Bounds for the Rank of a Permutation on a Tree

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Introduction

In this paper, we consider a permutation σ in S_n as acting on an arbitrary tree with n vertices (labeled $1, 2, \ldots, n$). Each edge $[a, b]$ of T corresponds to a transposition in S_n, and the set of all such transpositions forms a minimal generating set for S_n (and conversely). The permutation σ can be written as a product of these transpositions (a T-factorization of σ), and the minimum length of such a product is called the T-rank of σ. (For a general discussion see *e.g.*[1].)

If T is a path (two vertices of degree 1, all others of degree 2) or a star (one vertex of degree ≥ 2, all others of degree 1), then the T-rank of σ is easily computed (see [2],[3]), and there are algorithms to produce a minimal T-factorization. In [2], Edelman derives (for T a path or a star), exact upper and lower bounds for the T-rank of a permutation with k disjoint cycles.

For a general tree T, there is no straightforward algorithm for either producing a minimal T-factorization, or finding the T-rank. Indeed, it appears that good upper and lower bounds for the T-rank (in terms of reasonably computable quantities) are not known.

The primary purpose of this paper is to give such upper and lower bounds. Along the way, we find a special T-factorization of σ (the star-factorization) which is uniquely determined by σ and T. The star-factorization is of minimal length if T is a path or a star, but it need not be minimal otherwise.

The main results may be summarized as follows. Given the tree T, and a vertex x of T with $\sigma(x) \neq x$, there is a unique path $P(x, \sigma)$ in T between x and $\sigma(x)$. The length of this path, $|P(x, \sigma)|$, is the number of edges in it. We define the path-length of σ, denoted $PL(\sigma)$, to be the sum of all the lengths $|P(x, \sigma)|$ ($PL(\sigma)$ is shown to be an even number). In Section 2, we prove that if σ has $n - r$ fixed points, then

$$PL(\sigma)/2 \leq PL(\sigma) - [r/2][(r+1)]/2 \leq T\text{-rank}(\sigma).$$

In Section 3, we find a (uniquely determined) T-factorization of σ (the star-factorization) which has the form $\sigma = \beta_L \beta_{L-1} \ldots \beta_2 \beta_1$, where each β_i is a cycle, and we show that

$$T\text{-rank}(\sigma) \leq PL(\sigma) - L.$$

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Each of these bounds is best possible in the sense that for every (finite) tree T, there exists a permutation σ on T which attains the bound.

In Section 4, we apply our results to some particular cases. If σ is a 2-cycle, then T-rank(σ) = $PL(\sigma) - 1$, and conversely. If σ is a 3-cycle, then T-rank(σ) = $PL(\sigma) - 2$, and if σ is a 4-cycle, then T-rank(σ) = $PL(\sigma) - 3$. If σ is a product of two 2-cycles, then T-rank(σ) is either $PL(\sigma) - 2$ or $PL(\sigma) - 4$, and we give conditions for each.

Finally, we use the properties of the star-factorization of σ to show that if σ is a k-cycle, then T-rank(σ) $\leq PL(\sigma) - k + 1$, and from this, if σ is a product of t disjoint cycles, and σ has $n - r$ fixed points, then

$$T\text{-rank}(\sigma) \leq PL(\sigma) - r + t.$$

In Section 5, we give some open questions, and some examples.

1. Preliminaries.

In this section, we establish some basic notation and elementary results, including an easy lower bound for the rank of a permutation.

If n is a positive integer, S_n denotes the symmetric group on the set $\{1, 2, \ldots, n\}$. If $\alpha, \beta \in S_n$, then multiplication is composition "on the right", *i.e.* $(\alpha\beta)(x) = \alpha(\beta(x))$. We use the letter e to denote the identity. It is well-known that a set of transpositions of S_n, say

$$T = \{(a_i, b_i) \,|\, i = 1, 2, \ldots, k\}$$

is a minimal generating set for the group S_n if and only if $k = n - 1$, and the graph with vertex set $\{1, 2, \ldots, n\}$ and edge set T is a tree.

We assume throughout that $T = \{(a_i, b_i) \,|\, i = 1, 2, \ldots, n-1\}$ is a minimal generating set of transpositions, for S_n. We usually (abuse of notation) refer to T as a tree, and to $(a, b) \in T$ as an edge of T.

If $\sigma \in S_n$, then the least number m such that σ is equal to a product of m transpositions from T, is called the *rank of σ with respect to T*, or the *T-rank of σ*, or just the rank of σ if T is understood. If we have

$$\sigma = t_k t_{k-1} \ldots t_2 t_1, \quad t_i \in T,$$

we say that the right-hand side is a *T-factorization* (or a T-representation) of σ, of *length* k. If the length of a T-factorization of σ is the rank of σ, then we will say that the T-factorization is *minimal*.

From now on, we assume that T is a fixed but arbitrary finite tree.

Since T is a tree, given any two vertices x, y of T, there is a unique path in T between x and y, denoted $[x, y]$. If the vertices on this path are, in order from x to y : $x = a_1, a_2, \ldots, a_k = y$, we write

$$[x, y] = [a_1, a_2, \ldots, a_k].$$

The notation $[x, y) = [a_1, \ldots, a_{k-1}]$ and $(x, y] = [a_2, \ldots, a_k]$ is also convenient. The number $d_T(x, y)$ of edges in $[x, y]$ is called the *T-length* of $[x, y]$, or the *T-distance* from x to y, or just the length of $[x, y]$ ($d(x, y)$) if T is understood.

If $\sigma \in S_n$, and if $i \in \{1, \ldots, n\}$, the *σ-path of i* is $P(i, \sigma) = [i, \sigma(i)]$, and the length of this path is denoted by $|P(i, \sigma)|$. The sum of all the lengths of these paths is denoted by $PL(\sigma)$, and called the *path-length of σ*:

$$PL(\sigma) = \sum_{i=1}^{n} |P(i, \sigma)|.$$

The minimal disjoint connected components of T generated by the σ-paths, are called the *σ-components of T*, and the subtree of T spanned by the union of all these components is called the *span of σ*, denoted $T(\sigma)$.

Lemma 1.1. *Let $\sigma \in S_n$.*

(i) *If $[a, b]$ is any edge of T, then the number of σ-paths containing the edge $[a, b]$ is even.*

(ii) *$PL(\sigma)$ is even.*

Proof: If the edge $[a, b]$ is removed from T, the resulting configuration consists of two disjoint trees X and Y (the components of T determined by the edge $[a, b]$); say that $a \in X$, and $b \in Y$. Define the sets A and B by:

$$A = \{x \in X | \sigma(x) \in Y\}, \quad B = \{y \in Y | \sigma(y) \in X\}.$$

Then since σ is a permutation, these sets must have the same cardinality. Clearly, the edge $[a, b]$ is contained in the σ-path $P(i, \sigma)$ if and only if $i \in A$ or $i \in B$, so the number $Cr(a, b)$ of σ-paths containing $[a, b]$ is $|A \cup B| = 2|A|$, an even number. This proves (i). Since $PL(\sigma)$ is the total number of edges contained in the σ-paths (an edge is counted once for each path containing it) it is clear that

$$PL(\sigma) = \sum_{(a,b) \in T} Cr(a, b)$$

and so $PL(\sigma)$ must be an even number also. This proves (ii).

Definition 1.2. *The number $Cr(a, b)$ in Lemma 1.1, is called the crossover number of the edge $[a, b]$.*

In the next lemma it is shown that if a permutation is multiplied on the right by a transposition of T, the path-length can change only by 2, 0, or -2. (A similar argument gives the same result if the multiplication is on the left.) Then any product of k transpositions of T can have path-length at most $2k$, and this gives a lower bound for the rank of a permutation, in terms of its path-length.

Lemma 1.3. *Let $\sigma \in S_n$, and $t \in T$. Put $\tau = \sigma t$. Then $PL(\sigma) - PL(\tau)$ is either 2 or 0 or -2.*

Proof: Write $t = (a, b)$. Then if $i \neq a, b, P(i, \sigma) = P(i, \tau)$. There are only three other possibilities, and we consider them separately.

Case 1. Both paths contain $[a, b]$; $P(a\sigma) = [a, b, \ldots, \sigma(a)]$ and $P(b, \sigma) = [b, a, \ldots, \sigma(b)]$. Then since $\tau(a) = \sigma(b)$, and $\tau(b) = \sigma(a)$, we have

$$P(b, \tau) = [b, \ldots, \sigma(a)] \text{ and } P(a, \tau) = [a, \ldots, \sigma(b)],$$

and then $PL(\sigma) - PL(\tau) = 2$.

Case 2. The edge $[a, b]$ is contained in one of the paths, but not in the other; without loss of generality, suppose that $P(a, \sigma) = [a, b, \ldots, \sigma(a)]$ and $P(b, \sigma) = [b, c, \ldots, \sigma(b)]$, with $c \neq a$. Then

$$P(a, \tau) = [a, b, c, \ldots, \sigma(b)] \text{ and } P(b, \tau) = [b, \ldots, \sigma(a)]$$

and $PL(\sigma) - PL(\tau) = 0$.

Case 3. $P(a, \sigma) = [a, x, \ldots, \sigma(a)]$ and $P(b, \sigma) = [b, y, \ldots, \sigma(b)]$, and $[a, b]$ is not contained in either $P(a, \sigma)$ or in $P(b, \sigma)$. Then we have

$$P(a, \tau) = [a, b, y, \ldots, \sigma(a)] \text{ and } P(b, \tau) = [b, a, x, \ldots, \sigma(b)],$$

and $PL(\sigma) - PL(\tau) = -2$.

Corollary 1.4. *If $\sigma \in S_n$, then rank $\sigma \geq PL(\sigma)/2$.*

2. A lower bound for rank

Throughout this section, we will assume that $\sigma \in S_n$, that T is fixed, and that

$$\sigma = t_m \ldots t_1$$

is a fixed T-factorization of σ. We will derive a lower bound for the rank of σ.

Definition 2.1. *A walk in T is a finite sequence of vertices of T, say $W = \{x_1, x_2, x_3, \ldots, x_k\}$, where for each $i = 1, 2, \ldots, k-1$, either $x_i = x_{i+1}$, or else x_i and x_{i+1} are adjacent in T. The walk-length of W is the number of indices i such that $[x_i, x_{i+1}]$ is an edge in T (i.e. such that $x_i \neq x_{i+1}$).*

The following elementary lemma is easily established by induction, and we omit the proof.

Lemma 2.2. *If $W = \{x_1, x_2, x_3, \ldots, x_k\}$ is a walk in T, and if $x_1 = x_k$, then the walk-length L of W is even. If $a, b \in W$, then every vertex of the path in T between a and b is also in W.*

Definition 2.3. *Given a factorization $\sigma = t_m t_{m-1} \ldots t_2 t_1$. If x is a vertex of T, then the trajectory of x determined by this factorization of σ is the (ordered) sequence*

$$R(x, \sigma) = \{x = x_0, x_1, x_2, x_3, \ldots, x_m\}$$

where $x_i = t_i(x_{i-1}) = (t_i t_{i-1} \ldots t_2 t_1)(x)$ for $i = 1, 2, \ldots, m$. Then x_i and x_{i+1} are either equal or adjacent in T, and $R(x, \sigma)$ is a walk in T; its walk-length is called the σ-walk-length of x, denoted by $W(x, \sigma)$.

Remark: It is clear that for each $i = 0, 1, 2, \ldots, m-1$, precisely two of the trajectories of σ have unequal entries in the $i, i+1$ positions. Thus we have

Corollary 2.4.

$$\sum_{X \in T} W(x, \sigma) = 2m.$$

Lemma 2.5. *Let $\sigma \in S_n$, and suppose that $\sigma = t_m \ldots t_1$ is a T-factorization. Let $x \in T$, where the σ-path $P(x, \sigma) = [x = a_0, a_1, \ldots, a_k = \sigma(x)]$, and the trajectory for x, for this factorization, is $R(x, \sigma) = \{x = x_0, x_1, \ldots, x_m = \sigma(x)\}$. Then there exists a family of indices $\{f(j) | j = 0, 1, \ldots, k-1\}$ such that:*

(a) *If $i = f(j)$, then $x_i = a_j$, and $x_t \neq a_j$ for $i < t \leq m$, for all $j = 0, 1, \ldots, k-1$*
(b) $0 \leq f(1) < f(2) < f(3) < \cdots < f(k-1) \leq m-1$,
(c) *If $i = f(j)$, and $0 \leq j < k$, then $x_{i+1} = a_{j+1}$.*

Proof: (a) From Lemma 2.2, since $R(x, \sigma)$ is a walk in T and $x_0 = a_0$, and $x_m = a_m$, then $P(x, \sigma)$ must be a subset of $R(x, \sigma)$. For each $j = 0, 1, \ldots, k-1$, let $i = f(j)$ be the largest index such that $x_i = a_j$.

(b) For $0 \leq i \leq m-1$, let $R_i(x, \sigma) = [x_i, x_{i+1}, \ldots, x_m]$ be the trajectory of the permutation $\tau_i = \sigma t_1 t_2 \ldots t_i$ for x_i (*i.e.* $R_i(x, \sigma) = R(x_i, \tau_i)$). If $i = f(0)$, then $x_i = x = a_0$, and so $\tau_i(x) = \sigma(x)$. Then $R_i(x, \sigma)$ contains $P(x, \sigma)$, and in particular, for some $j, i < j \leq m$, we must have $x_j = a_1$. Thus $f(0) < f(1)$. Then (b) follows by induction.

(c) For $0 \leq j \leq k-1$, let $i = f(j)$. Then $a_j = x_i$, and a_j is not a member of $R_{i+1}(x, \sigma)$, while a_{j+1} is a member of $R_{i+1}(x, \sigma)$. The edge $[a_j, a_{j+1}]$ is the unique path in T from a_j to a_{j+1}, and so any walk in T from a_j to a_{j+1} must include this edge. In particular, $R_i(x, \sigma)$ must include this edge. Then we must have $x_{i+1} = a_{j+1}$.

Corollary 2.6. *(a)* $W(x,\sigma) \leq |P(x,\sigma)|$, *and* $W(x,\sigma) - |P(x,\sigma)|$ *is even.* *(b)* $W(x,\sigma) = |P(x,\sigma)|$ *if and only if* $d(x,x_i) \leq d(x,x_{i+1})$ *for all* $x_i \in R(x,\sigma), i = 0,1,\ldots,m-1$.

Definition 2.7. *Using the notation of Lemma 2.5, we say that the indices* $f(j)$ *are the path-indices for the trajectory* $R(x,\sigma)$. *If a fuller notation is needed, we write* $f(j) = f(x,j)$. *If* $i = f(0)$, *then* $[x_i, x_{i+1},\ldots,x_m]$ *is the final segment of* $R(x,\sigma)$, *denoted* $FS(x,\sigma)$.

Definition 2.8. *Let* x, y *be distinct elements of* T, *and suppose that* $R(x,\sigma)$ *and* $R(y,\sigma)$ *have two of their path-indices equal, that is, for some* j, k *we have* $i = f(x,j) = f(y,k)$. *Then we say that* x *and* y *meet at* t_i.

Definition 2.9. *Let* $x, y \in T$, *and suppose* $P(x,\sigma) \cap P(y,\sigma)$ *contains an edge* $[a,b]$ *of* T. *If* $P(x,\sigma) = [x,\ldots,a,b,\ldots,\sigma(x)]$ *while* $P(y,\sigma) = [y,\ldots,b,a,\ldots,\sigma(y)]$, *then we say that the paths* $P(x,\sigma)$ *and* $P(y,\sigma)$ *are in opposite directions (on* $[a,b]$*). Otherwise, they are in the same direction (on* $[a,b]$*).*

Remark. It should be emphasized that when we say that two paths have the same, or opposite, directions, then this implies that the paths intersect in an interval containing at least one edge of T.

Lemma 2.10. *(a) Let* $[a,b]$ *be an edge of* T, *and suppose that the crossover number* $Cr(a,b) = 2k$. *Then the transposition* (a,b) *must appear at least* k *times in every* T*-factorization of* σ.

(b) Suppose that $P(x,\sigma) \subset P(y,\sigma)$, *where* $x \neq y$. *If* $\sigma(x) = x$, *or if* $\sigma(x) \neq x$ *and* $P(x,\sigma)$ *and* $P(y,\sigma)$ *are in the same direction then* $W(x,\sigma) > |P(x,\sigma)|$.

Proof: (a) If $[a,b]$ is in $P(y,\sigma)$, then in $R(y,\sigma) = \{y_0,\ldots,y_m\}$, there must be some path-index i such that $y_i = a$, $y_{i+1} = b$; then $t_i = (a,b)$. If $Cr(a,b) = 2k$, then at least $2k$ of the trajectories must have consecutive entries a, b (or b, a), and these must occur for at least k different indices. This proves (a).

To see (b), write $R(x,\sigma) = \{x = x_0, x_1,\ldots,x_m\}$ and $R(y,\sigma) = \{y = y_0, y_1,\ldots,y_m\}$. If $R(x,\sigma)$ contains a vertex u which is not in $P(y,\sigma)$, then we are done, so we suppose that x_i is a vertex of $P(y,\sigma)$ for $i = 0,1,\ldots,m$. Since $P(x,\sigma) \subset P(y,\sigma)$, then $d(y_0,x_0) > d(y_0,y_0)$ and $d(y_0,x_m) < d(y_0,y_m)$. Then there is some least index i such that $d(y_0,x_i) > d(y_0,y_i)$, and $d(y_0,x_{i+1}) < d(y_0,y_{i+1})$. Then (since T is a tree, and $P(x,\sigma) \subset P(y,\sigma)$ are paths in T) it must be the case that $t_{i+1} = (x_i,x_{i+1}) = (y_i,y_{i+1})$, and $x_i = y_{i+1}, y_i = x_{i+1}$. Then $d(y_0,x_{i+1}) < d(y_0,x_i)$, and since $P(x,\sigma) \subset P(y,\sigma)$, then $d(x_0,x_{i+1}) < d(x_0,x_i)$. By Corollary 2.6, $W(x,\sigma) > |P(x,\sigma)|$.

Combining this with Corollaries 2.4 and 2.6 gives

Corollary 2.11. *Let* $\sigma = t_m \ldots t_1$ *be a* T*-factorization. Let* K *be the set of all fixed points* x *of* σ *such that for some* $y \neq x$, *we have* $x \in P(y,\sigma)$. *Let* J *be* *the set of all x such that $\sigma(x) \neq x$, and for some $y \neq x$, $P(x,\sigma) \subset P(y,\sigma)$, and the two paths are in the same direction. Then*

$$m \geq |K| + |J| + PL(\sigma)/2.$$

Theorem 2.12. *Let x, y be distinct elements of T, and let $M(x,y)$ be the number of distinct indices i such that x and y meet at t_i. Let $X = P(x,\sigma) \cap P(y,\sigma)$.*

(a) *If $|X| \leq 1$, then $M(x,y) = 0$.*
(b) *If $|X| > 1$, and $P(x,\sigma)$ and $P(y,\sigma)$ are in the same direction, then $M(x,y) = 0$.*
(c) *If $|X| > 1$ and $P(x,\sigma)$ and $P(y,\sigma)$ are in opposite directions, then $M(x,y) \leq 1$.*

Proof: Suppose that x and y meet at t_i. Then $R(x,\sigma)$ and $R(y,\sigma)$ have two of their path-indices equal, that is, for some j, k we have $i = f(x,j) = f(y,k)$. Since $R(x,\sigma) = \{x = x_0, \ldots, x_i, x_{i+1}, \ldots, x_m\}$, $R(y,\sigma) = \{y = y_0, \ldots, y_i, y_{i+1}, \ldots, y_m\}$, and $x_{i+1} = t_{i+1}(x_i) \neq x_i$, $y_{i+1} = t_{i+1}(y_i) \neq y_i$ (by Lemma 2.5(c)), then it must be that $t_{i+1} = (x_i, x_{i+1}) = (y_i, y_{i+1})$. Since $x_i = (t_i \ldots t_1)(x)$, $y_i = (t_i \ldots t_1)(y)$, and $x \neq y$, then $x_i \neq y_i$. Then $x_i = y_{i+1}$, and $x_{i+1} = y_i$.

Since $i = f(x,j) = f(y,k)$, then by Lemma 2.5(c), we have

$$P(x,\sigma) = [x \ldots x_i, x_{i+1}, \ldots \sigma(x)] \text{ and } P(y,\sigma) = [y, \ldots, y_i, y_{i+1}, \ldots, \sigma(y)]$$

and so $P(x,\sigma)$ and $P(y,\sigma)$ have the common edge $[x_i, x_{i+1}]$, and are in opposite directions on this edge. This proves (a) and (b).

To see (c): Suppose that x and y meet at t_i. The edge $[x_i, x_{i+1}]$ separates T into two disjoint components, say A containing $x_i = y_{i+1}$, and B containing $x_{i+1} = y_i$. The set $R_{i+1}(x,\sigma) = \{x_{i+1}, \ldots, x_m\}$ spans a subtree of T which does not contain x_i, but does contain x_{i+1}; so $R_{i+1}(x,\sigma)$ is a subset of B. Similarly, $R_{i+1}(y,\sigma)$ is a subset of A. Thus $R_{i+1}(x,\sigma)$ and $R_{i+1}(y,\sigma)$ are disjoint, and in particular, x and y cannot meet at t_k for any $k > i$. It follows that any two distinct elements x, y can meet at most once.

Definition 2.13. *For a T-factorization $\sigma = t_m \ldots t_1$, let $M(t_m \ldots t_1)$ be the total number of pairs $\{x,y\}$ which meet (i.e., x and y have a common path-index). Then define $M(\sigma)$ by*

$$M(\sigma) = \max\{M(t_m \ldots t_1) \mid t_m \ldots t_1 = \sigma \text{ is a } T\text{-factorization}\}.$$

Corollary 2.14. *If $\sigma = t_m \ldots t_1$ is a T-factorization, then*

$$PL(\sigma) - M(t_m \ldots t_1) \leq m.$$

Proof: For each x, consider the set of indices $F = \{f(x,j)\}$, and the trajectory $R(x,\sigma) = \{x_0 = x, \dots, x_m\}$. Suppose $P(x,\sigma) = [a_0 = x, a_1, \dots, a_k]$. For $0 \leq j \leq k-1$, letting $i = f(x,j)$, we have $[x_i, x_{i+1}] = [a_j, a_{j+1}]$, and so the length k of $P(x,\sigma)$ is equal to $|F|$. If we put $i = f(0)$, so that $FS(x,\sigma) = \{x_i, \dots, x_m\}$, then we must have $m - i \geq |F|$, and so $m - i \geq k$. By Theorem 2.12, if x and y are distinct, then $\{f(x,j)\} \cap \{f(y,j)\}$ contains at most one element, and so the set

$$\text{Ind} = \cup\{\{f(x,j)\} | x \in T\}$$

has cardinality equal to $PL(\sigma) - M(t_m \dots t_1)$. Since Ind is a subset of $\{0, 1, \dots, m-1\}$, it follows that

$$m \geq PL(\sigma) - M(t_m \dots t_1),$$

as required.

Corollary 2.15. *If m is the T-rank of σ, then $m \geq PL(\sigma) - M(\sigma)$.*

Theorem 2.16. *Let $\sigma \in S_n$, and suppose that σ has precisely $n-r$ fixed points. Then $M(\sigma) \leq [r/2][(r+1)/2]$ (where $[x]$ denotes the greatest integer $\leq x$).*

Proof: It is clear that if, for instance, $P(x,\sigma)$ and $P(y,\sigma)$ are in opposite directions, and also $P(x,\sigma)$ and $P(z,\sigma)$ are in opposite directions, then $P(y,\sigma)$ and $P(z,\sigma)$ cannot be in opposite directions. This is true for any directed paths in T.

Let F be a set of r directed paths in T, all of length ≥ 1;

$$F = \{P_i | i = 1, 2, \dots, r\}.$$

Let Z be the set of all ordered pairs (P_i, P_j) such that P_i, $P_j \in F$ and P_i and P_j are in opposite directions. We first prove that $|Z| \leq [r/2][(r+1)/2]$, by induction. If $|F| = 1$, then $Z = \emptyset$, and $|Z| \leq 0$ is true. If $|F| = 2$, then clearly $|Z| \leq 1$. Suppose the result is true for all families of cardinality less than r, and suppose that $|F| = r$. Choose any P_i in F, and let $A(i)$ be the set of all P_k in F such that P_i and P_k are in opposite directions. Let $B(i) = F - A(i)$.

Now suppose $P_j \in A(i)$. Then if P_k and P_j are in opposite directions, P_k must be a member of $B(i)$, and so if $|A(i)| > [r/2]$, then (for any $P_j \in A(i)$) we have $|A(j)| \leq |B(i)| \leq [r/2]$.

Then without loss of generality, we can assume that $|A(i)| \leq [r/2]$. The set Z contains precisely $|A(i)|$ ordered pairs in which one entry is P_i, and all the remaining ordered pairs have their entries in the set $F - \{P_i\}$, of cardinality $r-1$. By the induction assumption,

$$|Z| - |A(i)| \leq [(r-1)/2][r/2],$$

and so $|Z| \leq [(r-1)/2][r/2] + [r/2] = [r/2][(r+1)/2]$, as required.

Now suppose that

$$F = \{P(x,\sigma) | \sigma(x) \neq x\}.$$

If $P(x,\sigma)$ and $P(y,\sigma)$ meet at t_i, then $P(x,\sigma)$ and $P(y,\sigma)$ are in opposite directions, and so the number of pairs of such paths cannot exceed the cardinality of Z, and then $M(\sigma) \leq [r/2][(r+1)/2]$.

Corollary 2.17. *If $\sigma \in S_n$, and if m is the T-rank of σ, and if the number of fixed points of σ is $n-r$, then*

$$m \geq PL(\sigma) - [r/2][(r+1)/2]$$

3. An upper bound for rank

In this section, we establish an upper bound for the rank of σ. Along the way, we find an interesting family of invariants associated with σ, and a T-factorization of σ which is uniquely determined by σ and T. Throughout this section, n is fixed, T is fixed, and $\sigma \in S_n$.

We first define a function on T, associated with σ, which might be called a "next-point" function: for x in T, $f(x)$ is the first vertex after x, on the σ-path of x, which is NOT fixed by σ.

Definition 3.1. *Let $\sigma \in S_n$. Define a function $f : T \to T$ by: If $\sigma(x) = x$, then $f(x) = x$; if $\sigma(x) \neq x$, and if $P(x,\sigma) = [x = x_0, x_1, x_2, \ldots, x_k = \sigma(x)]$, then $f(x) = x_i$ where i is the least positive index such that $\sigma(x_i) \neq x_i$. A cycle of f is a sequence of distinct iterates of f, which returns to its starting point:*

$$\{a_0 = f(a_k), a_1 = f(a_0), a_2 = f(a_1), \ldots, a_k = f(a_{k-1})\}$$

(To avoid confusion in this section, a permutation which is a cycle will be called a permutation-cycle.) If $\{a_0, \ldots, a_k\}$ is a cycle of f, then the permutation-cycle $t = (a_k, a_{k-1}, \ldots, a_1, a_0)$ is the associated star of σ, or a σ-star. (Note that the σ-star is "opposite" to the corresponding cycle of f).

A cycle of f of length 1 corresponds to a fixed point of σ; it is called a trivial cycle of f.

Finally, if $[b_0, b_1, b_2, \ldots, b_m]$ is any path in T, then $C(b_0, b_m)$ denotes the T-factorization

$$(b_m, b_{m-1})(b_{m-1}, b_{m-2}) \ldots (b_2, b_1)(b_1, b_0)$$

of the permutation-cycle $(b_0, b_m, b_{m-1}, b_{m-2}, \ldots, b_1)$.

In the following lemma we state some obvious, but important, properties of the cycles of f, and the associated σ-stars. These follow almost immediately from the definitions, and we omit the proof.

Lemma 3.2. *(a) If $x \in T$, then the sequence of iterates $\{x, f(x), f(f(x)), \ldots\}$ must eventually repeat, and the periodic part of such a sequence is a cycle of f. Thus if σ is not the identity, the associated f has non-trivial cycles. Then σ has non-trivial σ-stars.*

(b) If A is a non-trivial cycle of f, let U be the subtree of T spanned by (the elements of) A. Then the outer vertices of U are precisely the members of A, and if u is a vertex of U which is not an outer vertex of U, then $\sigma(u) = u$ (i.e. the interior of U is fixed by σ).

(c) Let A and B be two distinct, non-trivial cycles of f. Then A and B are disjoint, and the corresponding σ-stars are disjoint permutation-cycles.

(d) The set of cycles of f is uniquely determined by σ.

(e) If $\alpha = (a_k, a_{k-1}, \ldots, a_1, a_0)$ is a σ-star, then the paths of σ and the paths of $\mu = \sigma\alpha$ are related as follows: If $x \neq a_i$ for any i, then $P(x,\mu) = P(x,\sigma)$, and otherwise,

$$
\begin{aligned}
P(a_0,\mu) &= P(a_k,\sigma) - [a_k, a_0) = [a_0, \ldots, \sigma(a_k)], \\
P(a_1,\mu) &= P(a_0,\sigma) - [a_0, a_1) = [a_1, \ldots, \sigma(a_0)], \\
&\ldots \\
P(a_k,\mu) &= P(a_{k-1},\sigma) - [a_{k-1}, a_k) = [a_k, \ldots, \sigma(a_{k-1})].
\end{aligned}
$$

Remark. Multiplying σ (on the right) by one of its σ-stars has the effect of shortening the paths in a uniform way. This is the basic idea behind the star-factorization, described below. We first prove some elementary properties.

Lemma 3.3. *Let $\sigma \in S_n$, and suppose that $\alpha_1, \alpha_2, \ldots, \alpha_k$ are all the distinct σ-stars. Put $\sigma_i = \sigma\alpha_i$, and $\tau = \sigma\alpha_1\alpha_2\ldots\alpha_k$. (a) $PL(\sigma_i) = PL(\sigma) - PL(\alpha_i)$, and if $j \neq i$, then α_j is a σ_i-star. (b) $PL(\tau) = PL(\sigma) - PL(\alpha_1) - PL(\alpha_2) - \cdots - PL(\alpha_k)$. (c) If β is a (non-trivial) τ-star, and for some i, β and α_i are not disjoint, then for every such α_i, β and α_i have precisely one common element.*

Proof: Statements (a) and (b) follow from Lemma 3.2 (c) and (e). For (c) suppose that β and $\alpha_i = (a_k, a_{k-1}, \ldots, a_1, a_0)$ have some common element, say a_0. Consider the tree $T(\alpha_i)$ (spanned by $a_0, \ldots, a_k$). By Lemma 3.2(b), the interior points of $T(\alpha_i)$ are all fixed by σ, and by Lemma 3.2(e), they are also fixed by τ. Removing the interior of $T(\alpha_i)$ divides T into at least $k+1$ disjoint components; let the component containing α_i be labeled $C(\alpha_i)$. By Lemma 3.2(e), $P(\alpha_i, \tau) \in C(a_0)$. Since β is a τ-star, then a_0 is an outer vertex of $T(\beta)$, and $T(\beta)$ is a subtree containing at least one edge of $P(\alpha_i, \tau)$. Thus $T(\beta) \subset C(a_0)$, and then if $j \neq 0$, a_j is not in $T(\beta)$.

Lemma 3.5. *Let $\sigma \in S_n$, and suppose that the associated function f has precisely one non-trivial cycle, say, $B = \{b_1, b_2, \ldots, b_k\}$. Let U be the subtree* *spanned by B, and for each i, let a_i denote the (unique) vertex of U which is adjacent to b_i. Then σ is a permutation-cycle, and*

$$(*) \qquad \sigma = C(a_k, b_1)C(a_{k-1}, b_k)C(a_{k-2}, b_{k-1})\ldots.C(a_2, b_3)C(b_1, b_2)$$

is a T-factorization of σ, of length $PL(\sigma) - k + 1$, which has $k - 1$ meetings.

Remark. The T-factorization (*) need not be minimal.

Definition 3.5. *Let $\sigma \in S_n$, $\sigma \neq e$, and suppose that α is a σ-star. Then α satisfies the conditions of Lemma 3.4, and the factorization (*) will be denoted by $S(\alpha)$. Define a sequence of permutations σ_i and τ_i as follows: $\sigma_0 = \sigma$; $\tau_0 = e$ (the identity); supposing that σ_i and τ_i have been defined, and if $\alpha_1, \ldots, \alpha_k$ are all the distinct stars of σ_i, then we let*

$$\sigma_{i+1} = \sigma_i S(\alpha_1) \ldots S(\alpha_k), \text{ and } \tau_i = S(\alpha_1) \ldots S(\alpha_k)$$

Theorem 3.6. *(a) The sequence $\{\sigma_0, \sigma_1, \ldots\}$ described in Definition 3.5 terminates in the identity; say $\sigma_m \neq e$, and $\sigma_{m+1} = e$.*

(b) The factorization $\sigma = t_m t_{m-1} \ldots t_1$ is a T-factorization of σ which requires no more than $PL(\sigma) - m$ transpositions.

(c) The factorization $\sigma = t_m t_{m-1} \ldots t_1$ is uniquely determined by σ, except for rearrangements of the factors $S(\alpha)$.

Proof: (a) By Corollary 3.3, each σ_i has path-length strictly less than that of σ_{i-1}; the path-lengths must decrease to 0, and the identity e is the only permutation with path-length 0 (and no non-trivial stars).

(b) The factorization $\sigma = t_m t_{m-1} \ldots t_2 t_1$ is a T-factorization by the definition of $S(\alpha)$. By Corollary 3.3, $PL(\sigma)$ is the sum

$$PL(\sigma) = \sum_{i=1}^{m} PL(\tau_i)$$

and the number of transpositions required for τ_i is less than $PL(\tau_i)$ by Lemma 3.4; the result follows.

(c) This is obvious from the fact that for any permutation, the set of its σ-stars is a uniquely determined set of permutations which act on disjoint subsets of T (and hence commute).

Definition 3.7. *The factorization $\sigma = t_m t_{m-1} \ldots t_2 t_1$ is called the star-factorization of σ. In view of (*), it is also a T-factorization of σ. If for each $i = 1, 2, \ldots, m$ the number of distinct stars for σ_i is k_i, then the sum*

$$L = \sum_{i=1}^{m} k_i$$

is called the star-length of σ*, and we can write*

$$(**) \quad \sigma = S(\beta_L)S(\beta_{L-1})\dots S(\beta_2)S(\beta_1) = t_k t_{k-1}\dots t_2 t_1 \quad (t_i \in T)$$

For $i = 1, 2, \dots, L$*, let* r_i *denote the number of* T*-transpositions in the expression (*) for* $S(\beta_i)$*, so that* $k = r_1 + \cdots + r_L$*, and let* $n_i = PL(\beta_i) - r_i$*.*

Corollary 3.8. *Using the notation of Definition 3.7,*

$$M(t_k \dots t_1) = \sum_{i=1}^{L} n_i$$

and

$$\text{rank}\, \sigma \leq PL(\sigma) - \sum_{i=1}^{L} n_i \leq PL(\sigma) - L.$$

4. Applications

In this section, we give some applications of the previous results. We return to the normal usage of the word "cycle" for a permutation cycle.

Theorem 4.1. *(a) Suppose that* T *is a path (i.e.* T *has two vertices of degree 1, and all others of degree 2). If* $\sigma \in S_n$*, then the star-factorization of* σ *is minimal.*

(b) Let T *be a star (i.e. one vertex has degree* ≥ 2*, and all others have degree 1). If* $\sigma \in S_n$*, then the star-factorization of* σ *is minimal.*

Proof: (a) Label the vertices of T from left to right, $1, 2, \dots, n$. A σ-star a is a transposition $(a, a+j)$ $(j > 0)$ such that $a \in P(a+j, \sigma)$, $a + j \in P(a, \sigma)$, and if $0 < i < j$, $\sigma(i) = i$. The representation (*) for α uses $2j - 1$ transpositions. It is trivial to check that σ_a has precisely $2j - 1$ fewer inversions than σ does, and since the rank of σ is the inversion number of σ, (a) follows.

(b) If T is a star, the σ-stars are precisely the disjoint cycles of σ, and the star-factorization is the same minimal factorization found in [3].

We now return to an arbitrary tree T with n vertices, and suppose that $\sigma \in S_n$.

Theorem 4.2. *The permutation* σ *is a 2-cycle if and only if rank* σ *is* $PL(\sigma) - 1$*; in this case the star-factorization is minimal.*

Proof: If σ is a 2-cycle, then σ has only two paths of length ≥ 1, and so by Corollary 2.17, rank $\sigma \geq PL(\sigma) - 1$. Since σ is an odd permutation, and $PL(\sigma)$ is even, then rank $\sigma = PL(\sigma) - 1$. For the converse, suppose that σ moves more than two points of T. If σ has more than one σ-star, or if σ has a σ-star that moves more than two points of T, then rank $\sigma < PL(\sigma) - 1$, by Corollary 3.8. Suppose that σ has only one σ-star, say α, which moves only two points of T. Then $\sigma\alpha$ cannot be the identity, since σ moves more than two points, and so $\sigma\alpha$ has a non-trivial star also. In all cases, the star-length of σ is more than one, and the result follows from Corollary 3.8.

4.3. *If σ is a 3-cycle, then rank $\sigma = PL(\sigma) - 2$.*

Proof: By Corollaries 2.17 and 3.8, $PL(\sigma) - 1 \geq$ rank $\sigma \geq PL(\sigma) - 2$. Since σ is an even permutation (or by Theorem 4.2), rank $\sigma = PL(\sigma) - 2$.

4.4. *If σ is a 4-cycle, then rank $\sigma = PL(\sigma) - 3$.*

Proof: By Corollaries 2.17 and 3.8, $PL(\sigma) - 1 \geq$ rank $\sigma \geq PL(\sigma) - 4$. By Theorem 4.2, rank $\sigma < PL(\sigma) - 1$, and since σ is an odd permutation, we must have rank $\sigma = PL(\sigma) - 3$.

Theorem 4.5. *Suppose σ is a product of two disjoint 2-cycles: $\sigma = (a, b)(c, d)$ with a, b, c, d all distinct.*

(i) *If $P(a) \cap P(c)$ does not contain an edge of T, then rank $\sigma = PL(\sigma) - 2$.*
(ii) *If $P(a) \subset P(c)$ (or vice versa), then rank $\sigma = PL(\sigma) - 2$.*
(iii) *Otherwise, rank $\sigma = PL(\sigma) - 4$.*

Proof: In all cases, by Corollaries 2.17 and 3.8, $PL(\sigma) - 2 \geq$ rank $\sigma \geq PL(\sigma) - 4$. The proof of (i) is like Theorem 4.2. (ii) follows from Theorem 4.1 (since in this case, σ is acting on a path); it can also be proved as follows. Suppose that $P(a) \subset P(c)$ are in the same direction . It is easy to show that if $\sigma(x) = x$, then $W(x, \sigma) \leq 2$ implies that x is an endpoint of some edge $[x, y]$ of T which is contained in no more than two of the paths of σ. Thus, if x is a fixed vertex of T between a and b, then we must have $W(x, \sigma) \geq 4$. By Lemma 2.10, $W(a, \sigma) \geq |P(a, \sigma)| + 2$, and similarly for $W(b, \sigma)$; and if x is a vertex of T between a and c, or between b and d, then $W(x, \sigma) \geq 2$. Then applying Corollary 2.4, we have rank $\sigma \geq PL(\sigma) - 2$.

In case (iii), it is always possible to arrange matters so as to have four meetings, by first judiciously "moving" two of the vertices a,b,c,d onto the common interval. We do one example to illustrate the method. Suppose that T has just four outer vertices, a,b,c,d; that $P(a)$ and $P(c)$ are in the same direction, and that $P(a) \cap P(c) = [z, \ldots, w]$ where $z, w \neq a,b,c,d$. Let $C(x, y)$ be defined as in Definition 3.1, and let $S(x, y)$ be the star-factorization of the transposition (x, y) as in Lemma 3.4. Then $\sigma = C(z, a)C(w, d)S(z, w)S(z, c)S(w, b)S(z, w)C(d, w)C(a, z)$ is a T-factorization with precisely $PL(\sigma) - 4$ transpositions. (The $C(a, z)$ on the right has the effect of "moving" a to the vertex z, and so on.)

In general, as in Case (iii) of Theorem 4.5, the star-factorization of σ need not be minimal. It is primarily useful in finding upper bounds for the rank of a permutation, as Theorem 4.10 (below) illustrates. The idea of the proof is quite simple, though the notation gets rather involved. Basically, we consider the set of "stars" in the star-factorization of σ as the vertices of a graph; each "star" has the effect of shortening some of the paths of σ. If one star affects a certain path of σ, the next star in line (reading from right to left of course) that affects the same path, will be declared adjacent to the first star. If such a graph has two or more disjoint connected components, then each component corresponds to a separate set of vertices of T; the permutation would have to have at least as many disjoint cycles as this graph has connected components.

In what follows, we assume we have, in the notation of Theorem 3.6,

$$\sigma = \tau_m \dots \tau_2 \tau_1 = S(\beta_L)S(\beta_{L-1}) \dots S(\beta_2)S(\beta_1).$$

Of course, we can also write σ as the product of cycles, $\sigma = \beta_L \dots \beta_2 \beta_1$, if the T-factorization is not in question.

The first lemma follows immediately from the properties of a star-factorization.

Lemma 4.6. *Let $x \in T$, and let the ordered sequence $\{x = x_0, x_1, \dots, x_L\}$ be defined by $x_i = \beta_i(x_{i-1}) = \beta_i \beta_{i-1} \dots \beta_2 \beta_1(x)$, for $i = 0, 1, 2, \dots, L-1$. Then $d(x_i, x_0) \leq d(x_{i+1}, x_0)$ for all $i = 0, 1, \dots, L-1$, and if k is the least index such that $x_k = \sigma(x)$, then for all $i < k$, $x_i \neq x_k$, and for all $i \geq k$, $x_i = x_k$ (i.e. $\beta_i(x_k) = x_k$).*

Definition 4.7. *We define a graph B with vertex set $V(B) = \{1, 2, \dots, L\}$ and edge set $E(B)$, where the edges (i,j) in $E(B)$ are determined as follows: Let $1 \leq i < j \leq L$. Then $(i,j) \in E(B)$ if and only if there exist $x, y \in T$ such that*

(1) $\beta_i(x) \neq x, \beta_j(x) \neq x$,
(2) $\beta_i \beta_{i-1} \dots \beta_2 \beta_1(y) = x \neq \sigma(y)$,
(3) *if* $i < k < j$, *then* $\beta_k(x) = x$.

Lemma 4.8. *If $\sigma(x) \neq x$, and if $I = \{i | \beta_i(x) \neq x\}$ contains more than one point, then I is a connected subset of B.*

Proof: Suppose that $\sigma(y) = x$, and that I contains more than one point. For the sequence $\{y = y_0, y_1, \dots, y_L\}$ as defined in Lemma 4.6, let t be the least index such that $y_t = x = \sigma(y)$. Then if $k < t$, $y_k \neq y_t$, and if $k \geq t$, $y_k = y_t = x$ and $\beta_k(x) = x$. Thus t is the largest index in I. The members of I can be ordered, say as $i_1 < i_2 < i_3 \cdots < i_r = t$. Suppose that $(i,j) = (i_1, i_2)$. The conditions (1) and (3) are clearly satisfied. If (2) were not satisfied, then we would have (by Lemma 4.6) $\beta_k(x) = x$ for all $k \geq i$. But since I has at least two members, then $j > i$, and $\beta_j(x) \neq x$. So (2) must hold, and (i_1, i_2) is an edge of B. Similarly, $(i_2, i_3), (i_3, i_4), \dots, (i_{r-1}, i_r)$ are all edges of B, and so I is connected.

Corollary 4.9. *If B has k disjoint connected components, then σ must have at least k disjoint cycles.*

Theorem 4.10. *Let σ be a cycle of length k. Then*

$$\text{rank}\, \sigma \leq PL(\sigma) - k + 1.$$

Proof: Write $\sigma = (a_1, a_2, \ldots, a_k)$, and as usual,

$$\begin{aligned}\sigma &= \tau_m \ldots \tau_2 \tau_1 \\ &= S(\beta_L) S(\beta_{L-1}) \ldots S(\beta_2) S(\beta_1) \\ &= \beta_L \beta_{L-1} \ldots \beta_1 .\end{aligned}$$

Let $\sigma_i = \beta_i \beta_{i-1} \ldots \beta_2 \beta_1$. For each $i = 1, 2, \ldots, L$, let K_i be the cycle-length of β_i, and let N_i be defined by:

$$N_i = |\{x \in T | \sigma_{i-1}(z) = x, \beta_i(x) = \sigma(z), x \neq \sigma(z)\}|$$

Then let $r_i = K_i - N_i$, so that r_i is the number of edges from $S(\beta_i)$ to some $S(\beta_j)$ with $j > i$. Since σ is a k-cycle, then B must be connected, and then B has at least $L - 1$ edges. By Lemma 4.6, each of the elements $a_1, a_2, \ldots, a_k$ appears in precisely one of the sets N_i, and so

$$\sum_{i=1}^{L} N_i = k, \text{ and } R = \sum_{i=1}^{L} r_i \geq L - 1.$$

The length of the T-factorization $\sigma = S(\beta_L) S(\beta_{L-1}) \ldots S(\beta_2) S(\beta_1)$ is (by Corollary 3.3)

$$\begin{aligned}PL(\sigma) - \sum_{i=1}^{L} K_i + L &= PL(\sigma) - \sum_{i=1}^{L} (N_i + r_i)) + L \\ &= PL(\sigma) - \sum_{i=1}^{L} N_i + L - \sum_{i=1}^{L} r_i \\ &\leq PL(\sigma) - k + 1.\end{aligned}$$

This completes the proof.

Corollary 4.11. *If σ has t disjoint cycles, and $n - r$ fixed points, then*

$$T\text{-rank}(\sigma) \leq PL(\sigma) - r + t.$$

Proof: Each of the paths of σ is a path of one of its disjoint cycles and conversely, so that $PL(\sigma)$ is the sum of the path-lengths of its disjoint cycles. Then the result follows from Theorem 4.10, and the fact that if $\sigma = \alpha\beta$, then rank$(\sigma) \leq$ rank$(\alpha)+$ rank(β).

5. Conjectures, questions, and examples

For any $\sigma \in S + n$, and a given tree T, we may identify σ with the family of its paths in T. It seems obvious that there must be some strong connections between the algebraic properties of σ, and the geometric behavior of its family of paths (including its T-rank). As usual with geometrical objects, some things seem clear to the intuition, and other things begin to seem very likely after much thought. We list here some open questions, and some examples.

Conjecture 1. If $\sigma \in S_n$, and T is a tree, and $T = A \cup B$, where A and B are disjoint subtrees of T obtained by removing one edge of T, and if $\sigma(A) = A$ and $\sigma(B) = B$, then

$$T\text{-rank}\,(\sigma) = A\text{-rank}\,(\sigma \mid A) + B\text{-rank}\,(\sigma \mid B).$$

Conjecture 2. If $\sigma \in S_n$, and t is a transposition of T, and $PL(\sigma t) = PL(\sigma) - 2$, then $T\text{-rank}(\sigma t) = T\text{-rank}(\sigma) - 1$.

Question 1. For a given T and σ, put $N(\sigma, T) = PL(\sigma) - T\text{-rank}\,(\sigma)$. Is $N(\sigma, T)$ independent of T?

Question 2. Suppose that a is an outer vertex of T, and that b is the unique vertex of T adjacent to a, and that $\sigma(a) = a$. Does there exist a minimal T-factorization of σ in which the transposition (a, b) does not appear? Does there exist a minimal T-factorization of σ in which the transposition (a, b) does appear?

Question 3. Suppose that a is an outer vertex of T, and that b is the unique vertex of T adjacent to a, and that $\sigma(a) \neq a$. Does there exist a minimal T-factorization of σ in which the transposition (a, b) appears exactly twice?

Question 4. Suppose that $[a, b]$ is an edge of T, and $\sigma \in S_n$, and the crossover number $Cr(a, b) = 2k$. Does there exist a minimal T-factorization of σ in which the transposition (a, b) appears exactly k times?

Question 5. Given σ and T, let M be the length of the star-factorization of σ, and let m be the T-rank of σ. Put $K(\sigma, T) = M - m$. What does the value of $K(\sigma, T)$ have to say about σ and T? In particular, given $\sigma \in S_n$, what is the maximum value of $K(\sigma, T)$? (It is easy to see the minimum value is 0.)

Question 6. Given σ and T, let B be the graph associated with the star-factorization of σ (see Section 4). Under what conditions is B connected? When is B itself a tree? If B is connected, or is a tree, what about $K(\sigma, T)$?

Counterexample 1. A permutation σ which is a cycle on the outer vertices of a tree T, and a transposition $t = (a, b)$ where the edge $[a, b]$ is in the interior of T, and $T\text{-rank}(\sigma t) < T\text{-rank}(\sigma)$.

Let T have 7 vertices, and edges: $[a_1, n]$, $[a_2, n]$, $[n, x]$,$[x, m]$,$[m, b_1]$, $[m, b_2]$, and let σ be the 4-cycle (a_1, b_1, a_2, b_2).

a_1 $\qquad\qquad\qquad\qquad$ b_1

$n \quad x \quad m$

a_2 $\qquad\qquad\qquad\qquad$ b_2

Then $PL(\sigma) = 16$, and by Theorem 4.4, T-rank$(\sigma) = 13$. We find the T-factorization

$$(n,x)(n,a_2)(n,x)(m,b_2)(a_1,n)(m,x)(m,b_1)\times$$
$$(n,x)(m,x)(m,b_2)(n,a_2)(n,x)(n,a_1)$$

Since σ fixes n and x, then $(n,x)\sigma$ has rank 12.

We have not found any examples of the following: σ is a cycle on outer vertices of T, and τ moves only elements fixed by σ, and τ is not a transposition of T, and $\tau\sigma t$ has smaller rank than σ.

Counterexample 2. A permutation σ with T-rank$(\sigma) > PL(\sigma) - M(\sigma)$.

In Counterexample 1, the given T-factorization of σ has four meetings (at the transpositions in positions 5,6,7,8, reading from the right). Since σ has just four paths, then no factorization can have more than four meetings, and so $M(\sigma) = 4$.

We have not found any examples where T-rank$(\sigma) - PL(\sigma) + M(\sigma) > 1$.

Counterexample 3. A k-cycle σ such that T-rank$(\sigma) < PL(\sigma) - k + 1$.

Let T be the path $\{1,2,3,4,5\}$, and let σ be the 5-cycle (1,4,2,5,3). Then $PL(\sigma) = 12$, and T-rank$(\sigma) = 6$ (the number of inversions), and $6 < 12 - 5 + 1$.

We thank Frederick Portier for many helpful discussions.

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