Balanced Arrays and Minkowski's Inequality

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Abstract. In this paper we obtain some inequalities on the existence of balanced arrays (B-arrays) of strength four in terms of its parameter by using Minkowski's inequality.

1. Introduction and Preliminaries

Balanced arrays (B-array) tend to unify various combinatorial areas of design of experiments, and have been found to be quite useful in the construction of symmetrical as well as asymmetrical fractional factorial designs. In this paper we concern ourselves with B-arrays of strength four and with two levels (symbols). For ease of reference, we recall the definition of a B-array.

Definition 1.1. A B-array T of strength 4 with two levels (symbols), m constraints (rows), N runs (columns, treatment-combinations), and index set $\bar{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$ is a matrix $T(m \times N)$ whose entries are the two symbols (say, 0 and 1) such that in every 4-rowed sub-matrix T_0 of T, every (4×1) vector $\bar{\alpha}$ of weight i (the number of 1's in $\bar{\alpha}$, $i = 0, 1, 2, 3, 4$) appears as a column of T_0 exactly μ_i times. The B-array is sometimes denoted by $(m, N, t = 4, s = 2; \mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$ If $\mu_i = \mu$ for each i, then T is an orthogonal array (0-array). It is quite evident that

$$N = \sum_{i=0}^{4} \binom{4}{i} \mu_i$$

For a given $\bar{\mu}'$, the problem of obtaining the maximum value of m is very important in combinatorial mathematics and statistical design of experiments. Such a problem for B-arrays and 0-arrays has been studied, among others, by Rao (1947), Bose and Bush (1952), Seiden and Zemach (1966), Rafter and Seiden (1974) etc. etc. To gain further insight into the importance of B-arrays to combinatorics and statistics, the interested reader may consult the list of references given at the end.

2. Minkowski's Inequality and Balanced Arrays

First of all we state Minkowski's Inequality for later use.

Minkowski's Inequality. For $x_i, y_i \geq 0$, and $p > 1$ we have

$$\left[\sum_{i=1}^{n} (x_i + y_i)^p\right]^{\frac{1}{p}} \leq \left(\sum x_i^p\right)^{\frac{1}{p}} + \left(\sum y_i^p\right)^{\frac{1}{p}}$$

The equality holds when either $p = 1$ or the sets $\{x\}$ and $\{y\}$ are proportional.

Next we state some results which are either quite obvious or easy to establish.

Lemma 2.1. *A B-array T of size $(4 \times N)$ and of strength four always exists.*

Lemma 2.2. *A B-array T of strength 4 with index set $\bar{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$ is also of strength 3,2 and 1 with index sets $\{\mu_i + \mu_{i+1}; i = 0,1,2,3\}$, $\{\mu_i + 2\mu_{i+1} + \mu_{i+2}; i = 0,1,2,\}$, and $\{\mu_i + 3\mu_{i+1} + 3\mu_{i+2} + \mu_{i+3}; i = 0,1\}$ respectively.*

Lemma 2.3. *Let $x_j (0 \leq j \leq m)$ denote the number of $(m \times 1)$ vectors of weight j in an array T of strength 4 with index set $\bar{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$. Then the following results hold:*

$$\sum x_j = N = \mu_0 + 4\mu_1 + 6\mu_2 + 4\mu_3 + \mu_4 = A_0 \quad \textit{(say)} \tag{2.1}$$

$$\sum j x_j = m = (\mu_1 + 3\mu_2 + 3\mu_3 + \mu_4) = A_1 \quad \textit{(say)} \tag{2.2}$$

$$\begin{aligned} \sum j^2 x_j &= m(m-1)(\mu_2 + 2\mu_3 + \mu_4) + m(\mu_1 + 3\mu_2 + 3\mu_3 + \mu_4) \\ &= A_2 \quad \textit{(say)} \end{aligned} \tag{2.3}$$

$$\begin{aligned} \sum j^3 x_j &= m(m-1)(m-2)(\mu_3 + \mu_4) + 3m(m-1)(\mu_2 + 2\mu_3 + \mu_4) \\ &\quad + m(\mu_1 + 3\mu_2 + 3\mu_3 + \mu_4) = A_3 \quad \textit{(say)} \end{aligned} \tag{2.4}$$

$$\begin{aligned} \sum j^4 x_j &= m(m-1)(m-2)(m-3)\mu_4 + 6m(m-1)(m-2)(\mu_3 + \mu_4) \\ &\quad + 7m(m-1)(\mu_2 + 2\mu_3 + \mu_4) + m(\mu_1 + 3\mu_2 + 3\mu_3 + \mu_4) \\ &= A_4 \quad \textit{(say)} \end{aligned} \tag{2.5}$$

Theorem 2.1. *Let T be a B-array of size $(m \times N)$ and index set $\bar{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$. Then we have*

$$A_1 + A_2 \leq N^{\frac{1}{3}} (A_3)^{\frac{1}{3}} \left[N^{\frac{1}{3}} + A_3^{\frac{1}{3}} \right] \tag{2.6}$$

where A_1, A_2, and A_3 are, as defined in the above lemma, polynomials in m and μ_i's.

Proof: In Minkowski's inequality, we choose $p = 3$, $x_i = (x_j)^{\frac{1}{3}}$ and $y_j = j(x_j)^{\frac{1}{3}}$. After raising both sides of the inequality to the power 3, and simplifying we obtain the above result.

Theorem 2.2. *Consider a B-array T of strength four with $m \geq 5$ and index set $\bar{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$. Then we have*

$$2A_1 + 3A_2 + 2A_3 \leq N^{\frac{1}{4}} A_4^{\frac{1}{4}} \left[2\sqrt{N} + 3N^{\frac{1}{4}} A_4^{\frac{1}{4}} + 2\sqrt{A_4} \right] \tag{2.7}$$

where A_i's are polynomials in the parameters of the array T and are given in lemma 2.3.

Proof: Here, to obtain the result, we select $p = 4$, $x_j = x_j^{\frac{1}{4}}$, $y_j = j x_j^{\frac{1}{4}}$ and the desired result is easily obtained after some simplification.

Remark: The above results (2.6) and (2.7) are quite useful in discussing the existence of B-arrays for a given m and $\bar{\mu}'$, and also in obtaining an upper bound on m for a B-array with a given $\bar{\mu}'$. It needs to be kept in mind that the above conditions are necessary. A computer program can be easily written to check conditions (2.6) and (2.7) for a given $\bar{\mu}'$ and $m \geq 5$. If for a given $\bar{\mu}'$ the above is contradicted first time for $m = m*(\geq 5)$ then an upper bound for m is $(m*-1)$ i.e. $m \leq m*-1$.

Next, we illustrate the application of (say) (2.7).

Example: Consider a B-array T with $\bar{\mu}' = (3, 3, 1, 1, 3)$. Here clearly $N = 28$. It can be easily checked that such an array exists for $m = 7$, and can be obtained by writing all the distinct (7×1) vectors of weight 2 and weight 6. Simple calculations will reveal that an upper bound on m cannot be obtained by using the results given in Chopra (1982). For this B-array, the polynomial inequality as given in Chopra and Dioś (1989) is $-9m^2 + 266m - 49 \geq 0$. From this we can easily obtain $m \leq 29$. If we use (2.7), we observe that it is contradicted for $m = 10$. Therefore we can say safely that $m \leq 9$ for the above B-array.

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