Finite Unitary Geometries and PBIBD's (II)

Wandi Wei
Department of Mathematics
Sichuan University
Chengdu, CHINA

Benfu Yang
Department of Mathematics
Chengdu Teacher's College
Chengdu, CHINA

Abstract. Let q be a prime power, $\mathbf{F}_{q^2}$ the finite field with q^2 elements, $U_n(\mathbf{F}_{q^2})$ finite unitary group of degree n over $\mathbf{F}_{q^2}$, and $UV_n(\mathbf{F}_{q^2})$ the n-dimensional unitary geometry over $\mathbf{F}_{q^2}$. It is proven that the subgroup consisting of the elements of $U_n(\mathbf{F}_{q^2})$ which fix a given (m,s)-type subspace of $UV_n(\mathbf{F}_{q^2})$, acts transitively on some subsets of subspaces of $UV_n(\mathbf{F}_{q^2})$. This observation gives rise to a number of PBIBD's.

1. Introduction

Let q be a prime power, and $\mathbf{F}_q$ be the finite field with q elements. $\mathbf{F}_{q^2}$, the finite field with q^2 elements, has a copy of $\mathbf{F}_q$ as one of its subfields. It is well known that $\mathbf{F}_{q^2}$ has an automorphism of order 2, namely $a \to \bar{a} = a^q$, which fixes this copy of $\mathbf{F}_q$. The set of $n \times n$ matrices $H = (h_{i,j})$ over $\mathbf{F}_{q^2}$ which satisfy $H\bar{H}^T = I$ forms a group, called the *unitary group of degree n* (defined by the identity matrix I) over $\mathbf{F}_{q^2}$, and denoted by $U_n(\mathbf{F}_{q^2})$.

Let $V_n(\mathbf{F}_{q^2})$ be an n-dimensional vector space over $\mathbf{F}_{q^2}$. Then $U_n(\mathbf{F}_{q^2})$ can be viewed as one of its transformation groups. $V_n(\mathbf{F}_{q^2})$ with $U_n(\mathbf{F}_{q^2})$ as its transformation group is called the *n-dimensional unitary space* or the *n-dimensional unitary geometry* over $\mathbf{F}_{q^2}$, and denoted by $UV_n(\mathbf{F}_{q^2})$.

Two vectors A and B of $UV_n(\mathbf{F}_{q^2})$ are said to be *orthogonal*, written $A \perp B$, iff $A\bar{B}^T = 0$. Let $P^* := \{V | U \perp V \text{ for all } U \in P\}$ be called the *conjugation* of P.

Let P be a m-dimensional subspace of $UV_n(\mathbf{F}_{q^2})$. A subspace P is said to be of *type* (m,s) iff the rank of the matrix $P\bar{P}^T$ is s. Clearly the type of a subspace is independent of the choice of the matrix whose rows span the subspace. Let $t := m - s \geq 0$ and $c := n + s - 2m$. We have proven in [6] that the conjugation of a subspace of type (m,s) is a subspace of type $(c+t, c)$, so $c \geq 0$.

Shen [1] constructed a number of PBIBD's from $UV_n\mathbf{F}_{q^2}$ by taking as vertices the subspaces of type $(m+1, 2)$ which contain a fixed subspace S of type $(m, 0)$. Yang and Wei [7] constructed a number of BIBD's and PBIBD's from $V_n(\mathbf{F}_{q_q})$ by taking as vertices the 1-dimensional subspaces which are not contained in a fixed m-dimensional subspace Q. Wei and Yang [4] constructed a number of PBIBD's from the 2ν-dimensional symplectic geometry $SV_{2\nu}(\mathbf{F}_{q_q})$ by taking as vertices the 1-dimensional subspaces which are orthogonal to a fixed subspace R of type (m, s), but not contained in R. In the present paper we show that it is possible to construct a number of PBIBD's from $UV_n(\mathbf{F}_{q^2})$ by taking as vertices the subspaces of type (1,0) which are orthogonal to a fixed subspace P of type (m, s), but not contained in P. The discussion in the orthogonal geometries appears in some other papers.

The concepts and notation used but not defined in this paper are adopted from [3] and [6].

Properties of $UV_n(\mathbf{F}_{q^2})$ and $U_n(\mathbf{F}_{q^2})$

Throughout this paper, P is a fixed subspace of type (m, s), and W is the set of subspaces of type (1,0) which are orthogonal to P but not contained in P. Let $V_1, V_2 \in W$. If $\dim\langle V_1, V_2\rangle \cap P = 1$, then V_1 and V_2 are said to be *first associates*. In this case, write $\langle V_1, V_2\rangle \cap P = \langle V\rangle$. Then $\langle V_1, V_2\rangle = \langle V_1, V\rangle$ and $V_1 \perp V$, and so $\langle V_1, V_2\rangle$ is a subspace of type (2,0). If $\langle V_1, V_2\rangle \cap P = \{0\}$ and $V_1 \perp V_2$ (i.e., $\langle V_1, V_2\rangle$ is again a subspace of type (2,0).), then V_1 and V_2 are said to be *second associates*. If $\langle V_1, V_2\rangle \cap P = \{0\}$ and V_1 is not orthogonal to V_2 (i.e., $\langle V_1, V_2\rangle$ is a subspace of type (2,2)), then V_1 and V_2 are said to be *third associates*. The symbol $(V_1, V_2) \in \Re_i$ will denote the fact that V_1 and V_2 are ith associates, $i = 1, 2, 3$.

In order to construct an association scheme, we need the following properties of the unitary geometry $UV_n(\mathbf{F}_{q^2})$ and the unitary group $U_n(\mathbf{F}_{q^2})$.

Theorem 1. *Let G be the subgroup consisting of the elements of $U_n(\mathbf{F}_{q^2})$ which fix P. Then G acts transitively on W, and G acts transitively on each of the sets $S_i = \{(V_1, V_2) | V_1, V_2 \in W, (V_1, V_2) \in \Re_i\}, i = 1, 2, 3$.*

The proof of this theorem is similar to that of Theorem 1 in [4], and is therefore omitted.

Theorem 2. *In $UV_n(\mathbf{F}_{q^2})$, a subspace P of type (m, s) intersects its orthogonal complement in a subspace of type $(t, 0)$.*

The proof of this theorem is similar to that of Theorem 3 in [5], and is therefore omitted.

Let $f(x) = q^{2x} - 1$, $F(x) = \prod_{i=1}^{x} f(i)$, $h(x) = q^x - (-1)^x$ and $H(x) = \prod_{i=1}^{x} h(i)$. Wan (see [2] or [3]) proved that the number of subspaces of type $(A+B, A)$ in a subspace of type $(C+D, C)$, denoted by $N(A+B, A; C+D, C)$, is given by:

$$N(A+B, A; C+D, C)$$
$$= \sum_{k=\max(0,[\frac{A+2B-C+1}{2}])}^{\min(D,B)} \frac{F(D)H(C)}{F(D-k)F(B-k)F(k)H(C-A-2B+2k)H(A)} \cdot q^{A(C-A-2B+2k)+2(A+B-k)(D-k)}$$

We are now in a position to prove:

Theorem 3. *With the definition of vertices and i th associates above, we obtain a 3-class association scheme with parameters:*

$$v = N(1,0;c+t,c) - N(1,0;t,0) = \frac{h(c)h(c-1)}{q^2-1}q^{2t},$$
$$n_1 = N(1,0;t+1,0) - N(1,0;t,0) - 1 = q^{2t} - 1,$$
$$n_2 = N(1,0;c+t-1,c-2) - N(1,0;t+1,0) = \frac{h(c-2)h(c-3)}{q^2-1}q^{2(t+1)},$$
$$p_{11}^1 = q^{2t} - 2, \quad p_{12}^1 = 0, \quad p_{22}^1 = n_2,$$
$$p_{22}^2 = N(1,0;c+t-2,c-4) - 2N(1,0;t+1,0) + N(1,0;t,0)$$
$$= q^{2t}(q^2-1) + \frac{h(c-4)h(c-5)}{q^2-1}q^{2(t+2)}.$$

(Note that $c > 1$, otherwise there are no vertices. $n_3 = v - n_1 - n_2 = 1 + q^{2c+2t-3} > 0$ always, but $n_2 = 0$ iff $c = 2$ or 3, and $n_1 = 0$ iff $m = s$. So this 3-class scheme may degenerate to a 2-class or 1-class scheme in some cases.)

Proof: By the transitivity in Theorem 1, we certainly obtain a 3^{ℓ}-class association scheme. We now evaluate its parameters.

The parameter v is the number of subspaces of type (1,0) which are contained in P^* but not contained in P. Since P^* is a subspace of type $(c+t,c)$ and $P \cap P^*$ is a subspace of type $(t,0)$, it follows that $v = N(1,0;c+t,c) - N(1,0;t,0)$.

Let V_1 be a given subspace of type (1,0) which is orthogonal to P but not contained in P. Let $V_2 \neq V_1$ be a subspace of type (1,0).

$(V_1,V_2) \in \Re_1$ iff $V_2 \subseteq (\langle P,V_1\rangle \cap \leq \langle P,V_1\rangle^* \backslash P$. Since $V_1 \subseteq P^*$, we have $V_1^* \supseteq P$. Thus $\langle P,V_1\rangle \cap \langle P,V_1\rangle^* \cap P = \langle P,V_1\rangle^* \cap P = P^* \cap V_1^* \cap P = P^* \cap P$. Noting that $\langle P,V_1\rangle$ is a subspace of type $(m+1,s)$ and $\langle P,V_1\rangle \cap \langle P,V_1\rangle^*$ is a subspace of type $(t+1,0)$, we have

$$n_1 = N(1,0;t+1,0) - N(1,0;t,0) - 1 = \frac{q^{2(t+1)-1}}{q^2-1} - \frac{q^{2t}-1}{q^2-1} - 1 = q^{2t} - 1.$$

$(V_1,V_2) \in \Re_2$ iff $V_2 \subseteq \langle P,V_1\rangle^* \backslash \langle P,V_1\rangle$. Since $\langle P,V_1\rangle^*$ is a $(c+t-1,c-2)$-type subspace and $\langle P,V_1\rangle^* \cap \langle P,V_1\rangle$ is a subspace of type $(t+1,0)$, we have $n_2 = N(1,0;c+t-1,c-2) - N(1,0;t+1,0)$.

Let $(V_1,V_2) \in \Re_1$. Then V_2 is a subspace of the subspace $\langle P,V_1\rangle$ of type $(m+1,s)$. Let $(V,V_i) \in \Re_1$, $i = 1,2$. Then $V \subseteq \langle P,V_1\rangle \cap \langle P,V_1\rangle^* \backslash P$. So $p_{11}^1 = q^{2t} - 2$. Since $p_{11}^1 = n_1 - 1$, this forces $p_{12}^1 = p_{13}^1 = 0$. Let $(V,V_i) \in \Re_2$. Then $V \subseteq \langle P,V_1\rangle^* \backslash \langle P,V_1\rangle$. Hence $P_{22}^1 = n_2$.

Now let $(V_1,V_2) \in \Re_2$, so $c > 3$. Let $(V,V_i) \in \Re_2$, $i = 1,2$. Then $V \subseteq \langle P,V_1,V_2\rangle^*$, $V \not\subseteq \langle P,V_i\rangle$, $i = 1,2$. Since $\langle P,V_1,V_2\rangle \subseteq \langle V_1\rangle^* \cap \langle V_2\rangle^*$, $P^* \cap \langle P,V_i\rangle = \langle P,V_i\rangle^* \cap \langle P,V_i\rangle$, $i = 1,2$, it follows that $\langle P,V_1,V_2\rangle^* \cap \langle P,V_i\rangle =$

$P^* \cap V_1^* \cap V_2^* \cap \langle P, V_i\rangle = P^* \cap \langle P, V_i\rangle = \langle P, V_i\rangle^* \cap \langle P, V_i\rangle$, $i = 1,2$. And since $\langle P, V_1\rangle \cap \langle P, V_2\rangle = P$, we have $[\langle P, V_1, V_2\rangle^* \cap \langle P, V_1\rangle] \cap [\langle P, V_1, V_2\rangle^* \cap \langle P, V_2\rangle] = [P^* \cap \langle P, V_1\rangle] \cap [P^* \cap \langle P, V_1\rangle] = P^* \cap P$. Since $P^* \cap P$ is a subspace of type $(t, 0)$, both $\langle P, V_i\rangle^* \cap \langle P, V_i\rangle$, $i = 1,2$, are subspaces of type $(t+1, 0)$, and the conjugation of the subspace $\langle P, V_1, V_2\rangle$ of type $(m+2, s)$ is a subspace of type $(c+t-2, c-4)$, we have

$$p_{22}^2 = N(1,0; c+t-2, c-4) - 2N(1,0; t+1, 0) + N(1,0; t, 0)$$

by the inclusion-exclusion principle. ∎

3. PBIBD's

In this section we will use the association schemes obtained in section 2 to construct a number of PBIBD's.

Theorem 4. *Consider the association scheme of Theorem 3. Let $c' \le c$. Take as blocks the subspaces of type (c', c') which are orthogonal to P, and define a vertex to be incident with a block if it is a subspace of that block. This gives a PBIBD with 3 classes and design parameters:*

$$b = N(c', c'; c+t, c) = \frac{H(c)}{H(c')H(c-c')} q^{c'(c-c'+2t)},$$

$$k = N(1,0; c', c') = \frac{h(c')h(c'-1)}{q^2-1},$$

$$r = \frac{bk}{v} = \frac{H(c-2)}{H(c'-2)H(c-c')} q^{c'(c-c'+2t)-2t},$$

$$\lambda_1 = 0,$$

$$\lambda_2 = \frac{b \cdot N(2,0; c', c') q^2 (q^2+1)}{v \cdot n_2} = \frac{H(c-4)}{H(c'-2)H(c-c')} q^{c'(c-c'+2t)-4t},$$

$$\lambda_3 = \frac{b \cdot N(2,2; c', c')}{N(2,2; c+t, c)} = \frac{H(c-2)}{H(c'-2)H(c-c')} q^{(c'-2)(c-c'+2t)}.$$

Proof: Let Q be a subspace of type (c', c') orthogonal to P. By theorem 2, $Q^* \cap Q = \{0\}$. Since $P \perp Q$ (that is, $P \subseteq Q^*$), it follows that $P \cap Q = \{0\}$, and $\binom{P}{Q}$ is an $(m + c') \times n$ matrix with rank $m + c'$. Without loss of generality, we may assume that

$$P\bar{P}^T = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}, \quad Q\bar{Q}^T = I_{c'}.$$

Noting that $P \perp Q$, we have

$$\begin{pmatrix} P \\ Q \end{pmatrix} \begin{pmatrix} \bar{P} \\ \bar{Q} \end{pmatrix}^T = \begin{bmatrix} I_s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{c'} \end{bmatrix}.$$

Similarly for $\bar{Q}$. Therefore there exists $T \in U_n(\mathbf{F}_{q^2})$ such that $\binom{P}{Q}T = \binom{P}{\bar{Q}}$, (see Theorem 4 in Chapter 3 of [2]), which implies that $PT = P$ (so $T \in G$) and $QT = \bar{Q}$. This implies that G is transitive on the blocks of this design. From this transitivity and the transitivity from Theorem 1, we certainly obtain a PBIBD.

The parameter b is the number of subspaces of type (c', c') which are contained in P^*, so $b = N(c', c'; c+t, c) \cdot k = N(1,0; c', c')$, trivially. $r = \frac{bk}{v} \cdot \lambda_1 = 0$.

To get λ_2, count triples (Q, V_1, V_2) with Q a block, $V_1, V_2 \in W$, $V_1, V_2 \subseteq Q$, $(V_1, V_2) \in \Re_2$ in two ways, we have $b \cdot N(2,0; c', c') \cdot (q^2+1)q^2 = v \cdot n_2 \cdot \lambda_2$, since the number of ordered pairs of subspaces of type (1,0) which are contained in a given subspace of type (2,0) is $(q^2+1)q^2$.

The parameter λ_3 is the number of blocks that contain a given subspace of type (2,2) which is orthogonal to P. Similarly count pairs (Q, V) with Q a block, V a subspace of type (2,2) orthogonal to P, and $V \subseteq Q$, to get $\lambda_3 \cdot N(2,2; c+t, c) = b \cdot N(2,2; c', c')$. ∎

(Note that there are special cases of this when the scheme is degenerate.)

Theorem 5. *Again consider the association scheme of Theorem 3. Let W be the set of blocks and let a vertex be incident with a block iff they are orthogonal as subspaces. Then this produces a PBIBD with parameters:*

$$b = v = \frac{h(c)h(c-1)}{q^2-1}q^{2t},$$

$$r = k = \lambda_1 = N(1,0; c+t-1, c-2) = q^{2t} + \frac{h(c-2)h(c-3)}{q^2-1}q^{2(t+1)},$$

$$\lambda_2 = N(1,0; c+t-2, c-4) = (q^2+1)q^{2t} + \frac{h(c-4)h(c-5)}{q^2-1}q^{2(t+2)},$$

$$\lambda_3 = N(1,0; c+t-2, c-2) = \frac{h(c-2)h(c-3)}{q^2-1}q^{2t}.$$

Proof: By the transitivity of G in Theorem 1, we certainly obtain a PBIBD. Clearly $b = v$ (and hence $r = k$). Let V_1 be a vertex and V a block. Then V_1 is incident with V iff $V \subseteq \langle P, V_1\rangle^* \backslash P$. Noting that $\langle P, V_1\rangle^* \cap P = P^* \cap P$, we have $r = N(1,0; c+t-1, c-2) - N(1,0; t, 0)$.

Let V_1 and V_2 be vertices with $(V_1, V_2) \in \Re_2$ and V_1, V_2 incident with a block V. Then $V \subseteq \langle P, V_1, V_2\rangle^* \backslash P$. Since $\langle P, V_1, V_2\rangle$ is a subspace of type $(m+2, s)$, that $\langle P, V_1, V_2\rangle^*$ is a subspace of type $(c+t-2, c-4)$, and that $\langle P, V_1, V_2\rangle^* \cap P = P^* \cap P$, we have $\lambda_2 = N(1,0; c+t-2, c-4) - N(1,0; t, 0)$.

Let V_1 and V_2 be vertices with $(V_1, V_2) \in \Re_3$. Then λ_3 is the number of subspaces of type (1,0) which are contained in $\langle P, V_1, V_2\rangle^* \backslash P$. Since $\langle P, V_1, V_2\rangle$ is a subspace of type $(m+2, s+2)$, $\langle P, V_1, V_2\rangle^*$ is a subspace of type $(c+t-2, c-2)$, and $\langle P, V_1, V_2\rangle^* \cap P = P^* \cap P$, we have $\lambda_3 = N(1,0; c+t-2, c-2) - N(1,0; t, 0)$. ∎

(Again we have the special cases when the scheme is degenerate.)

Theorem 6. *Consider the degenerate 2-class association scheme with* $t = 0$, $c > 3$. *Let* $c' := n + s' - 2m' \geq 0$ *and* $s' \geq m$. *Take as blocks the subspaces of type* (m', s') *that contain* P, *and define a vertex to be incident with a block iff the vertex is a subspace of the block. This gives a PBIBD with parameters:*

$$\begin{aligned} b &= N^T(m, m; m', s'), \\ r &= N^T(m+1, m; m', s'), \\ \lambda_1 &= N^T(m+2, m; m', s'), \\ \lambda_2 &= N^T(m+2, m+2; m', s'). \end{aligned}$$

Where $N^T(A, B; C, D; n)$ *is the number of subspaces of type* (C, D) *in* $UV_n(\mathbf{F}_{q^2})$ *that contain a given subspace of type* (A, B) *(see [6]).*

Proof: Let Q be a block. Then Q is a subspace of type (m', s') and can be written as $Q = \binom{P}{Y}$ for some $(m' - m) \times n$ matrix Y. Without loss of generality we may assume that

$$Q\bar{Q}^T = \begin{pmatrix} I_m & * \\ * & * \end{pmatrix}.$$

By suitably choosing Y we may further suppose that

$$Q\bar{Q}^T = \begin{pmatrix} I_m & 0 \\ 0 & Y\bar{Y}^T \end{pmatrix}.$$

Since Q is a subspace of type (m', s'), the rank of $Y\bar{Y}^T$ is $s' - m$. Therefore we may further assume that

$$Q\bar{Q}^T = \begin{pmatrix} I_s' & 0 \\ 0 & 0 \end{pmatrix}.$$

From this it can be seen that G acts transitively on the set blocks. And by the transitivity of G from Theorem 1, this certainly gives a PBIBD. The parameters are easy to compute. ∎

Theorem 7. *Again consider the degenerate 2-class scheme for* $t = 0$, $c > 3$. *Let* $c' := n + s' - 2m' \geq 0$ *and* $m' \leq c + t$. *Take as blocks the subspaces of type* (m', s') *which are orthogonal to* P *and define a vertex to be incident with a block iff it is a subspace of the block. This gives a PBIBD with parameters:*

$$\begin{aligned} b &= N(m', s'; c+t, c+t), \\ k &= N(1, 0; m's,'), \\ \lambda_1 &= \frac{b \cdot N(2, 0; m', s')}{N(2, 0; c+t, c+t)}, \\ \lambda_2 &= \frac{b \cdot N(2, 2; m', s')}{N(2, 2; c+t, c+t)}. \end{aligned}$$

Proof: As above G is transitive on the set of blocks, and Theorem 1 assures that this is a PBIBD.

The parameter b is the number of subspaces of type (m', s') which are contained in P^*. Since P^* is a subspace of type $(c+t, c+t)$, we have $b = N(m', s'; c+t, c+t)$.

Since the subspaces of type (1,0) which are in a block Q are orthogonal to P but not contained in P, we have $k = N(1,0; m', s')$.

Let V_1 and V_2 be two vertices with $(V_1, V_2) \in \Re_1$. Then $\langle V_1, V_2\rangle$ is a subspace of type (2,0) contained in P^*. The parameter λ_1 is the number of blocks which include such a subspace of type (2,0). Counting pairs (Q, V) of blocks and subspaces of type (2,0) with $V \subseteq P^*$ and $V \subseteq Q$, we have $b \cdot N(2,0; m', s') = N(2,0; c+t, c+t) \cdot \lambda_1$. Similarly for λ_2. ∎

Theorem 8. *Let $n = 2\nu + \delta, \delta = 0$ or 1. Consider the degenerate 2-class scheme with $n_2 = 0$ (so $c = 2$ or $3, t > 0$) and $s = 0$. Take as blocks the subspaces of type $(m, 0)$ which are orthogonal to P but not contained in P, and define a vertex to be incident with a block iff it is a subspace of the block. This gives a PBIBD with parameters:*

$$
\begin{aligned}
b &= N(m,0; \nu+1+\delta, 2+\delta) - N(m,0; \nu-1, 0) \\
&= (q^{1+2\delta}+1)q^{2(\nu-m)} \prod_{i=1}^{m-1} \left(\frac{q^{2(\nu-m+i)-1}}{q^{2i}-1}\right), \\
k &= N(1,0; m,0) - N(1,0; m-1, 0) = q^{2(m-1)}, \\
r &= \frac{(q^{1+2\delta}+1)}{(q+1)} \cdot \prod_{i=1}^{m-1} \left(\frac{q^{2(\nu-m+i)-1}}{q^{2i}-1}\right), \\
\lambda_1 &= \frac{(q^{1+2\delta}+1)}{(q+1)} \cdot \prod_{i=1}^{m-2} \left(\frac{q^{2(\nu-m+i)-1}}{q^{2i}-1}\right), \\
\lambda_2 &= 0.
\end{aligned}
$$

Proof: Since $n - 2m = 2$ or 3, we have $m = \nu - 1$. Let Q be a block, so Q is a subspace of type $(m, 0)$. Since there are no subspaces of type $(u, 0)$ in $UV_n(\mathbf{F}_{q^2})$ when $u > \nu$ in this degenerate case, it follows that $\langle P, Q\rangle$ is a subspace of type $(\nu, 0)$ and $P \cap Q$ is a subspace of type $(m-1, 0)$. Thus G is transitive on blocks, so we obtain a PBIBD as before.

The parameter b is the number of subspaces of type $(m, 0)$ which are contained in P^* but not in P. Since P^* is a subspace of type $(\nu+1+\delta, 2+\delta)$, and $P \subseteq P^*$, we have $b = N(m,0; \nu+1+\delta, 2+\delta) - N(m,0; \nu-1, 0)$.

Obviously $k = N(1,0; m,0) - N(1,0; m-1, 0)$ and $\lambda_2 = 0$. But then $r = \frac{bk}{v}$ and $\lambda_1 = \frac{r(k-1)-\lambda_2 n_2}{n_1}$. ∎

Acknowledgement

The authors would like to thank an anonymous referee for some very helpful suggestions.

References

1. Hao Shen, *Using Finite Geometries to Construct Block Designs*, J. Shanghai Jiao Tong University **21** (1987), 27–37.
2. Zhexian Wan, *Anzahl Theorems in Finite Singular Symplectic, Unitary and Orthogonal Geometries*. (submitted).
3. Zhexian Wan, Zhogduo Dai, Xuning Feng and Benfu Yang, in "Some Studies on Finite Geometries and Incomplete Block Designs", Science Press, Beijing, 1966.
4. Wandi Wei and Benfu Yang, *Finite Geometries and PBIB Designs (II)*, J. Math. Research and Exposition, No. 1 **11** (1991), 127–132.
5. Benfu Yang and Wandi Wei, *Finite Geometries and PBIB Designs (I)*, J. Math. Research and Exposition, No. 3 **10** (1990).
6. Benfu Yang and Wandi Wei, *Finite Unitary Geometries and PBIB Designs (I)*, JCMCC **6** (1989), 51–61.
7. Benfu Yang and Wandi Wei, *Finite Geometries and Block Designs*, in "Comb. Designs and Appl.", ed. by W.D. Wallis et. al., Marcel Dekker, INC., New York, 1990, pp. 183–200.