The Relationship Between an Edge Colouring Conjecture and A Total Colouring Conjecture

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Abstract. Chetwynd and Hilton made the following *edge-colouring conjecture*: if a simple graph G satisfies $\Delta(G) > \frac{1}{3}|V(G)|$, then G is Class 2 if and only if it contains an overfull subgraph H with $\Delta(H) = \Delta(G)$. They also made the following *total-colouring conjecture*: if a simple graph G satisfies $\Delta(G) \geq \frac{1}{2}(|V(G)| + 1)$, then G is Type 2 if and only if G contains a non-conformable subgraph H with $\Delta(H) = \Delta(G)$. Here we show that if the edge-colouring conjecture is true for graphs of even order satisfying $\Delta(G) > \frac{1}{2}|V(G)|$, then the total-colouring conjecture is true for graphs of odd order satisfying $\delta(G) \geq \frac{3}{4}|V(G)| - \frac{1}{4}$ and def $(G) \geq 2(\Delta(G) - \delta(G) + 1)$.

Introduction

An *edge colouring* of a graph G is a map ϕ: $E(G) \to C$, where C is a set of colours, such that no two edges with the same colour are incident with the same vertex. The *chromatic index* $\chi'(G)$ of G is the least value of $|C|$ for which G has an edge-colouring.

A well-known theorem of Vizing [26] states that, for G a simple graph,

$$\Delta(G) + 1 \geq \chi'(G) \geq \Delta(G),$$

where $\Delta(G)$ is the maximum degree of G. If $\chi'(G) = \Delta(G)$ then G is *Class 1*, and otherwise G is *Class 2*. If

$$|E(G)| > \Delta(G) \left\lfloor \frac{1}{2}|V(G)| \right\rfloor$$

then G is *overfull*. Since no colour can occur on more than $\lfloor\frac{1}{2}|V(G)|\rfloor$ edges, it is clear that if G is overfull, then G is Class 2.

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Chetwynd and Hilton [8] proposed the following edge colouring conjecture (now slightly modified).

Conjecture E: Let G be a simple graph with $\Delta(G) > \frac{1}{3}|V(G)|$. Then G is Class 2 if and only if G contains an overfull subgraph H with $\Delta(H) = \Delta(G)$.

The graph obtained from the Petersen graph by deleting one vertex shows that the figure $\frac{1}{3}$ in Conjecture E cannot be reduced. Conjecture E has been proved in papers by Plantholt [23,24], Chetwynd and Hilton [6,7,8,9,10], and Zhang Jiangxan, Zhang Zhongfu and Wang Jiangfong [27] for the case when $\Delta(G) \geq |V(G)|-3$. It has also been proved by Hilton [19] in the case when $|E(G)| = \Delta(G)\lfloor\frac{1}{2}|V(G)|\rfloor$ and $\Delta(G) \geq \frac{1}{4}(\sqrt{21}-1)(|V(G)|+1)+1$ (note that $\frac{1}{4}(\sqrt{21}-1) \simeq \cdot 896$). Further evidence for Conjecture E is that it implies (see [17]) the following further conjecture.

Conjecture R: Let G be a regular simple graph of degree $d(G)$ and of even order with $d(G) \geq \frac{1}{2}|V(G)|$. Then G is 1-factorizable (i.e. G is Class 1).

Conjecture R itself has been proved by Chetwynd and Hilton [3] in the case when $d(G) \geq \frac{1}{2}(\sqrt{7}-1)|V(G)|$ (note that $\frac{1}{2}(\sqrt{7}-1) \simeq \cdot 823$. All in all, it seems to be fair to say that Conjecture E is a very believable and intuitive conjecture. Various consequences of Conjecture E are given in [14,20,21,22].

A *total colouring* of a graph G is a map $\psi: E(G) \cup V(G) \to C$, such that no two incident elements of $E(G) \cup V(G)$ receive the same colour. The *total chromatic number* $\chi_T(G)$ of G is the least value of $|C|$ for which G has a total colouring.

In 1965 Behzad [1] conjectured that for a simple graph G,

$$\Delta(G) + 1 \leq \chi_T(G) \leq \Delta(G) + 2.$$

The lower bound is trivial, but the upper bound has so far been intractable. If $\chi_T(G) = \Delta(G) + 1$, then G is Type 1, and if $\chi_T(G) = \Delta(G) + 2$, then G is Type 2. So far no simple graphs have been found which are not Type 1 or Type 2.

Let

$$\operatorname{def}(G) = \sum_{v \in V(G)} (\Delta(G) - d_G(v)).$$

A *vertex-colouring* of a graph G is a map $\psi: V(G) \to C$ such that no two adjacent vertices receive the same colour. If G has a vertex colouring with $\Delta(G)+1$ colours such that the number of vertex colour classes of parity different from $|V(G)|$ is at most $\operatorname{def}(G)$, then G is *conformable*. In a total-colouring of G with $\Delta(G) + 1$ colours, for any colour which occurs on a set of vertices of parity different from $|V(G)|$, there is a vertex at which that colour does not occur (neither on the vertex itself, nor on an edge incident with the vertex). It follows that if G is non-conformable, then G isn't Type 1.

Chetwynd and Hilton [5] more recently proposed the following total colouring conjecture.

Conjecture T: Let G be a simple graph with $\Delta(G) \geq \frac{1}{2}(|V(G)| + 1)$. Then G is Type 2 if and only if G contains a non-conformable subgraph H with $\Delta(H) = \Delta(G)$.

The complete bipartite graph $K_{n,n}$ with n even is conformable but is Type 2, and so this shows that the lower bound in Conjecture T cannot be lowered any further (although the idea of Conjecture T can be extended to bipartite graphs—see [5,18]). Conjecture T has been proved in the case when $\Delta(G) = |V(G)| - 1$ [15]. It has also been proved [12] in the case when G is regular, of odd order, with $d(G) \geq \frac{\sqrt{7}}{3}|V(G)|$. These are the only non-trival cases where it has been proved, and so the fact shown in this paper that in certain cases Conjecture E (for which there is much more substantial evidence) implies Conjecture T lends a good deal more credibility to Conjecture T itself. We should note, however, that one counterexample to Conjecture T has been found very recently by Bor Liang Chen and Hung-Lin Fu. This is when $\Delta(G) = |V(G)| - 2, |V(G)|$ even. Possibly it is just an isolated case.

In this paper we prove the following theorem.

Theorem 1. *If Conjecture E is true for simple graphs G of even order satisfying $\Delta(G) > \frac{1}{2}|V(G)|$, then Conjecture T is true for simple graphs G of odd order satisfying $\delta(G) \geq \frac{3}{2}|V(G)| - \Delta(G)$ and $\operatorname{def}(G) \geq 2(\Delta(G) - \delta(G) + 1)$.*

We believe that the restriction $\operatorname{def}(G) \geq 2(\Delta(G) - \delta(G) + 1)$ reflects no more than the inadequacy of our argument. In fact we shall prove the following slight generalization of Theorem 1.

Theorem 2. *Let A be a constant, $\frac{1}{2} \leq A < 1$. If Conjecture E is true for simple graphs G of even order satisfying $\Delta(G) > A|V(G)|$ then Conjecture T is true for simple graphs G of order satisfying*

$$\Delta(G) \geq \frac{1}{2}(1 + A)|V(G)|, \quad \delta(G) \geq \frac{3}{2}|V(G)| - \Delta(G) - \frac{1}{2}$$
$$\textit{and} \quad \operatorname{def}(G) \geq 2(\Delta(G) - \delta(G) + 1).$$

Theorem 1 follows from Theorem 2 since if $A = \frac{1}{2}$ then the condition $\Delta \geq \frac{1}{2}(1 + A)|V(G)|$ becomes $\Delta \geq \frac{3}{4}|V(G)|$, and this is implied by the condition $\delta(G) \geq \frac{3}{2}|V(G)| - \Delta(G)$, since $\Delta(G) \geq \delta(G)$.

Since the necessity in each of the two conjectures is easy, in fact we only need to prove the following lemma.

Lemma 1. *Let A be constant, $\frac{1}{2} \leq A < 1$. Suppose that each Class 2 simple graph G of even order with $\Delta(G) > A|V(G)|$ contains an overfull subgraph H with $\Delta(H) = \Delta(G)$. Then each Type 2 simple graph G of odd order with $\Delta(G) \geq \frac{1}{2}(1 + A)|V(G)|$, $\delta(G) \geq \frac{3}{2}|V(G)| - \Delta(G) - \frac{1}{2}$ and $\operatorname{def}(G) \geq 2(\Delta(G) - \delta(G) + 1)$ contains a non-conformable subgraph H with $\Delta(H) = \Delta(G)$.*

2. Some Useful Lemmas

We give here some lemmas we shall use. The first is a well-known theorem of Dirac [13].

Lemma 2. *Let G be a simple graph whose minimum degree $\delta(G)$ satisfies*

$$\delta(G) \geq \frac{1}{2}|V(G)|.$$

Then G possesses a Hamiltonian circuit.

The next was proved by Chetwynd and Hilton [4].

Lemma 3. *If a simple graph G is overfull, then*

$$\operatorname{def}(G) \leq \Delta(G) - 2.$$

Lemma 4. *Let G be a simple graph with $|V(G)| = 2n+1$ and $\delta(G) \geq n$. Then G contains no overfull subgraph H with $\Delta(H) = \Delta(G)$ and $|V(H)| \leq 2n-1$.*

Proof: Suppose, to the contrary, that G does contain an overfull subgraph H with $\Delta(H) = \Delta(G)$ and $|V(H)| \leq 2n-1$. If $\Delta(H) = \delta(H)$ then H is a component of G. However $\delta(G \backslash H) \leq |V(G \backslash H)| - 1 = |V(G)| - |V(H)| - 1 \leq |V(G)| - (\Delta(H)+1) - 1 = |V(G)| - \Delta(H) - 2 = |V(G)| - \Delta(G) - 2 \leq (2n+1) - n - 2 = n - 1 < \delta(G)$, a contradiction. Hence $\Delta(H) \neq \delta(H)$.

Let $|V(G \backslash H)| = x$. Then $x = |V(G)| - |V(H)| \leq (2n+1) - (\Delta(H)+1) = (2n+1) - (\Delta(G)+1) \leq n$. Thus $x \leq n$. Also $|V(H)| \leq 2n-1$ so $x = |V(G)| - |V(H)| \geq 2$.

By Lemma 3, $\operatorname{def}(H) \leq \Delta(H) - 2$ and so at most $\Delta(H) - 2$ edges join vertices of H to vertices of $G \backslash H$. Therefore the least degree δ^* in G of the vertices of $G \backslash H$ satisfies

$$\delta(G) \leq \delta^* \leq x - 1 + \left\lfloor \frac{\operatorname{def}(H)}{x} \right\rfloor \leq x - 1 + \left\lfloor \frac{\Delta(H) - 2}{x} \right\rfloor.$$

Since $\delta(G) \geq n$ and since $\Delta(H) - 2 \leq |V(H)| - 3 \leq (2n-1) - 3 = 2n-4$, it follows that

$$n \leq x - 1 + \left\lfloor \frac{2n-4}{x} \right\rfloor \leq x - 1 + \frac{2n-4}{x},$$

from which it follows that

$$0 \leq x^2 - (n+1)x + 2n - 4.$$

The roots of $x^2 - (n+1)x + (2n-4)$ are $\frac{1}{2}(n+1) \pm \frac{1}{2}\sqrt{\{n^2 - 6n + 17\}}$, that is

$$\frac{1}{2}(n+1) \pm \frac{1}{2}\sqrt{\{(n-3)^2 + 8\}}.$$

Between the two roots the quadratic $x^2 - (n+1)x + 2n - 4$ is negative, and so it follows that either

$$x \leq \frac{1}{2}(n+1) - \frac{1}{2}\sqrt{\{(n-3)^2 + 8\}},$$

or

$$x \geq \frac{1}{2}(n+1) + \frac{1}{2}\sqrt{\{(n-3)^2 + 8\}}.$$

If $x \leq \frac{1}{2}(n+1) - \frac{1}{2}\sqrt{\{(n-3)^2\}}$, then $x < \frac{1}{2}(n+1) - \frac{1}{2}\sqrt{\{(n-3)^2 + 8\}} = \frac{1}{2}(n+1) - \frac{1}{2}(n-3) = 2$, a contradiction.

If $x \geq \frac{1}{2}(n+1) + \frac{1}{2}\sqrt{\{(n-3)^2 + 8\}}$, then $x > \frac{1}{2}(n+1) + \frac{1}{2}(n-3) = n-1$, so $x \geq n$. Since $n \geq x$ it follows that $x = n$. It follows that $n = x = |V(G)| - |V(H)| \leq 2n + 1 - (\Delta(H) + 1) = 2n + 1 - (\Delta(G) + 1) \leq n$, so that $\Delta(H) = \Delta(G) = \delta(G) = n$. But

$$\delta(G) \leq x - 1 + \left\lfloor \frac{\Delta(H) - 2}{x} \right\rfloor$$

and so we now have that $n \leq x - 1 + \lfloor \frac{n-2}{n} \rfloor$, so that $n \leq x - 1$, a contradiction.

This contradiction shows that, in fact, G contains no such overfull subgraph H, as required. ∎

The lower bound on $\delta(G)$ in Lemma 4 cannot be lowered any further. For if n is odd and G is the simple graph consisting of K_n and a disjoint graph obtained from K_{n+1} by removing a 1-factor, and if H is the K_n, then H is overfull, $|V(G)| = 2n + 1$ and $\delta(G) = \delta(H) = \Delta(G) = \Delta(H) = n - 1$.

Lemma 5. *Let G^* be a simple graph with $|V(G^*)| = 2n + 2$ and $\delta(G^*) \geq n + 1$. Then G^* contains no overfull subgraph H with $\Delta(H) = \Delta(G^*)$ and $|V(H)| \leq 2n - 1$.*

Proof: Suppose G^* contains an overfull subgraph H with $|V(H)| \leq 2n - 1$ and $\Delta(H) = \Delta(G^*)$. Then no vertex of $V(G^*) \backslash V(H)$ is joined to all vertices of degree Δ of H. Let $v \in V(G^*) \backslash V(H)$ and let $G = G^* \backslash \{v\}$. Then $|V(G)| = 2n + 1$, $\delta(G) \geq \delta(G^*) - 1 \geq n$ and $\Delta(G^*) = \Delta(G) = \Delta(H)$. But this contradicts Lemma 4. This contradiction proves Lemma 5. ∎

The lower bound on $\delta(G^*)$ in Lemma 5 can similarly not be lowered any further. For let n be even and let G^* consist of two copies of K_{n+1}, and let H be one of the K_{n+1}'s. Then H is overfull and $\delta(H) = \delta(G^*) = \Delta(H) = \Delta(G^*) = n$.

The restriction $|V(H)| \leq 2n-1$ in Lemma 5 cannot be removed. To see this note that an overfull graph H with $|V(H)| = 2n+1$ and $|E(H)| = \Delta(H)n+1$ has $\operatorname{def}(H) = (2n+1)\Delta - 2(\Delta n+1) = \Delta - 1$. Take any such H with $\Delta(H) \geq n+3$ and $\delta(H) \geq n$, which has x vertices of degree at most $\Delta(H)-1$ for some $x \in \{n+1, \ldots, \Delta(H)-2\}$. Form G^* by joining a further vertex v to each of these x vertices.

Lemma 6. *Let G be a conformable graph of odd order. Then G can be vertex-coloured with $\Delta(G)+1$ colours so that each colour is used on at least one vertex, and not more than* $\operatorname{def}(G)$ *colours are used on an even number of vertices.*

Proof: Since G is conformable and of odd order, G can be vertex-coloured with $\Delta(G)+1$ colours in such a way that the number of colours, say t, which occur on an even number of vertices is at most $\operatorname{def}(G)$. Let the vertex colour classes be $C_1, \ldots, C_{\Delta+1}$ and, for $1 \leq i \leq \Delta(G)+1$, let C_i be coloured with colour c_i.

If some of the colours are not actually used, then $|C_j| = 0$ for some j. In that case we can take some C_i with $|C_i| \geq 2$ and recolour two of its vertices with c_j if $|C_i| > 2$, or recolour one of its vertices if $|C_i| = 2$. In both cases the number of colours actually used increases by one. In the first case t remains the same, and in the second case t is reduced by one. Thus it remains true that $t \leq \operatorname{def}(G)$.

Since G is a simple graph it follows that $|V(G)| \geq \Delta(G)+1$, and so we can always continue with this procedure until all colours are actually used. ∎

3. Proof of Theorem 2

Proof of Lemma 1: Let G be a simple conformable graph of odd order $2n+1$ with $\Delta(G) \geq \frac{1}{2}(1+A)|V(G)|$, $\delta(G) \geq \frac{3}{2}|V(G)| - \Delta(G) - \frac{1}{2}$ and $\operatorname{def}(G) \geq 2(\Delta(G) - \delta(G) + 1)$. We shall assume that the edge-colouring conjecture is true for graphs J of even order satisfying $\Delta(J) > A|V(J)|$, and shall deduce that G is Type 1.

By Lemma 6, we may select a vertex-colouring with $\Delta(G)+1$ colours with the property that each colour is used on at least one vertex and not more than $\operatorname{def}(G)$ colours are used on an even number of vertices. For $1 \leq i \leq \Delta(G)+1$, let the set of vertices of colour c_i be C_i. We may suppose that $|C_1|, \ldots, |C_t|$ are even, and that $|C_{t+1}|, \ldots, |C_{\Delta(G)+1}|$ are odd. From G form a graph G^* by introducing a further vertex v^* and joining v^* to one vertex of each set C_i of odd cardinality, i.e., to one vertex of each of $C_{t+1}, \ldots, C_{\Delta(G)+1}$. Then the degree of v^* is $\Delta(G)+1-t$. The number of vertices not joined to v^* is $|C_1| + \cdots + |C_{s+t}| - s$ and is also $(2n+1) - d_{G^*}(v^*) = 2n - \Delta(G) + t$. It follows that

$$|C_1| + \cdots + |C_{s+t}| - s = 2n - \Delta(G) + t.$$

Let s be the number of C_i's for which $|C_i|$ is odd and greater than one. [The graph G^* is illustrated in Figure 1.]

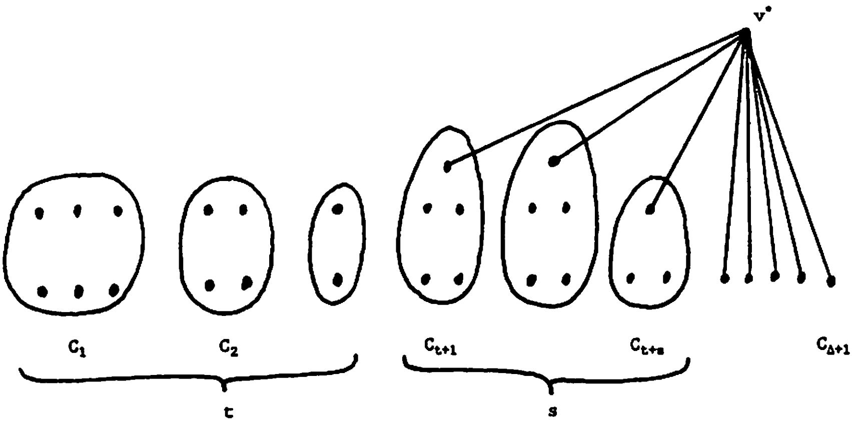

Figure 1. The graph G^*.

Then the number of C_i's of cardinality 1 is at most $|V| - 2t - 3s$ and is exactly $\Delta + 1 - s - t$. Therefore

$$\Delta(G) + 1 - s - t \leq |V(G)| - 2t - 3s$$

so that

$$|V(G)| \geq \Delta(G) + 1 + 2s + t. \tag{1}$$

Since $\Delta(G) \geq \frac{1}{2}(A+1)|V(G)|$ is follows that

$$|V(G)| \geq \frac{1}{2}(A+1)|V(G)| + 2s + t + 1$$

so that

$$2s + t \leq \frac{1}{2}(1 - A)|V(G)| - 1. \tag{2}$$

From (1) it also follows that

$$s + t \leq 2s + t \leq |V(G)| - \Delta(G) - 1. \tag{3}$$

Since $\delta(G) \geq \frac{3}{2}|V(G)| - \Delta(G) - \frac{1}{2}$ it follows that $\delta(G) + \Delta(G) \geq \frac{3}{2}|V(G)| - \frac{1}{2}$, so that $2\Delta(G) \geq \frac{3}{2}|V(G)| - \frac{1}{2}$, so that $\Delta(G) \geq \frac{3}{4}|V(G)| - \frac{1}{4}$. Since $2\Delta(G) \geq \frac{3}{2}|V(G)| - \frac{1}{2}$ it follows that $\frac{1}{2}|V(G)| \geq 2(|V(G)| - \Delta(G)) - \frac{1}{2}$. Therefore

$$\begin{aligned}
\delta(G) + \Delta(G) &\geq |V(G)| + \frac{1}{2}|V(G)| - \frac{1}{2} \\
&\geq |V(G)| + 2(|V(G)| - \Delta(G)) - 1 \\
&\geq |V(G)| + 2(|V(G)| - \Delta(G) - 1) + 1 \\
&\geq |V(G)| + 2s + 2t + 1,
\end{aligned}$$

by (3), and so

$$\delta(G) \geq |V(G)| - \Delta(G) + 2s + 2t + 1. \tag{4}$$

We now find edge-disjoint matchings $F_1, \ldots, F_{s+t}$ such that, in the graph $G^*\backslash(F_1 \cup \cdots \cup F_{i-1})$, for $t+1 \leq i \leq t+s$, F_i contains the edge joining v^* to the unique vertex of C_i joined to v^*, but, apart from that, for $1 \leq i \leq s+t$, no edge of F_i is incident with any vertex of C_i; moreover, for $1 \leq i \leq s+t$, there is an edge of F_i incident with every other vertex of G^*.

Let $C_1^*, \ldots, C_{\Delta+1}^*$ be $C_1, \ldots, C_{\Delta+1}$ reordered so that $|C_1^*| \geq |C_2^*| \geq \cdots \geq |C_{\Delta+1}^*|$, and, for $1 \leq i \leq s+t$ let the matching in $F_1, \ldots, F_{s+t}$ corresponding to C_i^* be denoted by F_i^*.

To see that such matchings exist, suppose that $F_1^*, \ldots, F_{i-1}^*$ have been found, and consider the graph $G^*\backslash(F_1^* \cup \cdots \cup F_{i-1}^*)$. Let

$$G_{i-1}^* = \begin{cases} (G^*\backslash(F_1^* \cup \cdots \cup F_{i-1}^*)) \setminus C_i^* & \text{if } C_i^* \in \{C_1, \ldots, C_t\}, \\ (G^*\backslash(F_1^* \cup \cdots \cup F_{i-1}^*)) \setminus (C_i^* \cup \{v^*\}) & \text{if } C_i^* \in \{C_{t+1}, \ldots, C_{t+s}\}. \end{cases}$$

Then, as we show below,

$$\delta(G_{i-1}^* \geq \frac{1}{2}|V(G_{i-1}^*)|.$$

To see this first note that

$$\delta(G_{i-1}^* \geq \delta(G) - (i-1) - |C_i^*|$$

and that

$$|V(G_{i-1}^*)| = |V(G)| - |C_i^*|.$$

Therefore

$$\delta(G_{i-1}^*) \geq \frac{1}{2}|V(G_{i-1}^*)| \quad \text{if}$$

$$\delta(G) - (i-1) - |C_i^*| \geq \frac{1}{2}|V(G)| - \frac{1}{2}|C_i^*|,$$

i.e.

$$\delta(G) \geq \frac{1}{2}|V(G)| + i - 1 + \frac{1}{2}|C_i^*|.$$

There are $\Delta(G) + 1 - s - t$ C_i^*'s with cardinality 1, so there are

$$|V(G)| - (\Delta(G) + 1 - s - t)$$

vertices in $C_1^* \cup \cdots \cup C_{s+t}^*$. There are at least $2(s+t-i)$ vertices in the sets $C_{i+1}^*, \ldots, C_{s+t}^*$, and so there are at most $|V(G)| - (\Delta(G) + 1 - s - t) - 2(s+t-i) =$

$|V(G)|-\Delta(G)-s-t-1+2i$ vertices in $C_1^*,\ldots,C_i^*$. Since $|C_1^*|\geq\cdots\geq|C_i^*|$, there are at most $\frac{1}{i}\{|V(G)|-\Delta(G)-s-t-1\}+2$ vertices in C_i^*.

Therefore $\delta(G_{i-1}^*)\geq\frac{1}{2}|V(G_{i-1}^*)|$ if

$$\delta(G)\geq\frac{1}{2}|V(G)|+i+\frac{1}{2i}\{|V(G)|-\Delta(G)-s-t-1\}$$

for $1\leq i\leq s+t$, i.e., if

$$0\geq 2i^2-(2\delta(G)-|V(G)|)i+|V(G)|-\Delta(G)-s-t-1.$$

This inequality is satisfied for i between the two roots of the quadratic $2i^2-(2\delta(G)-|V(G)|)i+|V(G)|-\Delta(G)-s-t-1$. These two roots are, writing $\delta=\delta(G),\Delta=\Delta(G)$ and $|V|=|V(G)|$,

$$\frac{1}{4}(2\delta-|V|)+\frac{1}{4}\sqrt{\{(2\delta-|V|)^2-8(|V|-\Delta-s-t-1)\}}$$

and

$$\frac{1}{4}(2\delta-|V|)-\frac{1}{4}\sqrt{\{(2\delta-|V|)^2-8(|V|-\Delta-s-t-1)\}}.$$

Since by (3) $s+t\leq|V|-\Delta-1$ it follows that if

$$|V|-\Delta-1\leq\frac{1}{4}(2\delta-|V|)+\frac{1}{4}\sqrt{\{(2\delta-|V|)^2-8(|V|-\Delta-2)\}}$$

and if $i\leq s+t$ then i is less than the upper root. But since $\Delta+\delta\geq\frac{3}{2}|V|-\frac{1}{2}$ it follows that $|V|-1+\delta\geq\frac{3}{2}|V|-\frac{1}{2}$, so that $\delta\geq\frac{1}{2}|V|+\frac{1}{2}$, so that $2\delta-|V|-1\geq 0$. Moreover $2\delta+\Delta=\delta+(\delta+\Delta)\geq(\frac{1}{2}|V|+\frac{1}{2})+(\frac{3}{2}|V|-\frac{1}{2})=2|V|$. Since $2\Delta+2\delta\geq 3|V|$ it follows that $2|V|-2\Delta\leq 2\delta-|V|$ so that $|V|-\Delta-1\leq\frac{1}{2}(2\delta-|V|)-1$. Therefore

$$\begin{aligned}|V|-\Delta-1&\leq\frac{1}{4}(2\delta-|V|)+\frac{1}{4}(2\delta-|V|-4)\\&=\frac{1}{4}(2\delta-|V|)+\frac{1}{4}\sqrt{\{(2\delta-|V|)^2-8(2\delta-|V|)+16\}}\\&\leq\frac{1}{4}(2\delta-|V|)+\frac{1}{4}\sqrt{\{(2\delta-|V|)^2-8(|V|-\Delta-2)\}}\end{aligned}$$

since $-8(2\delta-|V|)+16\leq-8(|V|-\Delta-2)$, as $2\delta+\Delta\geq 2|V|$. Therefore i is less than the upper root. Similarly the lower root satisfies

$$\begin{aligned}&\frac{1}{4}(2\delta-|V|)-\frac{1}{4}\sqrt{\{2\delta-|V|)^2-8(|V|-\Delta-s-t-1)\}}\\&\leq\frac{1}{4}(2\delta-|V|)-\frac{1}{4}\sqrt{\{2\delta-|V|)^2-8(|V|-\Delta-2)\}}\\&\leq\frac{1}{4}(2\delta-|V|)-\frac{1}{4}\sqrt{\{2\delta-|V|)^2-8(2\delta-|V|)+16\}}\\&=1.\end{aligned}$$

It follows that, for $1 \le i \le s+t$, i lies between the two roots. It follows therefore that, for $1 \le i \le s+t$, $\delta(G^*_{i-1}) \ge \frac{1}{2}|V(G^*_{i-1})|$, as asserted above.

It follows from Lemma 2 (Dirac's theorem) that G^*_{i-1} has a Hamiltonian circuit. For F^*_i we take alternate edges of this circuit; if $F^*_i \in \{F_{t+1}, \ldots, F_{s+t}\}$, we also include the edge from v^* to a vertex of C_i.

Now consider the graph $G^+_{s+t} = G^*\backslash(F_1 \cup \cdots \cup F_{s+t})$. This contains no overfull subgraph of maximum degree $\Delta(G) + 1 - (s+t)$. To see this, suppose first to the contrary that G^+_{s+t} contains an overfull subgraph H of maximum degree $\Delta(G^+_{s+t}) = \Delta(G) + 1 - (s+t)$ with $|V(H)| \le 2n-1$. It has order at least $\Delta(G)+2-(s+t)$. Since, by (4), $\Delta(G)+\delta(G) \ge |V(G)|+2s+2t$, it follows that $2\Delta(G) \ge |V(G)|+2s+2t$, so that $\Delta(G)-s-t \ge \frac{1}{2}|V(G)|$, so that $\Delta(G^+_{s+t}) \ge \frac{1}{2}|V(G)| + 1$. However, by assumption, $\delta(G) + \Delta(G) \ge \frac{3}{2}|V(G)| - \frac{1}{2}$, so $\delta(G) \ge \frac{1}{2}|V(G)| + (|V(G)| - \Delta(G) - 1) + \frac{1}{2} \ge \frac{1}{2}|V(G)| + s + t + \frac{1}{2}$, by (3). Therefore $\delta(G^+_{s+t}) \ge \delta(G) - (s+t-1) \ge \frac{1}{2}|V(G)| + \frac{3}{2}$. But this contradicts Lemma 4. Therefore G^+_{s+t} contains no overfull subgraph H of maximum degree $\Delta(G) + 1 - (s+t)$ with $|V(H)| \le 2n-1$.

Next consider the question of whether G^+_{s+t} contains an overfull subgraph H of maximum degree $\Delta(G) + 1 - (s+t)$ with $|V(H)| = 2n+1$.

Each vertex of G^+_{s+t} has degree at most $\Delta(G) + 1 - s - t$. We have

$$\sum_{v \in V(G)} d_G(v) = (2n+1)\Delta(G) - \operatorname{def}(G)$$

so

$$\begin{aligned}\sum_{v \in V(G^*)} d_{G^*}(v) &= (2n+1)\Delta(G) - \operatorname{def}(G) + 2d_{G^*}(v^*) \\ &= (2n+1)\Delta(G) - \operatorname{def}(G) + 2\Delta(G) + 2 - 2t.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{v \in V(G^+_{s+t})} d_{G^+_{s+t}}(v) &= (2n+3)\Delta(G) - \operatorname{def}(G) + 2 - 2t \\ &\quad - (|C_1| + \cdots + |C_{s+t}| - s)(s+t-1) \\ &\quad - (2n+2 - (|C_1| + \cdots + |C_{s+t}| - s))(s+t) \\ &= (2n+3)\Delta(G) - \operatorname{def}(G) + 2 - 2t \\ &\quad - (2n+2 - (\Delta(G) + 2 - t))(s+t-1) \\ &\quad - (\Delta(G) + 2 - t)(s+t) \\ &= (2n+2)(\Delta(G) + 1 - s - t) - \operatorname{def}(G) - t.\end{aligned}$$

Let w be a vertex of G^*. Then

$$\sum_{v \in V(G^+_{s+t})} d_{G^+_{s+t}}(v) - 2d_{G^+_{s+t}}(w)$$
$$= (2n+2)(\Delta(G) + 1 - s - t) - \operatorname{def}(G) - t - 2(d_{G^*}(w) - (s+t))$$
$$= 2n(\Delta(G) + 1 - s - t) + 2\Delta(G) - \operatorname{def}(G) - t + 2 - 2d_{G^*}(w).$$

Since $d_{G^*}(w) \geq \delta(G)$ and $\operatorname{def}(G) \geq 2(\Delta(G) - \delta(G) + 1)$ it follows that, if $w \neq v^*$, then

$$\sum_{v \in V(G^+_{s+t})} d_{G^+_{s+t}}(v) - 2d_{G^+_{s+t}}(w)$$
$$\leq 2n(\Delta(G) + 1 - s - t).$$

This also follows if $w = v^*$ since $t \leq \operatorname{def}(G)$ and $d_{G^*}(v^*) = \Delta(G) + 1 - t$. Therefore for any $w \in V(G^*)$, $G_{s+t}\backslash w$ is not an overfull subgraph with maximum degree $\Delta(G) + 1 - (s+t)$.

[We interject here that if we could be sure that we could choose t so that $t = \operatorname{def}(G)$, then we could remove the restriction that $\operatorname{def}(G) \geq 2(\Delta(G) - \delta(G) + 1)$; the only place where this restriction is needed is just above to show that $G^+_{s+t}\backslash w$ is not overfull.]

Using (2), it follows that

$$\begin{aligned}
\Delta(G) + 1 - (s+t) &\geq \Delta + 1 - (2s + t) \\
&\geq \Delta + 1 - \frac{1}{2}(1 - A)|V(G)| \\
&\geq \frac{1}{2}(A+1)|V(G)| - \frac{1}{2}(1-A)|V(G)| + 1 \\
&= A|V(G)| + 1 \\
&> A|V(G^*)|,
\end{aligned}$$

and so it follows that $G^+_{s+t} = G^*\backslash(F_1 \cup \cdots \cup F_{s+t})$ can be edge-coloured with $\Delta(G) + 1 - (s+t)$ colours.

Colour the edges of $G^*\backslash(F_1 \cup \cdots \cup F_{s+t})$ with colours $c_{s+t+1}, \ldots, c_{\Delta(G)+1}$. We may suppose that the edge joining the vertex v^* to the single vertex of the set C_i is coloured c_i. From this obtain a total colouring of G with colours $c_1, \ldots, c_{\Delta(G)+1}$ as follows. For $1 \leq i \leq s+t$, colour the vertices of C_i and the edges of $F_i \cap E(G)$ with colour c_i. For $s+t+1 \leq i \leq \Delta(G)+1$ colour the vertex of C_i with colour c_i, and colour those edges of G with c_i which have that colour in the edge colouring of G^*.

This proves Lemma 1.

Theorem 2 and Theorem 1 now follow.

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