Symmetric (31, 10, 3)-Designs With a Non-trivial Automorphism of Odd Order

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Abstract. We correct an omission by Mathon in his classification of symmetric (31, 10, 3)-designs with a non-trivial automorphism group and find that there are a further six such designs, all with an automorphism group of order 3.

1. Introduction

In recent years there have been attempts to classify symmetric (v, k, λ)-designs for some values of v, k and λ. In particular Denniston [1] has found all (25, 9, 3)-designs, Spence [4] has investigated (41, 16, 6) designs and between them Tonchev [2] and Mathon [3] appeared to have classified all (31, 10, 3)-designs with a non-trivial automorphism group. However, examination of Mathon's analysis shows up an omission which we shall rectify. We use the same notation as Mathon [3]. Let $D = (\mathcal{P}, \mathcal{B})$ be a symmetric (v, k, λ)-design with point set $\mathcal{P}$ and block set $\mathcal{B}$, $|\mathcal{P}| = |\mathcal{B}| = v$. An automorphism σ of prime order q induces partitions of $\mathcal{P}$ and $\mathcal{B}$ into f orbits of size 1 and w orbits of size q, so that $wq + f = v$. Choose a representative from each orbit of blocks and form an $(f + w) \times (f + w)$ matrix $T_\sigma = [\, T_{ij}\,]$, where T_{ij} is the number of points of the i-th point orbit incident with the j-th block-orbit representative. Mathon writes his incidence matrix A of D with the rows labelled by $\mathcal{P}$ and the columns labelled by $\mathcal{B}$. From the relation

$$AA^t = (k - \lambda) I + \lambda J, \tag{1}$$

he deduces that

$$\sum_{l=1}^{f+w} T_{il} s_l = k s_i, \quad \sum_{l=1}^{f+w} T_{il} T_{jl} s_l = \lambda s_i s_j, \tag{2}$$

where s_i is the size of the i-th point orbit $(1 \leq i, j \leq f + w, i \neq j)$. To determine all (31, 10, 3)-designs with a non-trivial automorphism, Mathon, following Tonchev [2], first observes that $q = 2, 3, 5$ or 7. Since the cases $q = 5, 7$ were investigated by Tonchev [2], he then goes on to consider $q = 2$ and 3 in turn, and determines the *orbit matrices* T_σ that satisfy (2). These are then used to find (31, 10, 3)-designs by a computer search. As we shall see, he omitted to find a solution of (2) in the case $q = 3$. This is done in the next section.

2. Automorphisms of order 3

We quote the following result from [3].

Lemma 1. *Let C be a fixed block of a symmetric $(31,10,3)$-design having an automorphism σ of order 3. Then C is incident with exactly one fixed point.*

An immediate corollary of this is that σ cannot have 10 fixed points, so the only possibilites are $f = 1$ or 4. When $f = 1$ Mathon finds eight orbit matrices T_σ satisfying (2), while in the case $f = 4$ he discovers only three. We examine this latter case more closely.

When $f = 4$ the incidence matrix of the fixed points and fixed blocks is isomorphic to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In order to embed this into the first four rows of an orbit matrix T_σ we require to find a $(0,1)$ matrix B of size 4×9 such that

$$BB^t = 6I + 3J, \quad B\mathbf{j} = 3\mathbf{j}, \tag{3}$$

where $\mathbf{j}$ is the all-one vector of size 4 and I and J have their usual meanings.

For $i = 0, 1, \ldots, 4$ let x_i denote the number of columns of B that contain i 1's. Then for B to satisfy (3) we require

$$\sum x_i = 9, \quad \sum i x_i = 12, \quad \sum i(i-1)x_i = 12.$$

These equations have solution sets

$$(x_0, x_1, x_2, x_3, x_4) \equiv (3,0,6,0,0), \quad (2,3,3,1,0), \quad (1,6,0,2,0).$$

Of these only the first two give solutions of (3) which can be embedded in an orbit matrix. They are unique, up to isomorphism:

$$B_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{4}$$

Since the dual of a (v,k,λ)-design is a (v,k,λ)-design we have proved

Proposition 2. *Let σ be an automorphism of order 3 of a $(31,10,3)$-design that fixes four points. The any orbit matrix T_σ takes the form*

$$\begin{bmatrix} P & B_i \\ 3B_j^t & * \end{bmatrix},$$

where P is a permutation matrix of order 4, and B_i, B_j are given by (4), ($1 \leq i$, $j \leq 2$).

In order to find all the orbit matrices satisfying (2) we first find for each pair (i, j) all the permutaton matrices P that give rise to non-isomorphic matrices of the form

$$\begin{bmatrix} P & B_i \\ 3B_j & 0 \end{bmatrix},$$

and then determine all non-isomorphic orbit matrices which correspond to the triple (P, B_i, B_j).

In the case $i = j = 1$ there is clearly only one permutation $P = I_4$, (since B_1 has S_4 as its automorphism group) and this gives two non-isomorphic matrices T_σ satisfying (2). In the case $i = j = 2$, however, since B_2 has S_3 as its group of automorphisms, there are two possible permutatons, $P_1 = I_4$, which was found by Mathon [3], and

$$P_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

which was omitted by Mathon. Each of these gives, up to isomorphism, a unique matrix T_σ satisfying (2). When $i = 1$ and $j = 2$ there is no solution.

The following are the four matrices T_σ, up to isomorphism, that satisfy (2). We write them in a form so that the submatrices corresponding to the non-fixed blocks and points are symmetric.

$$T_\sigma^{(1)} = \begin{array}{ccccccccccccc}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
3 & 3 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 1 & 1 \\
3 & 0 & 0 & 3 & 1 & 0 & 1 & 2 & 0 & 1 & 1 & 1 & 1 \\
3 & 0 & 3 & 0 & 0 & 1 & 1 & 1 & 2 & 0 & 1 & 1 & 1 \\
0 & 3 & 0 & 3 & 0 & 2 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 3 & 3 & 0 & 1 & 0 & 2 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 3 & 3 & 2 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 3 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 3 & 1
\end{array}$$

$$T_\sigma^{(2)} = \begin{matrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 0 \\ 3 & 0 & 3 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 2 \\ 3 & 0 & 0 & 3 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 2 & 1 \\ 0 & 3 & 3 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 2 & 1 \\ 0 & 3 & 0 & 3 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 3 & 3 & 1 & 1 & 1 & 1 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 2 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 2 & 0 & 1 & 1 & 2 \end{matrix}$$

$$T_\sigma^{(3)} = \begin{matrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 3 & 0 & 0 & 3 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 0 \\ 3 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 1 & 1 & 0 & 2 \\ 0 & 3 & 0 & 3 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 2 \\ 0 & 3 & 0 & 0 & 0 & 1 & 2 & 1 & 2 & 1 & 1 & 1 & 0 \\ 0 & 0 & 3 & 3 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 & 1 & 1 & 1 & 0 & 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 1 & 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 2 & 0 & 1 & 1 & 1 & 2 \end{matrix}$$

$$T_\sigma^{(4)} = \begin{matrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 3 & 0 & 0 & 3 & 1 & 1 & 0 & 1 & 1 & 0 & 2 & 1 & 1 \\ 3 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 3 & 3 & 1 & 1 & 1 & 0 & 2 & 1 & 0 & 1 & 1 \\ 0 & 0 & 3 & 0 & 1 & 1 & 1 & 2 & 0 & 0 & 1 & 1 & 2 \\ 0 & 3 & 0 & 3 & 1 & 0 & 2 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 3 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 2 & 1 & 2 & 0 & 1 \end{matrix}$$

The numbers of non-isomorphic $(31,10,3)$-designs that have these matrices as orbit matrices are $0,6,18,9$ respectively. Of the nine corresponding to $T_\sigma^{(4)}$, (the orbit matrix which Mathon missed) three were already accounted for, since these three (self-dual)-designs also have an automorphism of order 3 that fixes one point. However the remaining six are new. They comprise three pairs of dual designs, each with a full automorphism group of order 3.

Result 3. *There are precisely 44 symmetric* $(31,10,3)$*-designs having a non-trivial automorphism. They comprise eight self-dual designs and eighteen pairs of dual designs. All but two of these (a pair of dual designs) have an automorphism of order* 3.

In [3] Mathon specifically lists the two exceptional designs without an automorphism of order 3. In order to complete the classification we exhibit all the remaining designs in the Appendix, being content to show only one of a dual pair. All the self-dual designs possess a polarity and these polarities are shown by writing the incidence matrices in symmetric form. The following notation is used.

$$I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\quad V=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\quad W=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\quad J=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$$

$$\mathbf{I}=J-I,\quad \mathbf{V}=J-V,\quad \mathbf{W}=J-W,$$

and $\mathbf{j}$ denotes the all-one vector of order 3.

The designs are labelled with two numbers (x,y), the second of which corresponds to the numbering used by Mathon [3]. The first eight are the designs that possess a polarity, while numbers 23, 24 and 25 are the ones that Mathon missed. They possess 22, 1 and 4 ovals respectively and all have automorphism group of order 3.

References

1. R.H.F. Denniston, *Enumeration of symmetric designs (25,9,3)*, Annals of Discrete Math. **15** (1982), 111–127.
2. V.D. Tonchev, *Symmetric* 2-$(31,10,3)$ *designs with an automorphism of order seven*, "Combinatorial Design Theory", North-Holland, Amsterdam-New York, 1987, pp. 461–464.
3. R. Mathon, *Symmetric* $(31,10,3)$*-designs with non-trivial automorphism group*, Ars Combinatoria **25** (1988), 171–183.
4. E. Spence, *Symmetric* $(41,16,6)$*-designs with an automorphism of odd order*, to appear.

Appendix

1	j^t	j^t	j^t	0	0	0	0	0	0	0
j	I	0	0	I	*I*	*I*	*I*	*I*	*I*	0
j	0	0	I	0	*I*	*I*	*V*	*W*	I	*I*
j	0	I	0	*I*	*I*	*I*	*W*	*V*	0	I
0	I	0	*I*	0	W	*W*	*V*	*W*	0	I
0	*I*	*I*	*I*	V	*I*	0	0	0	V	V
0	*I*	*I*	*I*	*V*	0	*I*	0	*J*	*V*	*V*
0	*I*	*W*	*V*	*W*	0	0	*J*	*W*	*I*	*V*
0	*I*	*V*	*W*	*V*	0	*J*	*V*	0	*I*	*W*
0	*I*	I	0	0	W	*W*	*I*	*I*	I	0
0	0	*I*	I	I	W	*W*	*W*	*V*	0	0

(1,1)

1	j^t	j^t	j^t	0	0	0	0	0	0	0
j	I	0	0	I	*I*	*I*	*I*	*I*	*I*	0
j	0	0	I	0	*I*	*I*	*V*	*W*	I	*I*
j	0	I	0	*I*	*I*	*I*	*W*	*V*	0	I
0	I	0	*I*	0	V	*V*	*V*	*W*	0	I
0	*I*	*I*	*I*	W	*I*	0	0	0	W	W
0	*I*	*I*	*I*	*W*	0	*I*	0	*J*	*W*	*W*
0	*I*	*W*	*V*	*W*	0	0	*J*	*W*	*I*	*V*
0	*I*	*V*	*W*	*V*	0	*J*	*V*	0	*I*	*W*
0	*I*	I	0	0	V	*V*	*I*	*I*	I	0
0	0	*I*	I	I	V	*V*	*W*	*V*	0	0

(2,2)

1	j^t	j^t	j^t	0	0	0	0	0	0	0
j	I	0	0	I	*I*	*I*	*I*	*I*	*I*	0
j	0	0	I	0	*I*	*I*	*V*	*W*	I	*I*
j	0	I	0	*I*	*I*	*I*	*W*	*V*	0	I
0	I	0	*I*	0	I	*I*	*V*	*W*	0	I
0	*I*	*I*	*I*	I	*I*	0	0	0	I	I
0	*I*	*I*	*I*	*I*	0	*I*	0	*J*	*I*	*I*
0	*I*	*W*	*V*	*W*	0	0	*J*	*W*	*I*	*V*
0	*I*	*V*	*W*	*V*	0	*J*	*V*	0	*I*	*W*
0	*I*	I	0	0	I	*I*	*I*	*I*	I	0
0	0	*I*	I	I	I	*I*	*W*	*V*	0	0

(3,3)

1	j^t	j^t	j^t	0	0	0	0	0	0	0
j	I	0	0	I	*I*	*I*	*I*	*I*	*I*	0
j	0	0	I	0	*I*	*I*	*V*	*W*	I	*I*
j	0	I	0	*I*	*I*	*V*	*I*	*W*	0	I
0	I	0	*I*	0	I	*I*	*V*	*W*	0	I
0	*I*	*I*	*I*	I	*I*	0	0	0	I	I
0	*I*	*I*	*W*	*I*	0	*I*	0	*J*	*I*	*W*
0	*I*	*W*	*I*	*W*	0	0	*J*	*W*	*I*	*I*
0	*I*	*V*	*V*	*V*	0	*J*	*V*	0	*I*	*V*
0	*I*	I	0	0	I	*I*	*I*	*I*	I	0
0	0	*I*	I	I	I	*V*	*I*	*W*	0	0

(4,4)

1	j^t	j^t	j^t	0	0	0	0	0	0	0
j	I	0	0	I	*I*	*I*	*I*	*I*	*I*	0
j	0	0	I	0	*I*	*I*	*V*	*W*	I	*I*
j	0	I	0	*I*	*I*	*W*	*V*	*I*	0	I
0	I	0	*I*	0	I	*I*	*V*	*W*	0	I
0	*I*	*I*	*I*	I	*I*	0	0	0	I	I
0	*I*	*I*	*V*	*I*	0	*I*	0	*J*	*I*	*V*
0	*I*	*W*	*W*	*W*	0	0	*J*	*W*	*I*	*W*
0	*I*	*V*	*I*	*V*	0	*J*	*V*	0	*I*	*I*
0	*I*	I	0	0	I	*I*	*I*	*I*	I	0
0	0	*I*	I	I	I	*W*	*V*	*I*	0	0

(5,5)

1	j^t	j^t	j^t	0	0	0	0	0	0	0
j	0	*I*	*I*	0	*I*	0	*I*	I	*I*	I
j	*I*	*I*	0	I	I	*I*	*V*	*V*	0	0
j	*I*	0	*I*	*I*	0	I	*V*	0	I	*V*
0	0	I	*I*	I	*I*	0	*V*	0	V	*I*
0	*I*	I	0	*I*	0	W	0	W	*I*	*W*
0	0	*I*	I	0	V	I	*V*	*I*	*I*	0
0	*I*	*W*	*W*	*W*	0	*W*	*J*	*W*	0	*W*
0	I	*W*	0	0	V	*I*	*V*	0	*V*	I
0	*I*	0	I	W	*I*	*I*	0	*W*	0	W
0	I	0	*W*	*I*	*V*	0	*V*	I	V	0

(6,13)

1	j^t	j^t	j^t	0	0	0	0	0	0	0
j	0	*I*	*I*	0	*I*	0	*I*	I	*I*	I
j	*I*	*I*	0	I	I	*I*	*V*	*V*	0	0
j	*I*	0	*I*	*I*	0	I	*V*	0	I	*V*
0	0	I	*I*	I	*W*	0	*I*	0	I	*W*
0	*I*	I	0	*V*	0	I	0	I	*I*	*I*
0	0	*I*	I	0	I	I	*I*	*W*	*W*	0
0	*I*	*W*	*W*	*I*	0	*I*	*J*	*W*	0	*W*
0	I	*W*	0	0	I	*V*	*V*	0	*I*	I
0	*I*	0	I	I	*I*	*V*	0	*I*	0	I
0	I	0	*W*	*V*	*I*	0	*V*	I	I	0

(7,14)

1	j^t	j^t	j^t	0	0	0	0	0	0	0
j	0	*V*	*W*	0	*W*	0	*W*	V	*W*	V
j	*W*	0	*W*	*I*	0	W	*I*	0	W	*W*
j	*V*	*V*	0	V	V	*V*	*V*	*V*	0	0
0	0	*I*	W	*I*	*I*	W	0	V	*W*	0
0	*V*	0	W	*I*	I	*W*	*I*	0	0	V
0	0	V	*W*	V	*V*	0	*V*	0	V	*I*
0	*V*	*I*	*W*	0	*I*	*W*	*J*	*I*	*V*	0
0	W	0	*W*	W	0	0	*I*	I	*W*	*V*
0	*V*	V	0	*V*	0	W	*W*	*V*	0	V
0	W	*V*	0	0	W	*I*	0	*W*	W	*I*

(8,12)

1	j^t	j^t	j^t	0	0	0	0	0	0	0
j	*I*	*I*	0	I	I	*I*	*I*	*I*	0	0
j	*I*	0	*I*	*I*	0	W	*W*	0	I	*I*
j	0	*I*	*I*	0	*I*	0	*I*	I	*V*	I
0	I	*I*	0	0	W	*V*	0	*I*	W	*I*
0	I	0	*I*	I	0	0	*W*	W	*W*	*W*
0	*I*	V	0	*V*	0	I	*V*	*V*	0	V
0	*I*	*V*	*I*	0	*I*	*W*	*J*	*I*	*I*	0
0	*I*	0	I	*W*	W	*V*	*W*	0	0	I
0	0	I	*W*	I	*I*	0	*I*	0	I	*I*
0	0	*I*	I	*I*	*I*	V	0	W	*W*	0

(9,10)

1	j^t	j^t	j^t	0	0	0	0	0	0	0
j	*I*	*I*	0	I	I	*I*	*I*	*I*	0	0
j	*I*	0	*I*	*I*	*V*	*V*	0	0	I	I
j	0	*I*	*W*	0	0	*W*	I	W	*W*	*V*
0	I	*I*	0	*I*	0	V	W	0	*V*	*W*
0	I	0	*I*	*W*	*V*	0	*V*	V	V	0
0	*I*	V	0	0	W	0	*V*	*V*	*I*	V
0	*I*	*V*	*W*	V	0	*W*	0	V	0	V
0	*I*	0	W	0	V	V	*V*	*W*	0	*V*
0	0	I	*V*	*W*	*V*	V	0	*V*	I	0
0	0	*I*	W	V	*I*	0	V	0	*W*	*I*

(10,6)

1	j^t	j^t	j^t	0	0	0	0	0	0	0
j	*I*	*I*	0	I	I	*I*	*I*	*I*	0	0
j	*I*	0	*I*	*I*	*V*	*V*	0	0	I	I
j	0	*I*	*W*	0	0	*W*	I	W	*W*	*V*
0	I	*I*	0	*I*	0	V	W	0	*V*	*W*
0	I	0	*I*	*W*	*W*	0	*W*	V	I	0
0	*I*	V	0	0	V	0	*W*	*V*	*V*	V
0	*I*	*V*	*W*	V	0	*W*	0	V	0	V
0	*I*	0	W	0	V	V	*V*	*W*	0	*V*
0	0	I	*V*	*W*	*V*	V	0	*V*	I	0
0	0	*I*	W	V	*V*	0	I	0	*I*	*I*

(11,8)

1	0	0	0	j^t	j^t	j^t	0	0	0	0	0	0
0	1	0	0	j^t	0	0	j^t	j^t	0	0	0	0
0	0	1	0	0	j^t	0	j^t	0	j^t	0	0	0
0	0	0	1	0	0	j^t	0	j^t	j^t	0	0	0
j	j	0	0	0	*I*	*I*	*I*	*I*	*I*	I	*I*	0
j	0	j	0	*I*	0	*I*	*I*	*V*	*W*	*I*	0	I
j	0	0	j	*I*	*I*	0	*V*	*I*	*W*	0	I	*I*
0	j	j	0	*I*	*I*	*V*	0	*W*	*V*	0	V	*V*
0	j	0	j	*I*	*W*	*W*	*V*	0	*I*	*W*	0	I
0	0	j	j	*I*	*V*	*I*	*W*	*W*	0	I	*V*	0
0	0	0	0	I	*I*	0	0	*W*	V	V	*V*	*V*
0	0	0	0	*I*	0	I	V	0	*I*	*I*	W	*I*
0	0	0	0	0	I	*I*	*V*	V	0	*I*	*I*	W

(12,15)

1	0	0	0	j^t	j^t	j^t	0	0	0	0	0	0
0	1	0	0	j^t	0	0	j^t	j^t	0	0	0	0
0	0	1	0	0	j^t	0	j^t	0	j^t	0	0	0
0	0	0	1	0	0	j^t	0	j^t	j^t	0	0	0
j	j	0	0	0	*I*	*I*	*I*	*I*	*I*	I	*I*	0
j	0	j	0	*I*	0	*I*	*I*	*V*	*W*	*I*	0	I
j	0	0	j	*I*	*I*	0	*W*	*V*	*I*	0	I	*I*
0	j	j	0	*I*	*V*	*V*	0	*I*	*W*	0	V	*I*
0	j	0	j	*I*	*W*	*I*	*V*	0	*V*	*V*	0	W
0	0	j	j	*I*	*I*	*W*	*V*	*W*	0	V	*I*	0
0	0	0	0	I	*I*	0	0	*V*	I	W	*I*	*I*
0	0	0	0	*I*	0	I	V	0	*V*	*V*	W	*W*
0	0	0	0	0	I	*I*	*W*	I	0	*W*	*I*	V

(13,17)

0	0	0	1	j^t	j^t	j^t	0	0	0	0	0	0
0	0	1	0	j^t	0	0	j^t	j^t	0	0	0	0
0	1	0	0	j^t	0	0	0	0	j^t	j^t	0	0
1	0	0	0	0	j^t	0	j^t	0	j^t	0	0	0
j	j	j	0	*I*	*I*	*I*	*I*	0	0	*I*	*I*	*I*
j	0	0	j	*I*	0	*I*	*V*	*I*	*I*	*W*	I	0
j	0	0	0	*I*	*V*	*W*	0	V	*V*	*I*	0	I
0	j	0	j	*I*	*I*	0	*W*	*V*	*W*	0	*V*	I
0	j	0	0	0	*I*	V	*V*	I	*V*	*V*	*W*	0
0	0	j	j	0	*I*	*V*	*I*	*I*	*V*	I	0	*W*
0	0	j	0	*I*	*W*	*V*	0	*W*	I	0	I	*I*
0	0	0	0	*I*	I	0	*I*	*V*	0	I	V	*V*
0	0	0	0	*I*	0	I	I	0	*I*	*V*	*I*	V

(14,19)

0	0	0	1	j^t	j^t	j^t	0	0	0	0	0	0
0	0	1	0	j^t	0	0	j^t	j^t	0	0	0	0
0	1	0	0	j^t	0	0	0	0	j^t	j^t	0	0
1	0	0	0	0	j^t	0	j^t	0	j^t	0	0	0
j	j	j	0	*I*	*I*	*I*	*I*	0	0	*I*	*I*	*I*
j	0	0	j	*I*	0	*I*	*V*	*I*	*I*	*W*	I	0
j	0	0	0	*I*	*W*	*V*	0	V	*V*	*V*	0	V
0	j	0	j	*I*	*V*	0	*W*	*V*	*W*	0	*V*	V
0	j	0	0	0	*I*	I	*W*	V	*I*	*I*	*V*	0
0	0	j	j	0	*I*	*V*	*I*	*V*	*I*	I	0	*W*
0	0	j	0	*I*	*W*	*W*	0	*V*	W	0	W	*V*
0	0	0	0	*I*	W	0	*W*	*W*	0	I	V	*V*
0	0	0	0	*I*	0	I	W	0	*W*	*I*	*W*	I

(15,21)

0	0	0	1	j^t	j^t	j^t	0	0	0	0	0	0
0	0	1	0	j^t	0	0	j^t	j^t	0	0	0	0
0	1	0	0	j^t	0	0	0	0	j^t	j^t	0	0
1	0	0	0	0	j^t	0	j^t	0	j^t	0	0	0
j	j	j	0	*I*	*I*	*I*	*I*	0	0	*I*	*I*	*I*
j	0	0	j	*I*	0	*V*	*I*	*I*	*I*	*W*	I	0
j	0	0	0	*I*	*W*	*W*	0	V	*V*	*V*	0	W
0	j	0	j	*I*	*V*	0	*W*	*V*	*W*	0	*W*	W
0	j	0	0	0	*I*	I	*W*	W	*I*	*V*	*I*	0
0	0	j	j	0	*I*	*V*	*W*	*W*	*V*	V	0	*V*
0	0	j	0	*I*	*W*	*V*	0	*V*	I	0	W	*W*
0	0	0	0	*I*	W	0	*V*	*W*	0	I	V	*W*
0	0	0	0	*I*	0	V	I	0	*I*	*I*	*V*	I

(16,25)

0	0	0	1	j^t	j^t	j^t	0	0	0	0	0	0
0	0	1	0	j^t	0	0	j^t	j^t	0	0	0	0
0	1	0	0	j^t	0	0	0	0	j^t	j^t	0	0
1	0	0	0	0	j^t	0	j^t	0	j^t	0	0	0
j	j	j	0	*I*	*I*	*I*	*I*	0	0	*I*	*I*	*I*
j	0	0	j	*I*	0	*V*	*V*	*I*	*I*	*W*	V	0
j	0	0	0	*I*	*V*	*I*	0	I	*V*	*W*	0	I
0	j	0	j	*I*	*W*	0	*W*	*W*	*I*	0	*V*	W
0	j	0	0	0	*I*	W	*W*	W	*W*	*W*	*V*	0
0	0	j	j	0	*I*	*W*	*V*	*W*	*W*	V	0	*V*
0	0	j	0	*I*	*V*	*W*	0	*I*	I	0	I	*I*
0	0	0	0	*I*	V	0	*V*	*V*	0	W	I	*W*
0	0	0	0	*I*	0	I	V	0	*W*	*V*	*I*	I

(17,31)

0	0	0	1	j^t	j^t	j^t	0	0	0	0	0	0
0	0	1	0	j^t	0	0	j^t	j^t	0	0	0	0
0	1	0	0	j^t	0	0	0	0	j^t	j^t	0	0
1	0	0	0	0	j^t	0	j^t	0	j^t	0	0	0
j	j	j	0	*I*	*I*	*I*	*I*	0	0	*I*	*I*	*I*
j	0	0	j	*I*	0	*V*	*V*	*I*	*I*	*W*	V	0
j	0	0	0	*I*	*W*	*V*	0	I	*V*	*V*	0	V
0	j	0	j	*I*	*V*	0	*W*	*W*	*W*	0	*I*	I
0	j	0	0	0	*I*	W	*W*	I	*I*	*W*	*V*	0
0	0	j	j	0	*I*	*W*	*V*	*W*	*I*	W	0	*W*
0	0	j	0	*I*	*V*	*W*	0	*V*	V	0	I	*I*
0	0	0	0	*I*	V	0	*W*	*I*	0	W	I	*V*
0	0	0	0	*I*	0	V	W	0	*V*	*W*	*V*	I

(18,29)

0	0	0	1	j^t	j^t	j^t	0	0	0	0	0	0
0	0	1	0	j^t	0	0	j^t	j^t	0	0	0	0
0	1	0	0	j^t	0	0	0	0	j^t	j^t	0	0
1	0	0	0	0	j^t	0	j^t	0	j^t	0	0	0
j	j	j	0	*I*	*I*	*I*	*I*	0	0	*I*	*I*	*I*
j	0	0	j	*I*	0	*V*	*W*	*I*	*I*	*I*	I	0
j	0	0	0	*I*	*V*	*W*	0	I	*I*	*W*	0	W
0	j	0	j	*I*	*W*	0	*I*	*V*	*W*	0	*V*	I
0	j	0	0	0	*I*	I	*I*	V	*V*	*W*	*V*	0
0	0	j	j	0	*I*	*W*	*W*	*W*	*W*	W	0	*V*
0	0	j	0	*I*	*V*	*V*	0	*W*	I	0	V	*W*
0	0	0	0	*I*	V	0	*V*	*I*	0	I	V	*V*
0	0	0	0	*I*	0	V	I	0	*V*	*V*	*V*	V

(19,23)

0	0	0	1	j^t	j^t	j^t	0	0	0	0	0	0
0	0	1	0	j^t	0	0	j^t	j^t	0	0	0	0
0	1	0	0	j^t	0	0	0	0	j^t	j^t	0	0
1	0	0	0	0	j^t	0	j^t	0	j^t	0	0	0
j	j	j	0	*I*	*I*	*I*	*I*	0	0	*I*	*I*	*I*
j	0	0	j	*I*	0	*V*	*W*	*I*	*I*	*I*	I	0
j	0	0	0	*I*	*V*	*W*	0	I	*I*	*W*	0	W
0	j	0	j	*I*	*W*	0	*V*	*V*	*V*	0	*V*	V
0	j	0	0	0	*I*	I	*I*	V	*V*	*W*	*V*	0
0	0	j	j	0	*I*	*W*	*V*	*W*	*I*	W	0	*W*
0	0	j	0	*I*	*V*	*V*	0	*W*	I	0	V	*W*
0	0	0	0	*I*	V	0	*V*	*I*	0	I	V	*V*
0	0	0	0	*I*	0	V	V	0	*W*	*V*	*V*	I

(20,36)

0	0	0	1	j^t	j^t	j^t	0	0	0	0	0	0
0	0	1	0	j^t	0	0	j^t	j^t	0	0	0	0
0	1	0	0	j^t	0	0	0	0	j^t	j^t	0	0
1	0	0	0	0	j^t	0	j^t	0	j^t	0	0	0
j	j	j	0	*I*	*I*	*I*	*I*	0	0	*I*	*I*	*I*
j	0	0	j	*I*	0	*V*	*W*	*I*	*I*	*V*	V	0
j	0	0	0	*I*	*V*	*W*	0	W	*W*	*I*	0	I
0	j	0	j	*I*	*W*	0	*V*	*W*	*V*	0	*I*	I
0	j	0	0	0	*I*	V	*V*	W	*V*	*V*	*W*	0
0	0	j	j	0	*I*	*V*	*V*	*V*	*I*	V	0	*W*
0	0	j	0	*I*	*W*	*V*	0	*V*	I	0	I	*I*
0	0	0	0	*I*	W	0	*I*	*W*	0	I	I	*W*
0	0	0	0	*I*	0	V	I	0	*I*	*W*	*V*	W

(21,33)

0	0	0	1	j^t	j^t	j^t	0	0	0	0	0	0
0	0	1	0	j^t	0	0	j^t	j^t	0	0	0	0
0	1	0	0	j^t	0	0	0	0	j^t	j^t	0	0
1	0	0	0	0	j^t	0	j^t	0	j^t	0	0	0
j	j	j	0	*I*	*I*	*I*	*I*	0	0	*I*	*I*	*I*
j	0	0	j	*I*	0	*V*	*W*	*I*	*I*	*W*	W	0
j	0	0	0	*I*	*W*	*I*	0	W	*V*	*V*	0	I
0	j	0	j	*I*	*V*	0	*V*	*V*	*I*	0	*W*	V
0	j	0	0	0	*I*	V	*W*	I	*I*	*V*	*V*	0
0	0	j	j	0	*I*	*W*	*V*	*I*	*V*	I	0	*I*
0	0	j	0	*I*	*V*	*W*	0	*W*	I	0	W	*W*
0	0	0	0	*I*	V	0	*V*	*W*	0	V	I	*V*
0	0	0	0	*I*	0	I	V	0	*W*	*V*	*W*	W

(22,27)

1	0	0	0	j^t	j^t	j^t	0	0	0	0	0	0
0	0	1	0	j^t	0	0	0	0	j^t	j^t	0	0
0	1	0	0	j^t	0	0	j^t	j^t	0	0	0	0
0	0	0	1	0	j^t	0	j^t	0	j^t	0	0	0
j	j	j	0	0	*I*	*I*	*I*	*I*	*I*	*I*	*I*	0
j	0	0	j	*I*	*I*	0	*I*	*V*	0	I	*W*	*I*
j	0	0	0	*I*	0	*I*	*V*	*I*	I	0	*W*	I
0	0	j	j	*I*	*W*	*V*	0	V	*V*	0	*I*	*I*
0	0	j	0	*I*	*V*	*W*	V	0	0	*V*	*I*	I
0	j	0	j	*I*	0	W	*W*	0	*I*	*W*	*V*	*V*
0	j	0	0	*I*	W	0	0	*W*	*W*	*I*	*V*	V
0	0	0	0	*I*	*W*	*W*	*V*	*V*	*I*	*I*	*J*	0
0	0	0	0	0	*I*	I	*V*	V	*I*	I	0	*W*

(23,–)

1	0	0	0	j^t	j^t	j^t	0	0	0	0	0	0
0	0	1	0	j^t	0	0	0	0	j^t	j^t	0	0
0	1	0	0	j^t	0	0	j^t	j^t	0	0	0	0
0	0	0	1	0	j^t	0	j^t	0	j^t	0	0	0
j	j	j	0	0	*I*	*I*	*I*	*I*	*I*	*I*	*I*	0
j	0	0	j	*I*	*I*	0	*I*	*V*	0	I	*W*	*I*
j	0	0	0	*I*	0	*I*	*V*	*W*	V	0	*V*	V
0	0	j	j	*I*	*I*	*W*	0	V	*V*	0	*V*	*V*
0	0	j	0	*I*	*W*	*V*	W	0	0	*I*	*W*	I
0	j	0	j	*I*	0	W	*W*	0	*V*	*W*	*I*	*W*
0	j	0	0	*I*	I	0	0	*I*	*W*	*V*	*I*	W
0	0	0	0	*I*	*V*	*W*	*W*	*V*	*I*	*I*	*J*	0
0	0	0	0	0	*I*	W	*W*	I	*W*	W	0	*V*

(24,–)

1	0	0	0	j^t	j^t	j^t	0	0	0	0	0	0
0	0	1	0	j^t	0	0	0	0	j^t	j^t	0	0
0	1	0	0	j^t	0	0	j^t	j^t	0	0	0	0
0	0	0	1	0	j^t	0	j^t	0	j^t	0	0	0
j	j	j	0	0	*I*	*I*	*I*	*I*	*I*	*I*	*I*	0
j	0	0	j	*I*	*I*	0	*V*	*W*	0	W	*W*	*I*
j	0	0	0	*I*	0	*I*	*W*	*I*	I	0	*V*	V
0	0	j	j	*I*	*W*	*V*	0	V	*I*	0	*V*	*V*
0	0	j	0	*I*	*V*	*I*	W	0	0	*W*	*W*	I
0	j	0	j	*I*	0	I	*I*	0	*V*	*W*	*I*	*I*
0	j	0	0	*I*	V	0	0	*V*	*W*	*V*	*I*	I
0	0	0	0	*I*	*V*	*W*	*W*	*V*	*I*	*I*	*J*	0
0	0	0	0	0	*I*	W	*W*	I	*V*	V	0	*V*

(25,−)