On Optimal Weighing Designs

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Abstract. There are two criteria for optimality of weighing designs. One, which has been widely studied, is that the determinant of XX^T should be maximal, where X is the weighing matrix. The other is that the trace of $(XX^T)^{-1}$ should be minimal. We examine the second criterion. It is shown that Hadamard matrices, when they exist, are optimal with regard to the second criterion, just as they are for the first one.

1. Weighing designs.

Weighing designs were introduced by Yates [5], and refined by Hotelling [3], as an experimental design to be used when determining the weights of a number of objects, using a spring or beam (chemical) balance. The design is determined by a $p \times N$ matrix X with entries in $\{1, 0, -1\}$. The experiment consists of N weighings and determines the weights of p objects. In weighing j, the disposition of object i is decided as follows: it is placed in the left-hand pan if $x_{ij} = 1$, in the right-hand pan if $x_{ij} = -1$, and not used if $x_{ij} = 0$. (For spring balances, x_{ij} can only equal 1 or 0.) If y is the vector of observed weights, then

$$(XX^T)^{-1}Xy$$

is the least squares estimator of the weights, and is unbiased.

There are three main measures of efficiency of experimental designs, usually labeled A-, D- and E-efficiency. Banerjee [1] showed that E-efficiency is subsumed by D-efficiency among weighing designs. The A-efficiency was defined by Kishen as

$$\frac{p}{N \operatorname{tr}(XX^T)^{-1}}$$

and the D-efficiency by Mood [4] as

$$\frac{\det(XX^T)}{M}$$

where M is the maximum value of the determinant ZZ^T for $p \times N$ matrices Z over $\{1, 0, -1\}$.

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2. The square case; Hadamard matrices

Most attention has been paid to the symmetric case where $N = p$. (Since $N \geq p$ is a necessary condition for any estimates to be made, $N = p$ is in a sense the extreme case.) In this case we are naturally led to the discussion of Hadamard matrices.

An n by n matrix $H = (h_{ij})$ with all its entries $h_{ij} = 1$ or -1 is called a *Hadamard matrix* of order n if

$$HH^T = nI.$$

For any complex matrix $A = (a_{ij})$ of order n, Hadamard [2] showed that if $|a_{ij}| \leq 1$, for $i, j = 1, 2, \ldots, n$, then

$$|\det A| \leq n^{n/2}.$$

The bound is attained if and only if $AA^* = nI$ with $|a_{ij}| = 1$, for $i, j = 1, 2, \ldots, n$ (where $*$ denotes conjugate transpose). In particular, when all the entries of A are real, the bound is attained by A if and only if A is a Hadamard matrix.

Consequently Hadamard matrices provide D-optimal designs for all the symmetric cases where $N \equiv 0 \pmod 4$, so they are very important in the study of weighing designs. Our aim in this note is to show that they are equally important because of their contribution to A-efficiency.

3. The minimum trace theorem.

Theorem. *For any nonsingular complex matrix $H = (h_{ij})$ of order n with complex entries $|h_{ij}| \leq 1$ for $i, j = 1, 2, \ldots, n$,*

$$Tr((HH^*)^{-1}) \geq 1,$$

and equality holds if and only if $HH^* = nI$ with $|h_{ij}| = 1$ for $i, j = 1, 2, \ldots, n$.

Proof: From the well-known theorems of matrix theory, since HH^* is nonnegative definite, there exists a unitary matrix U such that

$$HH^* = U\text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2)U^* \tag{1}$$

and then

$$(HH^*)^{-1} = U\text{diag}(1/\sigma_1^2, 1/\sigma_2^2, \ldots, 1/\sigma_n^2)U^* \tag{2}$$

where σ_i^2's and $1/\sigma_i^2$'s are the eigenvalues of HH^* and $(HH^*)^{-1}$ respectively. As the trace of HH^* and $(HH^*)^{-1}$ are equal to the sum of eigenvalues of HH^* and $(HH^*)^{-1}$ respectively, we have

$$\text{Tr}(HH^*) = \sum_{i=1}^{n} \sigma_i^2, \qquad \text{Tr}((HH^*)^{-1}) = \sum_{i=1}^{n} 1/\sigma_i^2 \tag{3}$$

and then

$$\sum_{i=1}^{n}\sigma_i^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}|h_{ij}|^2 \le n^2 \tag{4}$$

where equality holds if and only if $|h_{ij}| = 1$ for $i, j = 1, 2, \ldots, n$.

Since a geometric mean is never greater than an arithmetic mean,

$$\sum_{i=1}^{n} 1/\sigma_i^2 \ge n\left(\prod_{i=1}^{n} 1/\sigma_i^2\right)^{1/n}, \qquad \sum_{i=1}^{n}\sigma_i^2 \ge n\left(\prod_{i=1}^{n}\sigma_i^2\right)^{1/n} \tag{5}$$

and equality holds if and only if $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_n^2$.

Thus, by (1), (2), (3), (4) and (5) we have

$$\begin{aligned}\mathrm{Tr}((HH^*)^{-1}) &= \sum_{i=1}^{n} 1/\sigma_i^2 \ge n\left(\prod_{i=1}^{n} 1/\sigma_i^2\right)^{1/n} = n\left(\prod_{i=1}^{n}\sigma_i^2\right)^{-(1/n)} \\ &\ge n^2\left(\sum_{i=1}^{n}\sigma_i^2\right)^{-1} = \frac{n^2}{\sum_{i=1}^{n}\sum_{j=1}^{n}|h_{ij}|^2} \ge 1.\end{aligned}$$

So

$$\mathrm{Tr}((HH^*)^{-1}) \ge 1,$$

and equality holds if and only if $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_n^2$ and $|h_{ij}| = 1$, for $i, j = 1, 2, \ldots, n$.

But this condition implies by (4) that $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_n^2 = n$(i.e. $HH^* = nI$) and $|h_{ij}| = 1$, for $i, j = 1, 2, \ldots, n$. ∎

In particular,

Theorem. *For any nonsingular real matrix $H = H(h_{ij})$ of order n with entries $|h_{ij}| \le 1$ for $i, j = 1, 2, \ldots, n$,*

$$\mathrm{Tr}((HH^*)^{-1}) \ge 1,$$

and equality holds if and only if H is a Hadamard matrix.

Thus Hadamard matrices also provide the A-optimal weighing designs.

References

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