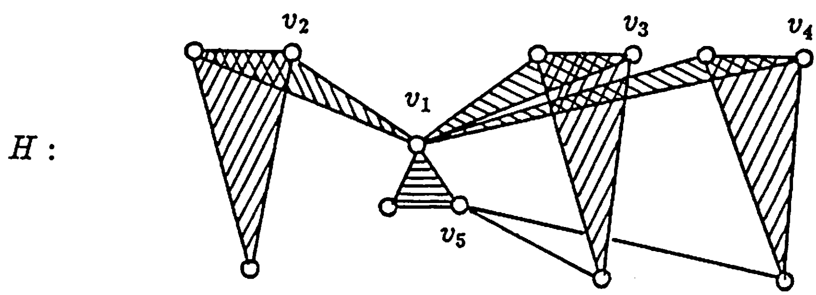


Figure 3

as an associated K_n-domination sequence the sequence 2,5,9,12, while the graph H shown in Figure 3 has as an associated total K_n-domination sequence the sequence 3,4. A nondecreasing sequence $a_2, a_3, ..., a_m$ of positive integers is said to be a *K_n-domination sequence* if it can be realized as the sequence of generalized domination numbers $\gamma_{K_2}(G), \gamma_{K_3}(G), \ldots, \gamma_{K_m}(G)$ of some graph G. A total *K_n-domination sequence* is analogously defined. (Sequences related to other generalized graphical parameters were characterized in [1], [6] and [9].)

It is shown that every nondecreasing sequence $a_2, a_3, ..., a_m$ of positive integers is a K_n-domination sequence (total K_n-domination sequence, if $a_2 \geq 2$, respectively). Further, the minimum order of a graph G with $a_2, a_3, ..., a_m$ as

K_n-domination sequence is determined. For $a_2 \geq 2$, we establish the existence of a K_m-connected graph with $a_2, a_3, \dots, a_m$ as K_n-domination sequence, for every nondecreasing sequence $a_2, a_3, \dots, a_m$ of positive integers.

We begin by stating a useful known result [7].

Lemma 1. *For $n \geq 2$, let D be a K_n-dominating set of a graph G. Then D is a minimal K_n-dominating set of G if and only if for each vertex $d \in D$, d has at least one of the following properties:*

$P(1, n)$: there exists a vertex $v \in V - D$ such that $N^{K_n}(v) \cap D = \{d\}$;

$P(2, n)$: d is K_n-adjacent to no vertex of D.

Next we establish that, for any given nondecreasing sequence $a_2, a_3, \dots, a_m$ of positive integers, there exists a graph G with $\gamma_{K_i}(G) = a_i$, for each $i = 2, 3, \dots, m$.

Theorem 1. *Every nondecreasing sequence $a_2, a_3, \dots, a_m$ of positive integers is a K_n-domination sequence.*

Proof: Let $a_2, a_3, \dots, a_m$ be a given nondecreasing sequence of positive integers. We show that $a_2, a_3, \dots, a_m$ can be realized as the sequence of generalized domination numbers $\gamma_{K_2}(G), \gamma_{K_3}(G), \dots, \gamma_{K_m}(G)$ of some graph G. If $a_2 = a_3 = \cdots = a_m$, then the empty graph $\bar{K}_{a_m}$ on a_m vertices is a graph G for which $\gamma_{K_i}(G) = a_i$ for each $i = 2, 3, \dots, m$. Hence in what follows, we may assume that $a_2 < a_m$.

Let k be the smallest integer with $3 \leq k \leq m$ for which $a_k = a_m$. If $k = 3$, then the graph $G' = K(1, a_3 - a_2) \cup \bar{K}_{a_2 - 1}$ if $a_2 > 1$ (otherwise $G' = K(1, a_3 - a_2)$ if $a_2 = 1$) is such that $\gamma(G') = a_2$ and $\gamma_{K_i}(G') = p(G') = a_i$ for each i with $3 \leq i \leq m$. Hence in what follows we assume that $k \geq 4$.

Let I denote the set of all values of i with $3 \leq i \leq k$ and such that $a_i > a_{i-1}$. We now construct a sequence of graphs as follows. Let H_1 denote a graph, isomorphic to K_{k-1}, and for each $i \in \{4, \dots, k\}$, let S_i be a set of $i - 3$ vertices of H_1. For each $i \in I$, let $H_i = \bar{K}_{a_i - a_{i-1}}$ and, if $a_2 > 1$, then let $H_2 = \bar{K}_{a_2 - 1}$.

If $a_2 > 1$ ($a_2 = 1$, respectively), then let G (H, respectively) be the graph obtained from $(\cup_{i \in I} H_i) \cup H_1 \cup H_2$ ($(\cup_{i \in I} H_i \cup H_1$, respectively) by the addition of a new vertex v and the insertion of an edge between v and each vertex of $(\cup_{i \in I} H_i) \cup H_1$, and, for each $i \in I$ with $4 \leq i \leq k$, the insertion of an edge between each vertex of H_i and each vertex of S_i. (The graph G for the case $2 \leq a_2 \leq a_3 \leq \cdots \leq a_k = \cdots = a_m$ is sketched in Figure 4.)

Observe that every vertex of H_2 is isolated in G, implying, necessarily, that $D_2 = \{v\} \cup V(H_2)$ is a dominating set of G with $\gamma(G) = |D_2| = a_2$. If $a_2 = 1$, however, then $D_2 = \{v\}$ is a dominating set of H with $\gamma(H) = |D_2| = a_2$. Observe further that for each $i \in I$, every vertex of H_i is K_i-isolated but not K_{i-1}-isolated in G (if $a_2 > 1$) or in H (if $a_2 = 1$). This implies that, for each $i \in I$, the set $D_i = (\bigcup_{j \in I, j \leq i} V(H_j)) \cup \{v\} \cup V(H_2)$ ($D_i = (\bigcup_{j \in I, j \leq i} V(H_j)) \cup \{v\}$, respectively) is a K_i-dominating set of G (H, respectively) with $\gamma_{K_i}(G) = |D_i| = a_i$ ($\gamma_{K_i}(H) = |D_i| = a_i$, respectively). Moreover, for each j with $3 \leq j \leq m$ such that $j \notin I$, let i be the smallest integer for which $a_i = a_j$; then the set D_i (as defined above) is necessarily a K_j-dominating set of G, if $a_2 > 1$, (of H, if $a_2 = 1$, respectively) with $\gamma_{K_j}(G) = |D_i| = a_j$ ($\gamma_{K_j}(H) = |D_i| = a_j$, respectively).

Hence if $a_2 > 1$ ($a_2 = 1$, respectively), then G (H, respectively) is a graph with $\gamma_{K_j}(G) = a_i$ ($\gamma_{K_i}(H) = a_i$, respectively) for each $i = 2, 3, \ldots, m$. This implies, however, that $a_2, a_3, \ldots, a_m$ is a K_n-domination sequence. ∎

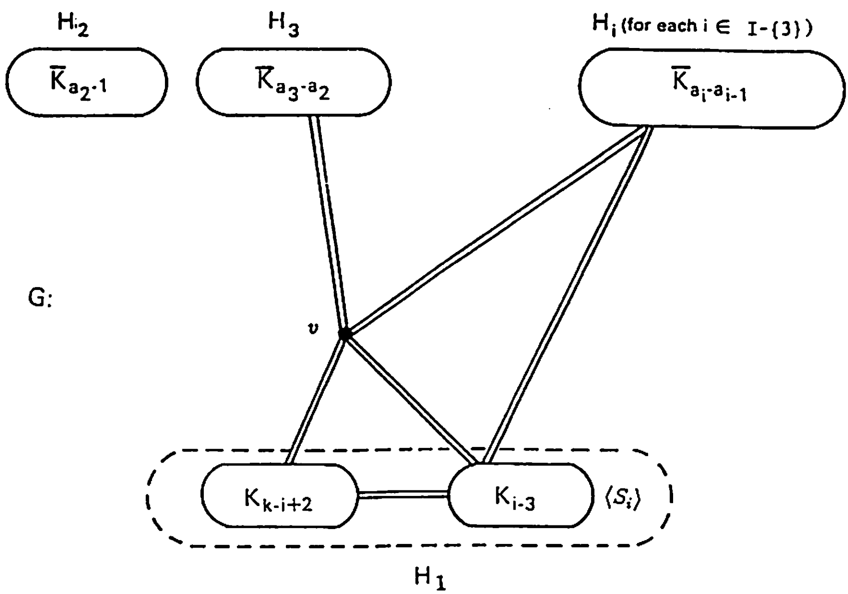


Figure 4 A graph G for which $\gamma_{K_i}(G) = a_i$ for $i = 2, 3, \ldots, m$ where $2 \leq a_2 \leq a_3 \leq \cdots \leq a_k = \cdots = a_m$ (===== denotes the join operation).

In what follows, let $a_2, a_3, \ldots, a_m$ be a given nondecreasing sequence of positive integers. The result of Theorem 1 establishes the existence of a graph G with $a_2, a_3, \ldots, a_m$ as K_n-domination sequence. Next we determine the minimum order of such a graph G.

If $a_2 = a_3 = \cdots = a_m$, then we observe that the empty graph $\bar{K}_{a_m}$ on a_m vertices is a graph G of minimum order (namely, a_m) for which $\gamma_{K_i}(G) = a_i$ for each $i = 2, 3, \ldots, m$. Hence in what follows, we consider the case $a_2 < a_m$.

Let G be a graph of minimum order with $a_2, a_3, \ldots, a_m$ as K_n-domination sequence. For $n \geq 2$, the definition of K_n-domination implies that $\gamma_{K_n}(H) \leq p(H)$ for every graph H. In particular, we note that $p(G) \geq \gamma_{K_m}(G) = a_m$. The next result establishes the order $p(G)$ of the graph G.

Theorem 2. *Let $S : a_2, \ldots, a_m$ be a nondecreasing sequence of positive integers, G a graph of minimum order with S as K_n-domination sequence and k the smallest integer with $3 \leq k \leq m$ for which $a_k = a_m$. If $k = 3$ or if $a_k - a_{k-1} \geq k-2$, then $p(G) = a_m$, while if $k \geq 4$ and $a_k - a_{k-1} < k-2$, then $p(G) = a_m + k - 1$.*

Proof: If $k = 3$, then the graph $G = G'$ constructed in the proof of Theorem 1 is a graph of order $a_m (= a_3)$ with S as K_n-domination sequence. This, together with the earlier observation that $p(G) \geq a_m$, implies that $p(G) = a_m$.

Suppose then that $a_k - a_{k-1} \geq k-2$, where $k \geq 4$. Let I denote the set of all values of i with $3 \leq i \leq k-1$ and such that $a_i > a_{i-1}$. We now construct a sequence of graphs as follows. Let $H_k = K_{k-3} + \bar{K}_{a_k - a_{k-1} - k + 3}$ and, for each $i \in I \backslash \{3\}$, let S_i denote a set of $i-3$ mutually adjacent vertices of H_k. For each $i \in I$, let $H_i = \bar{K}_{a_i - a_{i-1}}$ and, if $a_2 > 1$, then let $H_2 = \bar{K}_{a_2 - 1}$.

Let H be the graph obtained from the disjoin union of $(\bigcup_{i \in I \cup \{k\}} H_i)$ and H_2 (if $a_2 > 1$) by the addition of a new vertex v and the insertion of an edge between v and each vertex of $(\bigcup_{i \in I \cup \{k\}} H_i)$, and for each $i \in I \backslash \{3\}$, the insertion of an edge between every vertex of H_i and every vertex of S_i. (The graph H for the case $2 \leq a_2 < a_3 \leq \cdots \leq a_m$ is sketched in Figure 5.)

Observe that, for each $i \in I \cup \{k\}$, every vertex of H_i is K_i-isolated but not K_{i-1}-isolated in the graph H; however, for each such i, every vertex of H_i is K_{i-1}-dominated by the vertex v in H. Necessarily H is a graph of order a_m with S as K_n-domination sequence. This, together with the earlier observation that $p(G) \leq a_m$ implies that $p(G) = a_m$.

Next suppose that $a_k - a_{k-1} < k-2$, where $k \geq 4$. If $a_2 > 1$ ($a_2 = 1$, respectively), then the graph G (H, respectively) constructed in the proof of Theorem 1 is a graph of order $a_m + k - 1$ with S as K_n-domination sequence. This implies that $p(G) \leq a_m + k - 1$, where G is our graph of minimum order having S as K_n-domination sequence. It remains to be shown that $p(G) \geq a_m + k - 1$.

Let G be a graph with the given domination sequence S and let D_{k-1} be a K_{k-1}-dominating set of G with $\gamma_{K_{k-1}}(G) = |D_{k-1}| = a_{k-1}$. Then (cf. Lemma 1) each vertex of D_{k-1} has at least one of the two properties $P(1, k-1)$ and $P(2, k-1)$. We consider two cases.

Case 1: Suppose that some vertex $d \in D_{k-1}$ has property $P(1, k-1)$. Then there exists a vertex $v \in V(G) - D_{k-1}$ such that $N^{K_{k-1}}(v) \cap D_{k-1} = \{d\}$. This implies, however, that there exists a subgraph of $\langle V(G) - D_{k-1} \rangle$, isomorphic to K_{k-2}; in particular, we note therefore that $|V(G) - D_{k-1}| \geq k-2$. This implies that $K_k \prec G$; for otherwise, if $K_k \not\prec G$, then $V(G)$ is a minimal K_k-dominating set of G with $a_k = \gamma_{K_k}(G) = p(G) = |D_{k-1}| + |V(G) - D_{k-1}| \geq a_{k-1} + (k-2) > a_k$, producing a contradiction. Now let F denote a subgraph of G, isomorphic to K_k. Further, let w be a vertex of F and consider the set $D = V(G) - (V(F) - \{w\})$. Necessarily, D is a K_k-dominating set of G with $a_k = \gamma_{K_k}(G) \leq |D| = p(G) -$

$k+1$; consequently, $p(G) \geq a_k + k - 1 = a_m + k - 1$, as desired.

Case 2: Suppose that no vertex of D_{k-1} has property $P(1, k-1)$. Then (cf. Lemma 1) each vertex of D_{k-1} has property $P(2, k-1)$. However, since $a_{k-1} < a_k \leq p(G)$, there exists a vertex v in $V(G) - D_{k-1}$ such that v is K_{k-1}-adjacent to some vertex d in D_{k-1}. This implies, however, that there exists a subgraph of $\langle V(G) - D_{k-1} \rangle$, isomorphic to K_{k-2}. Proceeding then exactly as in Case 1, we show that $p(G) \geq a_m + k - 1$.

This completes the proof of the theorem. ∎

We note that all the graphs constructed in the proofs of Theorem 1 and Theorem 2 contain K_m-isolated vertices. However, given a nondecreasing sequence $a_2, a_3, \ldots, a_m$ of positive integers, does there exist a graph G, with $a_2, a_3, \ldots, a_m$ as K_n-domination sequence, that contains no K_m-isolated vertex? We answer this question in the affirmative (provided $a_2 \geq 2$) by showing the existence of a K_m-connected graph with the given domination sequence (cf. Theorem 3). That the restriction $a_2 \geq 2$ is unavoidable is shown in Lemma 2.

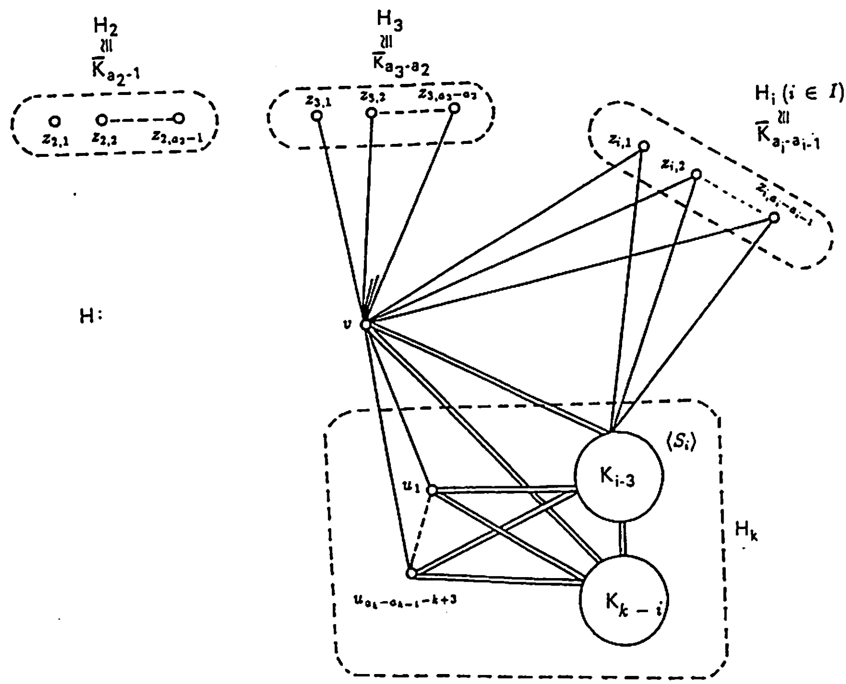


Figure 5. A graph H of order a_m $(= a_k)$ with $\gamma_{K_i}(G) = a_i$ for $i = 2, 3, \ldots, m$ where $2 \leq a_2 < a_3 \leq \cdots < a_k = \cdots = a_m$ (===== denotes the join operation).

Lemma 2. *For $m \geq 3$ an integer, if G is a graph with no K_m-isolated vertex* *and with $a_2, a_3, \ldots, a_m$ as K_n-domination sequence such that $a_2 = 1$, then $a_3 = \cdots = a_m = 1$.*

Proof: Since $\gamma(G) = a_2 = 1$, there exists a vertex v of G with $\deg v = p - 1$. We show that $\{v\}$ is a K_m-dominating set of G. Let w be a vertex of G, distinct from v. Since G contains no K_m-isolated vertex, there is necessarily a subgraph F of $G - v$, isomorphic to K_{m-1} that contains w. This implies, however, that $\langle V(F) \cup \{v\} \rangle \cong K_m$ and so v and w are K_m-adjacent vertices of G. Hence $\{v\}$ is a K_m-dominating set of G; consequently, $\gamma_{K_i}(G) = |\{v\}| = 1$ for each i with $2 \leq i \leq m$. ∎

Theorem 3. *Every nondecreasing sequence $a_2, a_3, \ldots, a_m$ of integers such that $a_2 \geq 2$ can be realized as the sequence of generalized domination numbers $\gamma_{K_2}(G), \gamma_{K_3}(G), \ldots, \gamma_{K_m}(G)$ of some K_m-connected graph G.*

Proof: Let $a_2, a_3, \ldots, a_m$ be a given nondecreasing sequence of integers such that $a_2 \geq 2$. Let $G_1, G_2, \ldots, G_{a_m}$ be a_m disjoint copies of K_m and v_i a vertex of $G_i (1 \leq i \leq a_m)$. Further, let u_1 be a vertex of $G_1 - v_1$. Let R denote a graph obtained by adding to $G_1 \cup G_2$ a new vertex v and joining v with an edge to each vertex of $G_i - v_i$ for $i = 1, 2$. If $a_m = 2$, then $G = R$ is a K_m-connected graph for which $\gamma_{K_i}(G) = |\{u_1, v_2\}| = a_i$, for each i with $2 \leq i \leq m$. Hence in what follows we assume that $a_m > 2$.

Let H be the K_m-connected graph obtained from $(\bigcup_{i=3}^{a_m} G_i) \cup R$ by the insertion of an edge between v_2 and every vertex of $G_i - v_i$, for each $i = 3, \ldots, a_m$. (The graph H is shown in Figure 6.) If $2 < a_2 = a_3 = \cdots = a_m$, then, for each i with $2 \leq i \leq m$, $D_i = \{u_1, v_2, v_3, \ldots, v_{a_i}\}$ is a K_i-dominating set of H with $\gamma_{K_i}(H) = |D_i| = a_i$. Hence in what follows we assume that $a_2 < a_m$.

For each i with $4 \leq i \leq m$, let S_i denote a set of $i - 2$ vertices of G_1 such that $u_1 \in S_i$ and $v_1 \notin S_i$. Further, let I denote the set of all values of i with $4 \leq i \leq m$ such that $a_{i-1} < a_i$. If $a_2 < a_3$, then let E_3 denote the set of edges in $\bar{H}$ between u_1 and all the vertices v_j with $a_2 + 1 \leq j \leq a_3$. Further, for each $i \in I$, let E_i denote the set of edges in $\bar{H}$ between S_i and all the vertices v_j with $a_{i-1} + 1 \leq j \leq a_i$.

We are now in a position to construct our graph G. Let G be the K_m-connected graph obtained from H by adding the edges of E_3, if $a_2 < a_3$, and, for each $i \in I$, adding the edges of E_i. (The graph G for the case $2 < a_2 < a_3 < \cdots < a_m$ is sketched in Figure 7.) Then, for each i with $2 \leq i \leq m$, $D_i = \{u_1, v_2, \ldots, v_{a_i}\}$ is a K_i-dominating set of G with $\gamma_{K_i}(G) = |D_i| = a_i$. This completes the proof of the theorem. ∎

Next we establish that, for any given nondecreasing sequence $a_2, a_3, \ldots, a_m$ of integers such that $a_2 \geq 2$, there exists a graph G with $\gamma^t_{K_i}(G) = a_i$ for each $i = 2, 3, \ldots, m$.

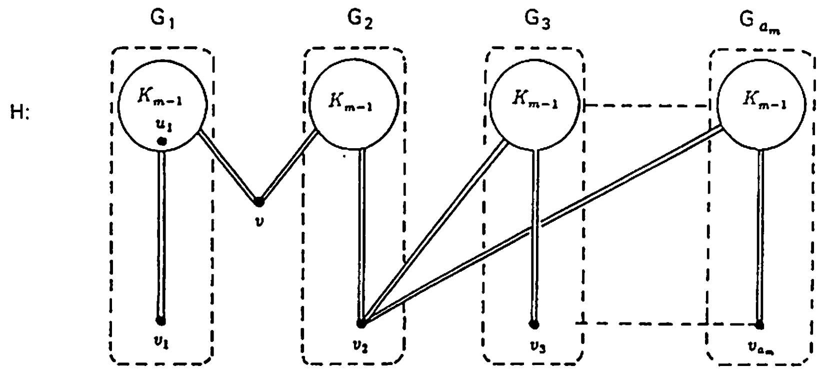


Figure 6. The graph H constructed in the proof of Theorem 3 (===== denotes the join operation).

Theorem 4. *Every nondecreasing sequence $a_2, a_3, \dots, a_m$ of integers such that $a_2 \geq 2$ is a total K_n-domination sequence.*

Proof: Let $a_2, a_3, \dots, a_m$ be a given nondecreasing sequence of positive integers such that $a_2 \geq 2$. We show that $a_2, a_3, \dots, a_m$ can be realized as the sequence of generalized total domination numbers $\gamma_t(G) = \gamma^t_{K_2}(G), \gamma^t_{K_3}(G), \dots, \gamma^t_{K_m}(G)$ of some graph G.

Suppose firstly that $a_2 = a_3 = \cdots = a_m = k$, say. If k is even, say $k = 2\ell$ for some positive integer, then $G = \ell K_m$ is a graph for which $\gamma^t_{K_i}(G) = 2\ell = a_i$ for each $i = 2, 3, \dots, m$. If k is odd, say $k = 2\ell + 1$ for some positive integer ℓ, then let R denote a graph obtained by adding to $K_m \cup K_m$ a new vertex v and inserting an edge between v and $m-1$ vertices in each copy of K_m Then the graph $G = R \cup (\ell - 1) K_m$ is such that $\gamma^t_{K_i}(G) = 2\ell + 1 = a_i$ for each $i = 2, 3, \dots, m$. Hence, in what follows, we may assume that $a_m > a_2 \geq 2$.

Let G_1 denote a graph, isomorphic to K_{m+2}, and $P : u_1 = u'_1, u'_2, u'_3, u'_4 = u_2$ a path of length 4 in G_1. Further, let $V(G_1) - V(P) = \{w_1, w_2, \dots, w_{m-2}\}$ and, for each i with $4 \leq i \leq m$, let $S_i = \{w_1, \dots, w_{i-3}\}$. Let G_2 be the graph obtained from G_1 by deleting the edges of the path P. Let $G_3, G_4, \dots, G_{a_m}$ be $a_m - 2$ disjoint copies of K_m and, for each i with $3 \leq i \leq a_m$, let u_i, v_i be two distinct vertices of G_i. Now let H be the K_m-connected graph obtained from $\bigcup_{i=2}^{a_m} G_i$ by the insertion of an edge between u_2 and every vertex of $G_i - v_i$, for each $i = 3, \dots, a_m$.

Let I denote the set of all values of i with $4 \leq i \leq m$ such that $a_{i-1} < a_i$. If $a_2 < a_3$, then let E_3 denote the set of edges of $\bar{H}$ between u_1 and all the vertices v_j with $a_2 + 1 \leq j \leq a_3$. Further, for each $i \in I$, let E_i denote the set of edges of $\bar{H}$ between $S_i \cup \{u_1\}$ and all the vertices v_j with $a_{i-1} + 1 \leq j \leq a_i$.

We are now in a position to construct our graph G. Let G be the K_m-connected graph obtained from H by adding the edges of E_3, if $a_2 < a_3$, and, for each $i \in I$,

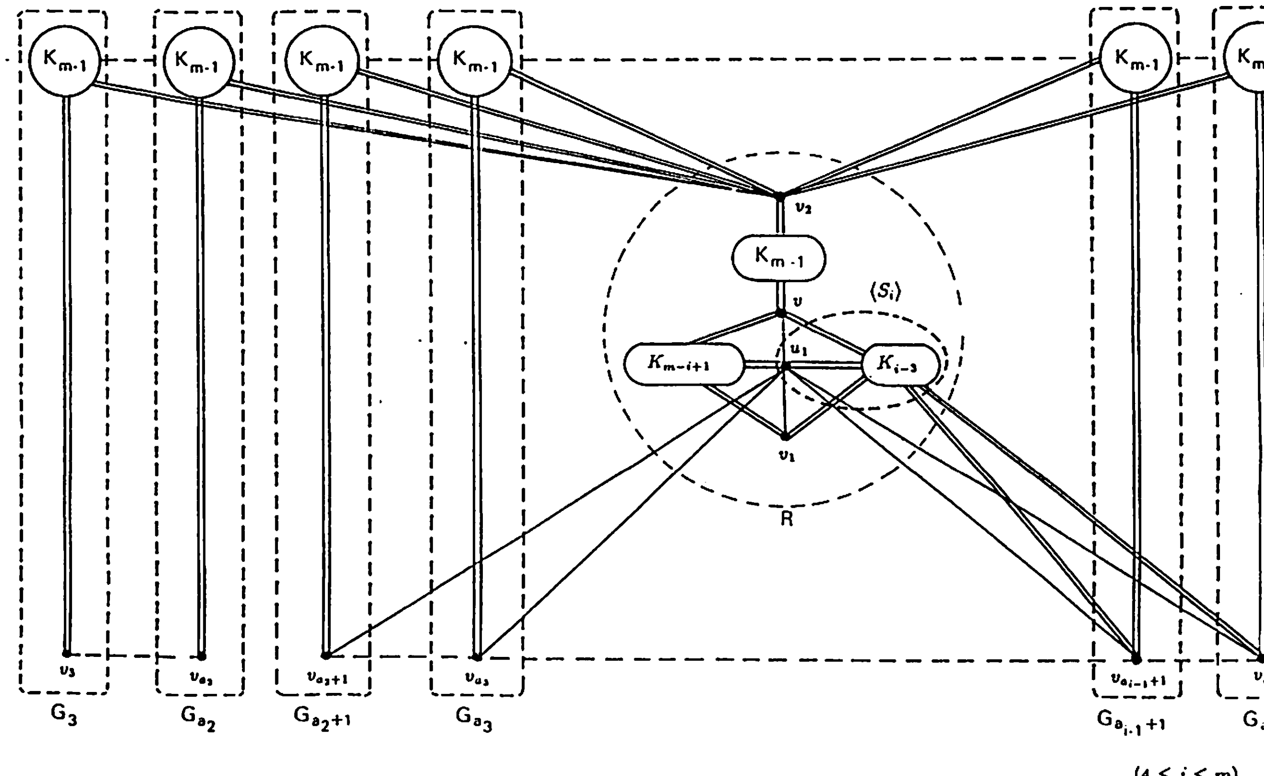


Figure 7. A K_m-connected graph G for which $\gamma_{K_i}(G) = a_i$ for $i = 2, 3, \ldots, m$ where $2 < a_2 < a_3 < \cdots < a_m$ (===== denotes the join operation).

adding the edges of E_i. (The graph G for the case $2 < a_2 < a_3 < \cdots < a_m$ is sketched in Figure 8.) Then, for each i with $2 \leq i \leq m$, $T_i = \{u_1, u_2, \ldots, u_{a_i}\}$ is a total K_i-dominating set of G with $\gamma^t_{K_i}(G) = |T_i| = a_i$. This completes the proof of the theorem. ∎

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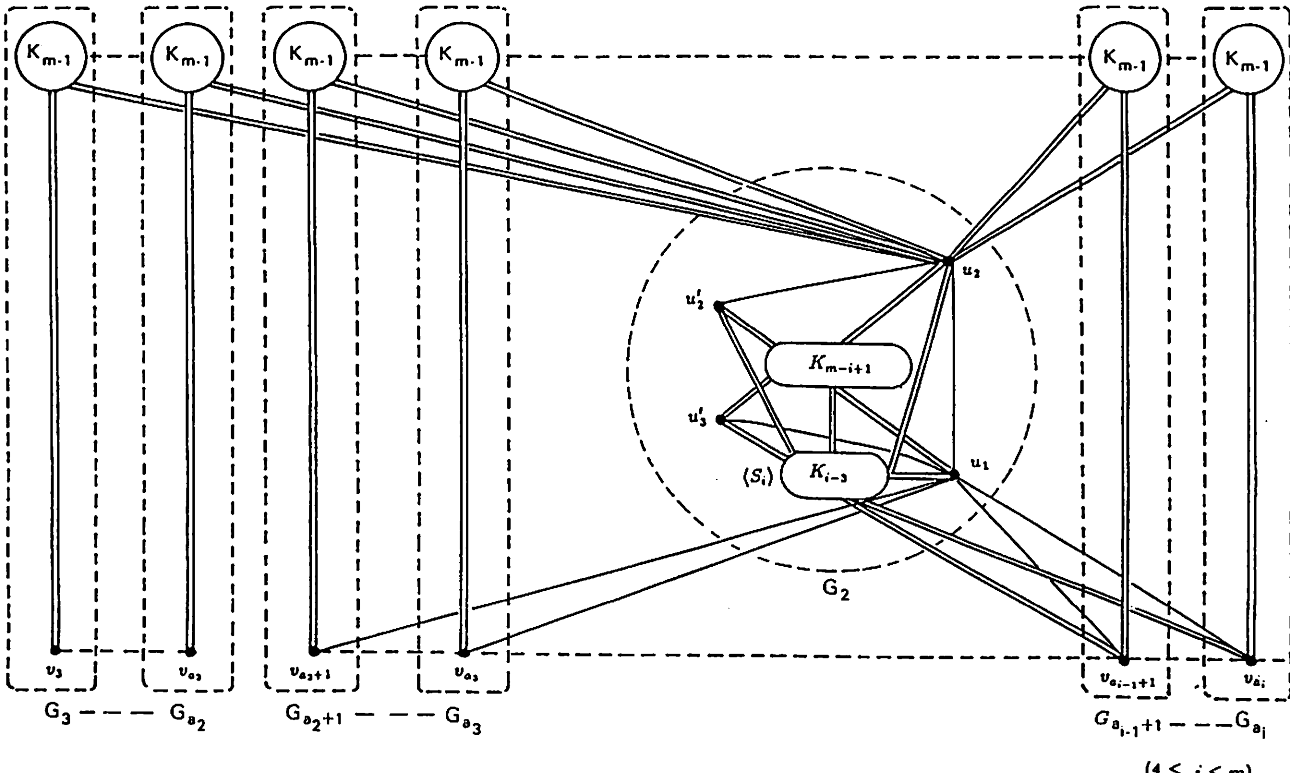

Figure 8. A graph G for which $\gamma^t_{K_i}(G) = a_i$ for $i = 2, \ldots, m$ where $2 < a_2 < a_3 < \cdots < a_m$ (===== denotes the join operation).