On Graphs with Prescribed Center and Periphery

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Abstract. A connected graph G is unicentered if G has exactly one central vertex. It is proved that for integers r and d with $1 \leq r < d \leq 2r$, there exists a unicentered graph G such that rad $G = r$ and diam $G = d$. Also, it is shown that for any two graphs F and G with rad $F = n \geq 4$ and a positive integer d $(4 \leq d \leq n)$, there exists a connected graph H with diam $H = d$ such that the periphery and the center of H are isomorphic to F and G respectively.

Let G be a connected graph. The *distance* $d(u, v)$ between two vertices u and v of G is the length of a shortest u–v path in G. Backley and Harary [2] have written a book devoted to the topic of distance in graphs. Terms not defined here may be found in this book. The *eccentricity* of a vertex v in G is defined by $e(v) = \max\{d(v, w) \mid w \in V(G)\}$. The *radius* rad G of G is the minimum eccentricity, while the *diameter* diam G of G is the maximum eccentricity. A vertex of minimum eccentricity is called a *central vertex*. The *center* $C(G)$ of G is the subgraph induced by the central vertices. It is well known that the radius and the diameter of a connected graph G are related by the inequalities rad $G \leq$ diam $G \leq 2$ rad G. Ostrand [5] proved the sharpness of these inequalities that is, for integers r and d with $1 \leq r \leq d \leq 2r$, there exists a connected graph G such that rad $G = r$ and diam $G = d$. In order to strengthen Ostrand's result, we first introduce an additional term.

A connected graph G is *unicentered* if G has exactly one central vertex.

Theorem 1. *If r and d are integers with $1 \leq r < d \leq 2r$, then there exists a unicentered graph G such that* rad $G = r$ *and* diam $G = d$.

Proof: We consider two cases.

<u>Case 1.</u> *Assume that* $d = 2r$.

Let $G = P_{2r+1}: v_0, v_1, \ldots, v_{2r}$. Then $e(v_i) = \max\{i, 2r - i\}$. Therefore rad $G = e(v_r) = r$, diam $G = e(v_0) = e(v_{2r}) = d$, and $C(G) = \{v_r\}$.

<u>Case 2.</u> *Assume that* $d < 2r$.

Let $n = 2d - 2r + 2$. Since $d > r$, it follows that $n \geq 4$. Let H be a graph consisting of n copies of P_r. Denote H by $Q_1 \cup Q_2 \cup \cdots \cup Q_n$, where $Q_i: v_{i1}, v_{i2}, \ldots, v_{ir}$ $(1 \leq i \leq n)$ is a path of order r. We construct the graph G by adding a new vertex v to H and the edges $v_{ir}v_{i+1,r}$, $1 \leq i \leq n-1$, together with the edges joining v with all vertices v_{i1}, $1 \leq i \leq n$ (see Figure 1). Then $e(v) = r$. Consider two vertices v_{ij} and v_{km}. Without loss of generality, we assume that $i < k$. Then the vertices v_{ij} and v_{km} lie on the cycle $C\colon v, v_{i1}, \ldots, v_{ir}, v_{i+1\,r}, \ldots, v_{kr}, v_{kr-1}, \ldots, v_{k1}, v$. Therefore,

$$d(v_{ij}, v_{km}) \le \left\lfloor \frac{|V(C)|}{2} \right\rfloor = \left\lfloor \frac{2r+k-i}{2} \right\rfloor$$
$$\le \left\lfloor \frac{2r+n-1}{2} \right\rfloor = \left\lfloor \frac{2r+2d-2r+2-1}{2} \right\rfloor = d.$$

On the other hand, since $d(v_{ij}, v_{t,r-j+1}) = r+1$ where $t = i+2 \pmod{n}$ it follows that $r+1 \le e(v_{ij}) \le d$ for all $1 \le i \le n$ and $1 \le j \le r$. Finally, since $d(v_{1t}, v_{ns}) = d$, where $t = \lfloor \frac{d}{2} \rfloor$ and $s = \lceil \frac{d}{2} \rceil$, it follows that $\operatorname{rad} G = r$, $\operatorname{diam} G = d$ and $C(G) = \{v\}$. ∎

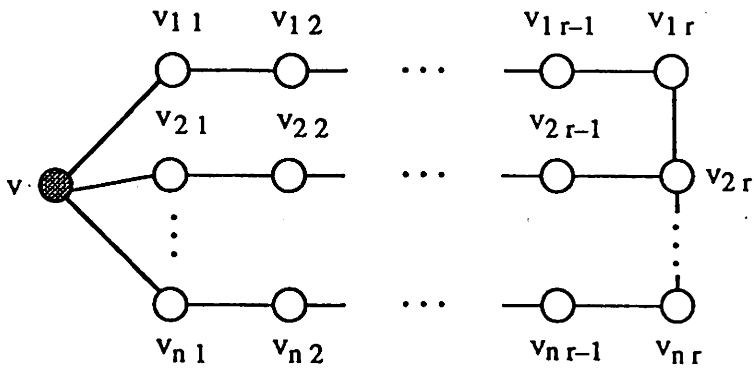

Figure 1

Ostrand [5] also proved that if r and d are integers with $2 \le r < d \le 2r - 1$, then the minimum order of a connected graph of radius r and diameter d is $r + d$. It is interesting to know the minimum order of unicentered graphs of radius r and diameter d. If $d = 2r$, then since a path of length d is a unicentered graph of radius r and diameter d, the minimum order in this special case is $d + 1$. However, a general answer to this problem is unknown.

We are now prepared to present a result concerning the embedding problem. Kopylov and Timofeev mentioned in [6] and Hedetniemi (see [3]) proved that for every graph G, there exists a connected graph G such that $C(H) \cong G$. The graph H so constructed by Hedetniemi has radius 2 and diameter 4. Consequently, in his construction, the subgraph of H induced by those vertices with eccentricity 2 is isomorphic to G. An extension of Hedetniemi's result follows immediately from Theorem 1.

Corollary 2. *For every graph G and for integers r and d with $2 \le r < d \le 2r$, there exists a connected graph H with* $\operatorname{rad} H = r$, $\operatorname{diam} H = d$ *and* $C(H) \cong G$.

Proof: By Theorem 1, there exists a unicentered graph H_0 such that rad $H_0 = r$ and diam $H_0 = d$. We construct the graph H by replacing the central vertex v of H_0 by G and joining each vertex of G with all vertices adjacent to v in H_0. Clearly,, rad $H = r$, diam $H = d$ and $C(H) \cong G$. ∎

The center is an interpretation of the "middle" of a connected graph. A common definition of the "exterior" of a connected graph G is the periphery of G. Formally, the *periphery* $P(G)$ of a connected graph G is the subgraph of G induced by those vertices with maximum eccentricity. Bielak and Syslo [1] proved that a graph G is isomorphic to the periphery of some connected graph if and only if either G is a complete graph of $\Delta(G) \leq p(G) - 2$, where $\Delta(G)$ represents the maximum degree of the vertices of G. Chartrand, Johns and Oellermann [4] extended this characterization of the periphery of a graph. They showed that a graph G is isomorphic to the periphery of a graph having diameter n if and only if rad $G \geq n$. Our next result extends Bielak and Syslo's result even further.

Theorem 3. *Let F be a connected graph with* rad $F = n \geq 4$. *Then there exists a unicentered graph G with* diam $G = d$ *such that $P(G) \cong F$ for all $4 \leq d \leq n$.*

Proof: Let $k = \lceil \frac{d}{2} \rceil$. We first construct a preliminary graph G_1 by attaching each vertex of F with a copy of P_k (see Figure 2(a)). For $v \in V(F)$, we denote such an attaching path in G_1 by P_v: $v, \ldots, w_v$. Let G_2 be the graph obtained from G_1 by joining all the vertices w_v, $v \in V(F)$, to a new vertex w (see Figure 2(b)). If n is even, then we let $G = G_2$. It is straightforward to show that $d_G(u,v) \leq d$ for all $u, v \in V(F)$. For $u \in V(F)$, let v be an eccentric vertex of u in F. Then, $d_F(u,v) = e(u) \geq d$. Observe that if a shortest u–v path P in G contains the vertex w, then the length of P is $2k = d$. Therefore, $d_G(u,v) = d$ so that $e_G(u) \geq d$ for all $u \in V(F)$. Clearly, the distance between a vertex of F and a vertex not in F is less than d. Therefore, $e_G(u) = d$ for all $u \in V(F)$. For $u, v \in V(G) - V(F)$, it follows that

$$d(u,v) \leq d(u,w) + d(w,v) \leq 2(k-1) = d - 2 < d.$$

Therefore, diam $G = d$ and $P(G) \cong F$. Clearly, $e(w) = k = \lfloor \frac{d}{2} \rfloor$. For $x \in V(P_u) - \{u\}$, where $u \in V(F)$, let v be an eccentric vertex of u in F. Then

$$\begin{aligned} d_G(x,v) &= \min\{d_G(x,u) + d_G(u,v), d_G(x,w) + d_G(w,v)\} \\ &\geq \min\{e(u) + 1, k + 1\} \\ &= k + 1 > k = \frac{d}{2} \end{aligned}$$

so x is not a central vertex of G. Therefore, $C(G) = \{w\}$.

Now suppose that d is odd. We construct the connected graph G from G_2 by adding the edges $w_u w_v$ for all $u, v \in V(F)$ with $d_F(u,v) \geq d - 1$ (see Figure 2(c)). Clearly, $e(w) = k < d$. Let $u, v \in V(F)$. If $d_F(u,v) \geq d - 1$, then

$$d_G(u,v) \le d_G(u,w_u) + d_G(w_u,w_v) + d_G(w_v,v) = (k-1) + 1 + (k-1) = d.$$

If $d_F(u,v) < d-1$, then $d_G(u,v) \le d_F(u,v) < d-1 < d$. Thus $d_G(u,v) \le d$ for all $u, v \in V(F)$. Let $u \in V(F)$ and let v be an eccentric vertex of u in F. Then $d_F(u,v) = e(u) \ge d$. If a shortest u–v path P contains a vertex not in F, then the length of P is at least $2k-1 = d$. Therefore, $d_G(u,v) = d$. Consider vertices $x \in V(F)$ and $y \in V(G) - V(F) - \{w\}$. Suppose, without loss of generality, that $y \in V(P_v) - \{v\}$. If $d_F(x,v) < d-1$, then $d_G(x,y) \le d(x,w_x) + d(w_x,w_v) + d(w_v,y) \le k-1+1+k-2 = 2k-2 < d$. If $d_F(x,v) \ge d-1$, then $d_G(x,y) \le \min\{d(x,v)+d(v,y), d(x,w_x) + d(w_x,w_v) + d(w_v,y)\} \le \min\{d-2+d(v,y), k+1+d(w_v,y)\} = \min\{d-2+d(v,y), k+1+(k-1-d(v,y))\} = \min\{d-2+d(v,y), d+1-d(v,y)\} \le d-1$. Therefore

$$d(x,y) \le d-1 \quad \text{for } x \in V(F) \text{ and } y \in V(G) - V(F) - \{w\}.$$

Combining the above, we see that $e(v) = d$ for all $v \in V(F)$. Let $x, y \in V(G) - V(F)$. It follows that

$$d(x,y) \le d(x,w) + d(w,y) \le k-1+k-1 < 2k-2 = d-1.$$

Therefore, diam $G = d$ and $P(G) \cong F$.

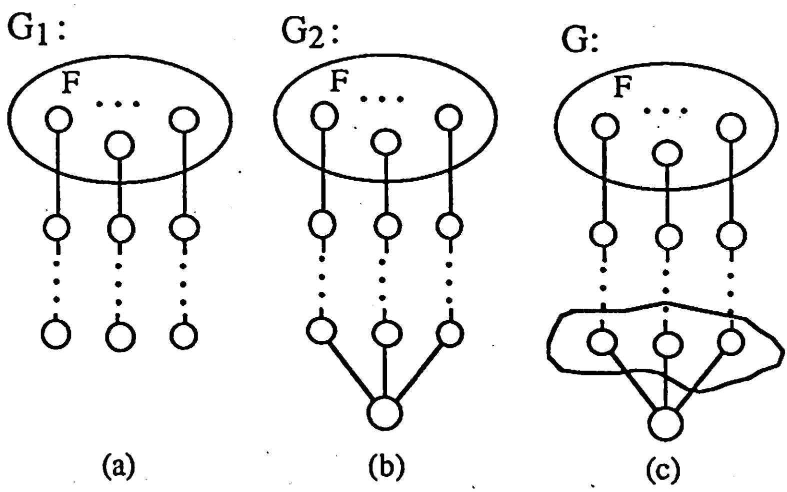

Figure 2

To prove that $C(G) = \{w\}$, it suffices to show that $e(x) > e(w)$ for all $x \in V(G) - V(F) - \{w\}$. Suppose that $x \in V(P_u) - \{w_u\}$ for some $u \in V(F)$. If $x \neq w_u$, then we consider an eccentric vertex, say v, of u in F. So, $d_F(u,v) = e(u) \geq d$. Therefore,

$$\begin{aligned} e(x) &\geq d_G(x,y) \\ &= \min\{d_G(x,u) + d_F(u,v), d_G(x,w_u) + d_G(W_u,w_v) + d_G(w_v,v)\} \\ &\geq \min\{1 + e(u), 1 + 1 + k - 1\} \\ &= k + 1 > k = e(w). \end{aligned}$$

If $x = w_u$, then let v be a vertex such that $d_F(u,v) = d-2$. Then $w_u w_v \notin E(G)$. Therefore,

$$\begin{aligned} e(x) = e(w_u) &\geq d(w_u,v) \\ &= \min\{d(w_u,u) + d_F(u,v), d(w_u,w) + d(w,v)\} \\ &\geq \min\{k - 1 + d - 2, 1 + k\}. \end{aligned}$$

Since $d \geq 4$, it follows that $e(x) \geq k + 1 > e(w)$. This completes the proof. ∎

The following interesting result is an immediate corollary of Theorem 3.

Corollary 4. *Let F be a connected graph with* rad $F = n \geq 4$. *For every graph G and integer d ($4 \leq d \leq n$), there exists a connected graph H with* diam $H = d$, $P(H) \cong F$ *and* $C(H) \cong G$.

Proof: By Theorem 3, there exists a unicentered graph H_0 such that diam $H_0 = d$ and $P(H_0) \cong F$. Let H be the graph obtained by replacing the central vertex v of H_0 by G and joining each vertex of G with all vertices adjacent to v in H_0. Then, diam $H = d$, $P(H) \cong F$ and $C(H) \cong G$. ∎

It appears difficult to determine whether a given connected graph of radius at least 3 is isomorphic to the periphery of some unicentered graph of diameter 3.

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