On the ℓ-Connectivity Function of Caterpillars and Complete Multipartite Graphs

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Abstract. For an integer $\ell \geq 2$ the ℓ-connectivity (ℓ-edge-connectivity) of a graph G of order p is the minimum number of vertices (edges) that need to be deleted from G to produce a disconnected graph with at least ℓ components or a graph with at most $\ell - 1$ vertices. Let G be a graph of order p and ℓ-connectivity κ_ℓ. For each $k \in \{0, 1, \ldots, \kappa_\ell\}$ let s_k be the minimum ℓ-edge-connectivity among all graphs obtained from G by deleting k vertices from G. Then $f_\ell = \{(0, s_0), \ldots, (\kappa_\ell, s_{\kappa_\ell})\}$ is the ℓ-connectivity function of G. The ℓ-connectivity functions of complete multipartite graphs and caterpillars are determined.

1. Introduction

We follow the graph theory terminology of [3]. It is well-known that the *connectivity* $\kappa(G)$ (*edge-connectivity* $\lambda(G)$) of a graph G is the minimum number of vertices (edges) whose deletion produces a graph with at least two components or the trivial graph. These two parameters are frequently used to measure the reliability of networks which one can model naturally with a graph. While these parameters have the advantage that they can be computed efficiently, there are instances where they provide insufficient information about the reliability of a network. For example, the star $K_{1,m}$ and the path P_{m+1} ($m \geq 3$) are both graphs of order $m+1$ and size m that have connectivity 1, but the deletion of a cut-vertex from $K_{1,m}$ produces m components whereas the deletion of a cut-vertex from P_{m+1} always produces exactly two components. So in some sense $K_{1,m}$ is less reliable than P_{m+1} for $m \geq 3$.

A measure of reliability was introduced in [2] that differentiates between the reliability of these graphs. In particular, for $\ell \geq 2$, the *ℓ-connectivity* $\kappa_\ell(G)$ (*ℓ-edge-connectivity*) $\lambda_\ell(G)$ of a graph G of order $p \geq \ell - 1$ is defined as the minimum number of vertices (edges) that are required to be deleted from G to produce a graph with at least ℓ components or with fewer than ℓ vertices. So $\kappa_2(G) = \kappa(G)$ and $\lambda_2(G) = \lambda(G)$. Since the problem of determining whether the independence number $\beta(G)$ of a graph G, of order $p \geq \ell$, is at least ℓ is NP-complete and since $\beta(G) \geq \ell$ if and only if $\kappa_\ell(G) \neq p - \ell + 1$, it follows that the problem of determining whether $\kappa_\ell(G) \neq p-\ell+1$ is NP-complete. A graph is (n, ℓ)*-connected* if $\kappa_\ell(G) \geq n$. So n-connected graphs are the $(n, 2)$-connected graphs. For a graph G of order p, the sequence of numbers $\kappa_2(G), \kappa_3(G), \ldots, \kappa_p(G)$ is called the *sequence of connectivity numbers* of G. Sequences of non-negative integers that are sequences of connectivity numbers of graphs are characterized in [2]. Unfortunately there are no known efficient algorithms for computing $\kappa_\ell(G)$ or $\lambda_\ell(G)$ for a graph G. In [2] and [6] sharp bounds for $\kappa_\ell(G)$ are established.

It is well-known that with the aid of Menger's Theorem, Whitney [8] showed that a graph G is n-connected if and only if for every pair u, v of distinct vertices of G, there exist at least n internally disjoint u–v paths in G. It was pointed out in [5] and [6] that no analogous characterization of (n, ℓ)-connected graphs exists. It is well-known that if G is a graph of order p, and n is an integer such that $1 \leq n \leq p-1$, then if $\delta(G) \geq (p+n-1)/2$, the graph G is n-connected. So for such graphs G, Whitney's theorem implies that for every pair u, v of vertices of G there exist at least n internally disjoint u–v paths. Hedman [4] actually showed that for such graphs G and every pair u, v of distinct vertices of G there exist at least n internally disjoint u–v paths each of length at most 2. An analogue of this result is established in [6]. For a set S of at least two vertices of a graph G an *S-path* is a path between a pair of vertices of S whose internal vertices do not belong to S. Two S-paths are *internally disjoint* if they have no internal vertices in common.

In [6] it is shown that for a graph G of order $p \geq 2$, and integers $\ell \geq 3$ and $n (1 \leq n \leq p-\ell+1)$, if

$$\delta(G) \geq \frac{p+(n-2)(\ell-1)}{\ell}$$

then for each set S of ℓ vertices of G there exist at least n internally disjoint S-paths each of length at most 2.

The problem of disconnecting a graph into at least two components by the deletion of both vertices and edges was first considered by Beineke and Harary [1]. These concepts were extended in [7]. Let G be a graph with ℓ-connectivity $\kappa_\ell = \kappa_\ell(G)$. If $k \in \{0, 1, \ldots, \kappa_\ell(G)\}$, then let s_k be the minimum ℓ-edge-connectivity among all subgraphs obtained by removing k vertices from G. The *ℓ-connectivity function* of G is defined by $f_\ell(k) = s_k$ for $0 \leq k \leq \kappa_\ell(G)$. So for $\ell = 2$, the ℓ-connectivity function of a graph is its connectivity function, which has been characterized by Beineke and Harary [1]. For $\ell \geq 3$ no characterizations of the ℓ-connectivity function of a graph are known and it appears to be a difficult problem to characterize such functions. In [7] several necessary conditions for a function to be an ℓ-connectivity function of a graph are established and the ℓ-connectivity function of the complete graph is derived. We study here the ℓ-connectivity function of certain types of trees and the complete n-partite graphs.

2. The ℓ-connectivity function of certain classes of graphs

In [7] the following formula for the ℓ-connectivity function of a complete graph is established.

Theorem A. *Let $p, \ell \geq 2$ be integers with $p \geq \ell$ and suppose that $G \cong K_p$. Then the ℓ-connectivity function of G is given by*

$$f_\ell(k) = \begin{cases} 0 & \text{if } k = \kappa_\ell(G) \\ (\ell-1)(p-\ell-k+1) + \binom{\ell-1}{2} & \text{for } 0 \leq k < \kappa_\ell(G) \end{cases}$$

We now extend this result to complete n-partite graphs.

Theorem 1. *Suppose $G \cong K_{m_1,m_2,\ldots,m_n}$ where $m_1 \leq m_2 \leq \cdots \leq m_n$ and $n \geq 2$. Let $P = \sum_{i=1}^{n} m_i$ and let k be an integer with $0 \leq k \leq \kappa_\ell(G)$. If $s = \min\{m_{n-1}, \sum_{i=1}^{n-1} m_i - k\}$, then the ℓ-connectivity function of G is given by*

$$f_\ell(k) = \begin{cases} 0 & \text{if } k = \kappa_\ell(G) \\ (\ell-1)(p-m_n-k) & \text{if } k \neq \kappa_\ell(G) \text{ and } \ell \leq m_n - s + 2 \\ (\ell-1)(p-m_n-k) - \binom{\ell-m_n+s-1}{2} & \text{if } k \neq \kappa_\ell(G) \text{ and } \ell > m_n - s + 2. \end{cases}$$

To prove this result we begin by establishing a series of lemmas.

Lemma 1. *Let $G = K_{r_1,r_2,\ldots,r_t}$ be a complete t-partite graph ($t \geq 2$) of order p and let ℓ be an integer, $2 \leq \ell \leq p$. There exists a set of $\lambda_\ell(G)$ edges of G, say E_ℓ, such that $G - E_\ell$ has ℓ components, at most one of which is non-trivial.*

Proof: Let $V_1, V_2, \ldots, V_t$ be the partite sets of G with $|V_i| = r_i$ for $i = 1, 2, \ldots, t$. There exists a set F_ℓ of $\lambda_\ell(G)$ edges of G such that $G - F_\ell$ has ℓ components. Of all such sets F_ℓ let E_ℓ be one such that $G - E_\ell$ has as few non-trivial components as possible. We shall show that $G - E_\ell$ has at most one non-trivial component.

Assume, to the contrary, that $G - E_\ell$ has at least two non-trivial components, G_1 and G_2, with $V(G_1) = A$ and $V(G_2) = B$. For $i = 1, 2, \ldots, t$, let $A \cap V_i = A_i$, $B \cap V_i = B_i$, $|A_i| = a_i$ and $|B_i| = b_i$. Then there exist $i_1, i_2, j_1, j_2 \in \{1, 2, \ldots, t\}$ such that $i_1 \neq i_2, j_1 \neq j_2$ and $a_{i_1}, a_{i_2}, b_{j_1}, b_{j_2} \geq 1$. Letting $H = \langle A \cup B \rangle_G$, we note that for $v \in A_i \cup B_i (i \in \{1, 2, \ldots, t\})$

$$\deg_H v = a + b - a_i - b_i. \tag{1}$$

Furthermore, the set $[A, B]$ of all edges in H with one end vertex in A, the other in B, has cardinality

$$|[A, B]| = \sum_{i=1}^{t} a_i(b - b_i) = \sum_{i=1}^{t} b_i(a - a_i). \tag{2}$$

It follows from our choice of E_ℓ that isolating a single vertex of H requires the removal of more edges than separating the components G_1 and G_2 in H; i.e., for $v \in V(H)$, $\deg_H v > |[A,B]|$. Hence, for every $i \in \{1,2,\ldots,t\}$ such that $a_i + b_i \geq 1$,

$$\begin{aligned} 2\sum_{\substack{j=1\\ j\neq i}}^{t}(a_j + b_j) &= 2(a + b - a_i - b_i) \\ &> \sum_{j=1}^{t} a_j(b - b_j) + \sum_{j=1}^{t} b_j(a - a_j). \end{aligned} \tag{3}$$

Assuming (without loss of generality) that $a_t + b_t \geq 1$, we obtain from (3) with $i = t$

$$\sum_{j=1}^{t-1} a_j(b - b_j - 2) + \sum_{j=1}^{t-1} b_j(a - a_j - 2) + a_t(b - b_t) + b_t(a - a_t) < 0. \tag{4}$$

Since $a - a_j, b - b_j \geq 1$ for all $j \in \{1,2,\ldots,t\}$, it follows from (4) that there exists $j \in \{1,2,\ldots,t-1\}$ such that $a_j \geq 1$ and $b - b_j - 2 < 0$ or $b_j \geq 1$ and $a - a_j - 2 < 0$; say $b_1 \geq 1$ and $a - a_1 < 2$. Then $a - a_1 = 1$ and there exists $m \in \{2,3,\ldots,t\}$ such that $a_m = 1$ and $a_j = 0$ for all $j \in \{2,3,\ldots,t\} - \{m\}$. We note that $a_1 \geq 1$.

Since $|[A,B]| < \deg_H v$ for $v \in A$, it follows from (1) and (2) that

$$a_1(b - b_1) + a_m(b - b_m) < a - a_1 + b - b_1 = 1 + b - b_1;$$

hence

$$(a_1 - 1)(b - b_1) + b - b_m < 0$$

which, with $a_1 - 1 \geq 0$, $b - b_1 \geq 1$, $b - b_m \geq 1$, yields a contradiction, thus establishing the validity of the lemma. ∎

For a vertex v in a graph G, let the set of edges of G incident with v be denoted by $E_G(v)$.

Lemma 2. *Let $G = K_{r_1,r_2,\ldots,r_t}$ with $r_1 \leq r_2 \leq \cdots \leq r_t, t \geq 2, p = p(G) = \sum_{i=1}^{t} r_i$ and $\ell \in \{2,3,\ldots,p\}$. Let $V_1, V_2, \ldots, V_t$ be the partite sets of G with $|V_i| = r_i$. The following algorithm yields a set E_ℓ of edges of G such that $|E_\ell| = \lambda_\ell(G)$ and $G - E_\ell$ has ℓ components, at least $\ell - 1$ of which are trivial:*

1. *Let $H_1 = G$ and let v_1 be a vertex of minimum degree in H_1. (i.e., $v_1 \in V_t$). Let $E_2 = E_{H_1}(v_1)$ and $H_2 = H_1 - v_1$.*
2. *For $i \in \{2,\ldots,\ell-1\}$, let v_i be a vertex of minimum degree in H_i and let $E_{i+1} = E_{H_i}(v_i) \cup E_i$, $H_{i+1} = H_i - v_i$.*

Proof: The validity of the lemma for $\ell = 2$ is an immediate consequence of Lemma 1. Further, the lemma follows if $\ell = p$, in which case $|E_\ell| = q(G) = \lambda_p(G)$. Suppose that the lemma does not hold and let m be the smallest value of ℓ for which the algorithm yields a set E_ℓ that does not satisfy the requirements of the lemma; so $2 < m < p$. Since $G - E_m$ certainly contains m components, $m-1$ of which are trivial, it follows that $|E_m| > \lambda_m(G)$. Let F_m be a set of edges of G such that $|F_m| = \lambda_m(G)$, $G - F_m$ contains m components of which $m-1$ are trivial.

Let $W = \{w_1, w_2, \ldots, w_{m-1}\}$ denote the set of $m-1$ isolated vertices in $G - F_m$ and, for $w_k \in W$ let $G_k = G - (W - \{w_k\})$. Let $i = i(F_m)$ be such that $v_1, \ldots, v_{i-1} \in W$ and $v_i \notin W$. Choose F_m such that $i(F_m)$ is as large as possible. Suppose $v_s = w_s$ for $1 \le s \le i-1$. Let $W' = W - \{v_1, \ldots, v_{i-1}\}$ and let $v_i \in V_j$; then $V_j \cap W' = \emptyset$, since otherwise, if $w_k \in V_j \cap W'$, the set of edges of F_m incident with w_k in G_k, namely $E_{G_k}(w_k)$, may be replaced by $E_{G_k}(v_i)$ to yield a set F'_m of edges of G with $|F'_m| = \lambda_m(G)$ such that $G - F'_m$ has m components, $m-1$ of which are trivial and $i(F'_m) > i(F_m)$, contrary to our choice of F_m. Hence the only vertices which are adjacent to v_i in H_i and not to v_i in G_{m-1} are those in $W' - \{w_{m-1}\}$. Consequently $\deg_{G_{m-1}} v_i = \deg_{H_i} v_i - (m-2-i+1)$. Furthermore, $\deg_{G_{m-1}} w_{m-1} \ge \deg_{H_i} w_{m-1} - (m-2-i+1)$; so, since $\deg_{H_i} w_{m-1} \ge \deg_{H_i} v_i$, it follows that $\deg_{G_{m-1}} w_{m-1} \ge \deg_{G_{m-1}} v_i$. Hence, replacing the subset $E_{G_{m-1}}(w_{m-1})$ of F_m by $E_{G_{m-1}}(v_i)$, we obtain a set F''_m of edges of G with $|F''_m| \le |F_m| = \lambda_m(G)$ such that $G - F''_m$ has m components, $m-1$ of which are trivial, and $i(F''_m) > i(F_m)$.

Thus the validity of the lemma is established. ∎

Let $G = K_{m_1, m_2, \ldots, m_n}$ with $m_1 \le m_2 \le \cdots \le m_n (n \ge 2)$ and partite sets $V_1, \ldots, V_n$ where $|V_i| = m_i$ for $i = 1, 2, \ldots, n$; $p = \sum_{i=1}^{n} m_i$. Let S be a proper subset of $V(G)$ such that $|S| = k \in \{0, 1, \ldots, \kappa_\ell(G)\}$ and $k < p - m_n$, where we note that

$$\kappa_\ell(G) = \begin{cases} p - m_n = \sum_{i=1}^{n-1} m_i & \text{if } \ell \le \beta(G) = m_n, \\ p - \ell + 1 & \text{if } \ell > m_n, \end{cases}$$

then $G - S$ is a complete multipartite graph, say $K_{r_1, \ldots, r_t}$. It is an immediate consequence of Lemma 2 that S may be chosen to yield $G - S$ of minimum ℓ-edge connectivity, namely $\lambda_\ell(G - S) = f_\ell(k)$, by letting S consist of k vertices of maximum degree in G, i.e., for some $j \in \{1, 2, \ldots, m-1\}$, $S = \cup_{i=1}^{j} V'_i$, where $V'_i = V_i$ if $i < j$ and $V'_j \subseteq V_j$. Then $E_\ell \subseteq E(G - S)$ may be obtained as prescribed by Lemma 2 to produce $G - S - E_\ell$ containing ℓ components, $\ell - 1$ of which are trivial.

If $\ell > m_n$ or $k = p - m_n$, then $f_\ell(k) = 0$, obviously. Hence we have the following lemma

Lemma 3. *If $G = K_{m_1, \ldots, m_n}$ with $m_1 \le m_2 \le \ldots \le m_n (n \ge 2)$, and partite sets $V_1, \ldots, V_n$ such that $|V_i| = m_i$ for $i = 1, \ldots, n$, then, for $2 \le \ell \le p$ and*

$0 \leq k \leq \kappa_\ell$, *there exist* $S \subseteq V(G)$ *and* $E_\ell \subseteq E(G-S)$ *such that* $|S| = k$, $|E_\ell| = f_\ell(k)$, *and such that* $G - S - E_\ell$ *contains at least* ℓ *components, at least* $\ell - 1$ *of which are trivial and, for some* $j \in \{1, \ldots, n\}$, $S = \cup_{i=1}^{j} V_i'$, *where* $V_i' = V_i$ *for* $i \leq j$ *and* $V_j' \subseteq V_j$.

Proof of Theorem 1: Clearly if $k = \kappa_\ell(G)$, then $f_\ell(k) = 0$. If $\ell \leq m_n - s + 2$ then, since the degrees of vertices in V_{n-1} exceed those of vertices in V_n by $m_n - s$ in $G-S$, the $\ell-1$ vertices isolated in $G-S-E_\ell$ occur in V_n. (We note that, for $i \in \{1, \ldots, \ell-2\}$, if $w \in V_{n-1} - S$ and $z \in V_n$, then in $G-S-\{v_1, \ldots, v_i\}$, $\deg w \geq \deg z$.) In this case it is obvious that $|E_\ell| = (\ell-1)(p-m_n-k)$.

If $\ell \geq m_n - s + 2$ then, applying the algorithm in Lemma 2 to $G - S$, we note that $v_1, \ldots, v_{m_n-s+1}$ may be chosen from V_n and that their isolation requires the removal of $(p-m_n-k)(m_n-s+1)$ edges. The isolation of $v_{m_n-s+2}, \ldots, v_{\ell-1}$ requires the removal, successively, of $p-m_n-k-1, p-m_n-k-2, \ldots, [(p-m_n-k)-(\ell-m_n+s-2)]$ edges. Hence, in this case,

$$|E_\ell| = (p-m_n-k)(m_n-s+1) + \sum_{i=1}^{\ell-m_n+s-2} (p-m_n-k-i)$$
$$= (\ell-1)(p-m_n-k) - \binom{\ell-m_n+s-1}{2} \quad \text{if } k \neq \kappa_\ell(G).$$

∎

It is not difficult to see that Theorem A follows as a corollary to Theorem 1.

We next turn our attention to the ℓ-connectivity function of caterpillars. Recall that a caterpillar is a tree that is either isomorphic to K_1 or K_2 or has the property that if its end-vertices are deleted then a path is produced. For a graph G of order p and an integer k, $0 \leq k < p$, let $c_\kappa(G)$ be the maximum number of components that are produced when k vertices are deleted from G. Note that if $\ell \geq 2$ is an integer and T is a tree with independence number $\beta(T) \geq \ell$, then $f_\ell(k) = (\ell-1) - c_k(T)$ for $0 \leq k < \kappa_\ell(T)$. Let $\delta_\beta(T) = \min\{k \mid c_\kappa(T) = \beta(T)\}$. The following algorithm finds for a given caterpillar T and every k, $0 \leq k \leq \delta_\beta(T)$, a set V_k of k vertices such that $k(T - V_k) = c_k$.

Algorithm 1. *Let* $T \not\cong K_1, K_2$ *be a caterpillar.*

1. (a) $F_0 \leftarrow T$.
   (b) $V_0 \leftarrow \emptyset$.
   (c) $S_0 \leftarrow \{v \in V(F_0) \mid \deg_{F_0} v = \Delta(F_0)\}$.
   (d) $H_0 \leftarrow \langle S_0 \rangle_{F_0}$.
   (e) $n \leftarrow 0$.
   (f) *Let* $P: u_1, \ldots, u_r$ *be the path produced by deleting the end-vertices of* T.
2. *Let* $T_1, T_2, \ldots, T_s$ *be the components of* H_n *and* $a_i = \lceil \frac{p(T_i)}{2} \rceil$.
   Let $U_n = \{w_1^n, w_2^n, \ldots, w_{\beta_n}^n\}$ *be a maximum independent set of vertices of* H_n

(with $\beta_n = \beta(H_n)$) chosen as follows: The vertices $w_1^n, w_2^n, \ldots, w_{a_1}^n$ belong to T_1. If $s > 1$, then for $i = 2, \ldots, s$, the vertices $w_{a_1+\cdots+a_{i-1}+1}^n, \ldots, w_{a_1+\cdots+a_{i-1}+a_i}^n$ belong to T_i and if $w_m^n = u_{i_1}$ and $w_r^n = u_{i_2}$ belong to some T_i and $m < r$, then $i_1 < i_2$. Further, $w_{a_1+\cdots+a_{i-1}+1}^n$ is an end-vertex of T_i for $2 \leq i \leq s$ and w_1^n is an end-vertex of T_1.

3. *(a) $F_{n+1} \leftarrow F_n - U_n$*
   (b) $n \leftarrow n + 1$
   (c) $S_n \leftarrow \{v \in V(F_n) \mid \deg_{F_n} v = \Delta(F_n)\}$
   (d) $H_n \leftarrow \langle S_n \rangle_{F_n}$
   (e) If $\Delta(F_n) > 1$, return to Step 2;
   otherwise let $\delta_\beta \leftarrow \sum_{i=1}^{n-1} |U_i|$ and continue.
4. *For $k = 1, 2, \ldots, \delta_\beta$ let $v_1, v_2, \ldots, v_k$ denote, in order, the first k vertices in the sequence $w_1^1, w_2^1, \ldots, w_{a_1}^1, w_1^2, \ldots, w_{a_2}^2, \ldots$, and define $V_k = \{v_1, v_2, \ldots, v_k\}$.*

Theorem 2. *Suppose Algorithm 1 is applied to a caterpillar $T \ncong K_1$ or K_2. Then*

$$k(T - V_k) = c_k(T) \quad \text{for } 0 \leq k \leq \delta_\beta.$$

Proof: Suppose the theorem does not hold. Let k be the smallest integer such that $k(T - V_k) < c_k$. Let $Z = \{z_1, z_2, \ldots, z_k\} \subseteq V(T)$ be such that $k(T - Z) = c_k$. If $v_1 \in Z$, let j be the smallest integer such that $v_{j+1} \notin Z$ otherwise let $j = 0$. Among all sets $Z \subseteq V(T)$ satisfying $k(T - Z) = c_k$, choose Z such that j is as large as possible. For $i = 1, 2, \ldots, k$, let $Z_i = Z - \{z_i\}$ and suppose the vertices of Z have been labelled in such a way that if $j \geq 1$, then $z_s = v_s$ for $1 \leq s \leq j$. By our choice of Z, it follows that for $i = j+1, j+2, \ldots, k$ the vertex z_i cannot be replaced by v_{j+1} in Z to form $Z_i' = Z_i \cup \{v_{j+1}\}$ with $k(T - Z_i') = c_k$. Hence

$$\deg_{T-Z_i} v_{j+1} < \deg_{T-Z_i'} z_i.$$

However,

$$\deg_{T-\{v_1, v_2, \ldots, v_j\}} v_{j+1} \geq \deg_{T-\{v_1, v_2, \ldots, v_j\}} z_i.$$

Therefore v_{j+1} has a neighbour in $\{z_{j+1}, \ldots, z_k\} - \{z_i\}$, say z_m is such a neighbour. Similarly, v_{j+1} has a neighbour in $\{z_{j+1}, \ldots, z_k\} - \{z_m\}$; say z_n.

Note that every vertex of $Z \cup V_k$ lies on the path P described in Step 1 (f). Let a and b be neighbours different from v_{j+1} of z_m and z_n, respectively. We show next that $a, b \notin Z$. Suppose $v_{j+1} \in U_t$. Then $\deg_{F_t} v_{j+1} \geq \deg_{F_t} z_m$. Suppose $a \in Z$. Then a lies on P. Therefore

$$k\left(T - (Z_m \cup \{v_{j+1}\})\right) \geq k(T - Z) = c_k,$$

which contradicts our choice of Z. So $a \notin Z$, and similarly $b \notin Z$.

Suppose $\deg_{F_t} z_m < \deg_{F_t} v_{j+1} = \Delta(F_t)$. Then once again it follows that

$$k\left(T - (Z_m \cup \{v_{j+1}\})\right) \geq k(T - Z),$$

which contradicts our choice of Z. Hence $\deg_{F_t} z_m = \Delta(F_t)$. Similarly $\deg_{F_t} z_n = \Delta(F_t)$. If $\deg_{F_t} a$ and $\deg_{F_t} b$ are less than $\Delta(F_t)$, then z_m and z_n are end vertices of a component of H_t, which contradicts our choice of V_k. Hence $\deg_{F_t} a = \Delta(F_t)$. If $\deg_{F_t} b < \Delta(F_t)$, then by the choice of V_k it follows since z_n is an end vertex of a component of H_t, not in V_k, a must be v_j. This is impossible since $a \notin Z$. Otherwise, if $\deg_{F_t} b = \Delta(F_t)$, then a or b is v_j which once again produces a contradiction. This completes the proof of the validity of Algorithm 1. ∎

With the aid of Algorithm 1 and Theorem 2 we are now able, in the next two theorems, to characterize the ℓ-connectivity functions of caterpillars.

Theorem 3. *For an integer $\ell \geq 2$, a function $f_\ell\colon \{0, \ldots, \kappa_\ell\} \to \mathrm{N} \cup \{0\}$ is the ℓ-connectivity function of a caterpillar with independence number at least ℓ if and only if*

(i) f_ℓ is decreasing,

(ii) $f_\ell(0) = \ell - 1$ and $f_\ell(\kappa_\ell) = 0$, and

(iii) if $\kappa_\ell \geq 2$, then $f_\ell(k) - f_\ell(k+1) \geq f_\ell(k+1) - f_\ell(k+2)$ for $0 \leq k \leq \kappa_\ell - 2$.

Proof: Suppose first that f_ℓ is the ℓ-connectivity function of a caterpillar T. Then $f_\ell(k) = \ell - c_k(T)$ for $0 \leq k < \kappa_\ell(G) = \kappa_\ell$. Since $c_k(T) < c_{k+1}(T)$ for $0 \leq k < \kappa_\ell$, it follows that f_ℓ is decreasing.

Since every edge of a tree is a bridge, $\ell - 1$ edges must be deleted from a tree to produce ℓ components. Hence $f_\ell(0) = \ell - 1$. Since $\ell \leq \beta(T)$, it follows that there exists a set of $\kappa_\ell(T)$ vertices whose deletion produces a graph with at least ℓ components. Hence $f_\ell(\kappa_\ell) = 0$. Hence (ii) holds.

Observe that if $\kappa_\ell \geq 2$, then $f_\ell(k) - f_\ell(k+1) = c_{k+1}(T) - c_k(T)$ and $f_\ell(k+1) - f_\ell(k+2) = c_{k+2}(T) - c_{k+1}(T)$. Let $v_1, v_2, \ldots$ be as in Step 4 of Algorithm 1. Suppose $v_{k+1} \in U_r$ and $v_{k+2} \in U_s$. Then $r \leq s \leq r+1$ and $\deg_{F_r} v_{k+1} \geq \deg_{F_s} v_{k+2}$. Since $c_{k+1}(T) - c_k(T) = \deg_{F_r} v_{k+1} - 1$ and $c_{k+2}(T) - c_{k+1}(T) = \deg_{F_s} v_{k+2} - 1$, condition (iii) follows.

For the converse suppose that $f_\ell\colon \{0, \ldots, \kappa_\ell\} \to \mathrm{N} \cup \{0\}$ is a function that satisfies conditions (i), (ii) and (iii) of Theorem 2. Construct a caterpillar T as follows. Begin with a path $v_1, u_1, v_2, u_2, \ldots, u_{\kappa_\ell - 1}, v_{\kappa_\ell}$. Next join $f_\ell(0) - f_\ell(1)$ new vertices to v_1 and for $2 \leq i \leq v_{\kappa_\ell - 1}$ join $f_\ell(i-1) - f_\ell(i) - 1$ new vertices to v_i. Finally join $f_\ell(\kappa_\ell - 1) - f_\ell(\kappa_\ell)$ new vertices to v_{κ_ℓ}. Let T be the resulting caterpillar. Then it can be shown that T has independence number at least ℓ and its ℓ-connectivity function is f_ℓ. ∎

The next result characterizes ℓ-connectivity functions of caterpillars whose independence numbers are less than ℓ.

Theorem 4. *For an integer $\ell \geq 2$ a function $f_\ell\colon \{0, 1, \ldots, \kappa_\ell\} \to \mathrm{N} \cup \{0\}$ is the ℓ-connectivity function of a caterpillar T of order $p \geq \ell$, independence number $\beta = \beta(T) < \ell$ and $m = \delta_\beta(T)$ if and only if*

(i) $f_\ell(0) = \ell - 1, f_\ell(\kappa_\ell) = 0$,

(ii) $f_\ell(k+1) < f_\ell(k)$ *for* $0 \le k \le m-1$ *and* $f_\ell(m) = f_\ell(m+1) = \cdots = f_\ell(\kappa_\ell - 1) = \ell - \beta$.

(iii) $f_\ell(k) - f_\ell(k+1) \ge f_\ell(k+1) - f_\ell(k+2)$ *for* $0 \le k < \kappa_\ell - 2$,

(iv) (a) if $f_\ell(m-1) - f_\ell(m) > 1$, *then* $m < \kappa_\ell \le 2m - f_\ell(m) + 2$, *otherwise*

(b) let s *be the largest positive integer such that* $f_\ell(t) - f_\ell(t+1) = 1$ *for* $m - s \le t \le m-1$, *then* $m < \kappa_\ell \le 2m - f_\ell(m) - s + 2$.

Proof: Suppose f_ℓ is the ℓ-connectivity function of a caterpillar with independence number $\beta = \beta(T)$ and $m = \delta_\beta(T)$. Then condition (i) clearly holds. As in Theorem 3 $f_\ell(k) = \ell - c_k(T)$ for $0 \le k < \kappa_\ell$. Since $c_k(T) < c_{k+1}(T)$ for $0 \le k < \delta_\beta(T) = m$ it follows that $f_\ell(k+1) < f_\ell(k)$ for $0 \le k \le m-1$. Since $c_k(T) = \beta$ for $m = \delta_\beta(T) \le k \le \kappa_\ell - 1$, $f_\ell(m) = f_\ell(m+1) = \cdots = f_\ell(\kappa_\ell - 1) = \ell - \beta$. Hence condition (ii) holds.

Since $f_\ell(k+1) - f_\ell(k+2) = 0$ and $f_\ell(k) - f_\ell(k+1) \ge 0$ for $m - 1 \le k < \kappa_\ell - 2$, condition (iii) holds for $m-1 \le k < \kappa_\ell - 2$. Suppose now that $0 \le k \le m-2$. Then, as in the proof of Theorem 3, $f_\ell(k) - f_\ell(k+1) \ge f_\ell(k+1) - f_\ell(k+2)$. Thus condition (iii) holds.

Let m_1 be the smallest integer so that if S consists of the first m_1 vertices selected by Algorithm 1, then the components of $T - S$ are all paths. (Note possibly $m_1 = m$.) For each of the $m - m_1$ vertices $v_i \in \{v_{m_1+1}, \ldots, v_m\}$ removed next by the algorithm there exists a vertex w_i isolated by the removal of v_i. Let P be a longest path in T. Let $T_0 = T$ and for $i = 1, 2, \ldots, m_1 - 1$ let $T_i = T - \{v_1, \ldots, v_i\}$. Observe that if vertex v_j is deleted from $T_{j-1} (1 \le j \le m_1)$, the number of components is increased by $f_\ell(j-1) - f_\ell(j)$. Hence at least $f_\ell(j-1) - f_\ell(j) - 1$ vertices not on P are isolated in the process. Let there be k vertices v_j for which $f_\ell(j-1) - f_\ell(j)$ vertices not on P are isolated when v_j is deleted from T_{j-1}. Then v_j is adjacent with a vertex from the set $\{v_1, v_2, \ldots, v_{j-1}\}$. Thus there are exactly $\sum_{j=1}^{m_1} (f_\ell(j-1) - f_\ell(j) - 1) + k = f_\ell(0) - f_\ell(m_1) - m_1 + k$ vertices of T not on P. Let S_1 denote the set of these vertices and $S_2 = \{v_1, v_2, \ldots, v_{m_1}\}$. Further, let $S_3 = \{v_{m_1+1}, v_{m_1+2}, \ldots, v_m\} \cup \{w_{m_1+1}, w_{m_1+2}, \ldots, w_m\}$. Note that each component of $T_m = T - \{v_1, v_2, \ldots, v_m\}$ is isomorphic to K_1 or K_2. Let S_4 be the set of vertices that belong to components isomorphic to K_2 in T_m. Then $|S_4| \le 2(m_1 + 1 - k)$. To see this note that the deletion of the vertices of S_2 from T produces a tree with at most $m_1 + 1 - k$ nontrivial components. If Algorithm 1 is now applied to $T - S_2$ to delete the next $m - m_1$ vertices and thus to produce T_m, each of the nontrivial components of $T - S_2$ corresponds to at most one K_2 of T_m. Thus

$$
\begin{aligned}
p &= |S_1| + |S_2| + |S_3| + |S_4| \\
&\le f_\ell(0) - f_\ell(m_1) - m_1 + k + m_1 + 2(m - m_1) + 2(m_1 + 1 - k) \\
&= 2m - f_\ell(m_1) + 2.
\end{aligned}
$$

Since $\kappa_\ell = p - \ell + 1 = p - f_\ell(0)$, it follows that $\kappa_\ell \le 2m - f_\ell(m_1) + 2$.

Clearly $m < \kappa_\ell$. Now if $f_\ell(m-1) - f_\ell(m) > 1$, then $m_1 = m$ so that (iv) (a) follows. Otherwise, $s = m - m_1$ and $f_\ell(m_1) = f_\ell(m) + s$. Hence, in this case $\kappa_\ell \leq 2m - f_\ell(m) - s + 2$; thus (iv) (b) follows.

For the converse suppose f_ℓ: $\{0, 1, \ldots, \kappa_\ell\} \to \mathrm{N} \cup \{0\}$ is a function that satisfies conditions (i)–(iv). Let $p = \kappa_\ell + f_\ell(0)$. Let P: $u_1, v_1, u_2, v_2, \ldots, u_m, v_m, u_{m+1}$. Join v_i to $f_\ell(i-1) - f_\ell(i) - 1$ new vertices for $1 \leq i \leq m$ and let T' be the resulting caterpillar. Observe that the caterpillar constructed thus far has order $f_\ell(0) - f_\ell(m) + m + 1$. Since $f_\ell(m) \geq 1$ it follows by (iv) that $p' = p - (f_\ell(0) - f_\ell(m) + m + 1) = \kappa_\ell - m + f_\ell(m) - 1 \geq 0$. If $p' = 0$, then it can be shown that $T = T'$ has f_ℓ as its ℓ-connectivity function and independence number β and $\delta_\beta(T) = m$. If $p' > 0$, then $p' \leq m + 1$ if $f_\ell(m-1) - f_\ell(m) > 1$ and $p' \leq m - s + 1$ if $f_\ell(m-1) - f_\ell(m) = 1$. Suppose first that $f_\ell(m-1) - f_\ell(m) > 1$. In this case, if $p' \leq m$, subdivide the edges $u_i v_i$ exactly once for $1 \leq i \leq p'$ to obtain T; otherwise subdivide the edges $u_i v_i$ for $1 \leq i \leq m$ and the edge $v_m u_{m+1}$ exactly once to obtain T. Suppose now that $f_\ell(m-1) - f_\ell(m) = 1$. Now subdivide the edges $u_i v_i$ exactly once $(1 \leq i \leq p')$ to obtain T. In both cases it can be seen that the corresponding f_ℓ is the ℓ-connectivity function of T. ∎

The complex characterizations of the ℓ-connectivity functions of caterpillars given in Theorems 3 and 4 lead one to believe that the problem of characterizing the ℓ-connectivity functions of trees in general is a difficult task. In closing we remark that it also remains an open problem to characterize the ℓ-connectivity functions of the n-cube.

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