Packings in Septuples

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1. Introduction.

We shall be looking at small packings in this paper, using small in the following technical sense. We have v elements and we wish to determine the packing number D(2,k,v). This is the cardinality of the largest family of k-sets chosen from the v elements in such a way that no pair occurs more than once. Saying that v is "small" shall mean that v is less than or equal to $k^2 - k + 1$.

The reason for this restriction is that, if a projective geometry with k points per line does exist, then it provides a perfect packing of all pairs selected from the $v = k^2 - k + 1$ points in exactly $b = k^2 - k + 1$ blocks. For v values that exceed $k^2 - k + 1$, the ordinary Fisher-Yates counting process provides the bound $bk \le Rv$ (R being the maximum replication number for any element in the packing). This bound is, of course, equally valid for $v \le k^2 - k + 1$, but it is usually far from providing an accurate answer; we shall use the weight algorithm described in [7] to provide better bounds.

The values of D(2,k,v) are known for small v when k = 3, 4, 5. 6 (cf. [7]). In this paper, we shall consider the case k = 7. This is particularly interesting, since the BIBD (43, 43, 7, 7, 1) does not exist and so we can not employ the conic bounds used in [7].

We summarize the concept of the weight of a design. Since we only consider packing designs in this paper, we restrict our definitions to that case, although they can be more general (cf. [6]). We define the weight of a block B to be

$$w(B) = (b - 1) - \Sigma(r_i - 1),$$

where the summation is over all elements in the block B. It is easy to see that, in a packing design, w(B) is also equal to x_0, the number of blocks that are disjoint from B, and so is a non-negative quantity. The weight of the whole design is then found by summing w(B) over all blocks and so is

$$w(D) = b(b - 1) - \Sigma r_i(r_i - 1),$$

where the summation is now over all varieties in the design; w(D) is likewise non-negative. It is essential to note that, for a fixed b, the maximum value of w(D) occurs when the frequencies r_i are as nearly equal as possible (cf [4]). So, for a fixed b, we have bk elements in the packing array, and can compute

$$bk = av + t,$$

where $t < v$. Then the design of maximum weight in b blocks will occur when there are t elements of frequency (a + 1) and v - t elements of frequency a. Any change in the frequencies will increase the value of $\Sigma r_i(r_i - 1)$ and consequently will decrease the weight of the design.

2. Packings in Septuples: the Early Cases.

We start with the BIBD (8, 28, 7, 2, 1), which is the unique set of all pairs from 8 elements. Dualize this design to give a packing of 28 elements in 8 blocks of length 7. By Theorem A of [5], this gives the packing number D(2,7,28) = 8. It is useful to write down this design in the following array.

1,2,3,4,5,6,7	1,8,9,10,11,12,13
2,8,14,15,16,17,18	3,9,14,19,20,21,22
4,10,15,19,23,24,25	5,11,16,20,23,26,27
6,12,17,21,24,26, 28	7,13,18,22,25,27,28

This dual array thus gives us

Lemma 1. The packing number D(2,7,28) = 8, and there is a unique packing array.

By stripping away elements from this array, one at a time, we obtain bounds for the packing numbers D(2,7,v) for $7 \le v \le 27$. It is easy to show that these bounds are met by simply computing the weight of a design that contains one more block. For example, stripping away elements 27 and 28 shows that D(2,7,26) is at least 5; if one could obtain a packing in 6 blocks, it would contain 42 elements and the packing array of maximum weight would contain 16 elements of frequency 2, 10 of frequency 1. But this maximal-weight array would have w(D) = 6(5) - 16(2) - 10(0) < 0; hence it does not exist.

These results can be summarized in

Lemma 2. D(2,7,v) = 6 for v = 27; D(2,7,v) = 5 for v = 26 and 25; D(2,7v) = 4 for v = 24, 23, 22; D(2,7,v) = 3 for v = 21, 20, 19, 18; D(2,7,v) = 2 for v = 17, 16, 15, 14, 13; D(2,7,v) = 1 for v = 12, 11, 10, 9, 8, 7.

Of course the procedure just illustrated works for all values of k, not just k = 7. So we can really restrict ourselves to the interesting cases which are those in which v ranges from $v = 1 + (k+1)k/2$ up to $k^2 - k + 1$.

3. Septuple Packings for v between 29 and 37.

The next natural place to start building septuple packings is at v = 35. Here we know that the triple system (15, 35, 7, 3, 1) exists and so we again apply Theorem A to give the result stated in Lemma 3.

Lemma 3. The packing number D(2,7,35) = 15 and there are 80 distinct packings obtained by dualizing the 80 triple systems on 15 elements.

This packing has 35 elements of frequency 3; delete one of these and we get a packing of 34 elements in 12 blocks. If there were a packing possible in 13 blocks, then the maximal weight would be w(D) = 13(12) - 23(6) - 11(2) < 0. Hence there is no packing in 13 blocks and we have

Lemma 4. D(2,7,34) = 12.

Lemma 4 is a special case of Theorem B from [5].

Now consider v = 33. A packing in 12 blocks would have negative weight, but the maximal-weight packing in 11 blocks has weight 0 and contains 11 elements of frequency 3, 22 of frequency 2. Each block must contain 3 elements of frequency 3, 4 elements of frequency 2. The design is readily constructed by cycling modulo 11 on the initial block $(1_1,2_1,4_1,1_2,5_2,1_3,6_3)$. Thus we have

Lemma 5. D(2,7,33) = 11.

For v = 32, a design in 11 blocks would have negative weight. For 10 blocks, a design of maximal weight has w(D) = 2 and contains 6 elements of frequency 3, 26 of frequency 2. By Balance Lemma 2 [4], we could also have a design with weight zero; it would contain either an element of frequency 4, 4 of frequency 3, 27 of frequency 2; or it would contain 7 elements of frequency 3, 24 of frequency 2, one of frequency 1. We construct the first of these designs of weight zero.

There must be 4 blocks containing the element of frequency 4, 6 blocks containing 2 elements of frequency 3 (all other elements in the blocks have frequency 2). Hence we specify the first blocks as (1,2,3,4,5,6,7), (1,8,9,10,11,12,13), (1,14,15,16,17,18,19), (1,20,21,22,23,24,25). The other 6 blocks can be taken as starting with (2,8,14,20), (3,9,15,21), (4,10,16,22), (5,11,17,23), (6,12,18,24), (7,13,19,25). Then it is easy to complete these 6 blocks by filling in with 6 triples from the blocks of a Fano geometry on elements 26, 27, ..., 32. Thus we have

Lemma 6. D(2,7,32) = 10.

For v = 31, a design in 10 blocks has negative weight. So there can be at most 9 blocks. We first look at the case v = 30 and note that a design in 9 blocks with v = 30 has 3 elements of frequency 3, 27 of frequency 2, weight zero. So every block contains exactly one element of frequency 3. Such a design is easily written down as:

1,2,3,4,5,6,7	1,8,9,10,11,12,13	1,14,15,16,17,18,19
29,2,8,14,20,21,22	29,3,9,15,23,24,25	29,4,10,16,26,27,28
30,5,11,17,20,23,26	30,6,12,18,21,24,27	30,7,13,19,22,25,28

This array, together with our remark on design weights, proves the result of

Lemma 7. D(2,7,30) = D(2,7,31) = 9.

Finally, we look at the case v = 29. A design in 9 blocks would have maximum weight 9(8) - 5(6) - 24(2) < 0. So the design on 28 elements can be used and we have

Lemma 8. D(2,7,29) = 8.

4. The Case v = 36.

If we try 17 blocks, we get a design of negative weight. So we try b = 16; then the maximum weight is 240 - 4(12) - 32(6) = 0, and so we must search for a design with 4 elements of frequency 4, 32 of frequency 3. Each block meets every other block and each block contains one element of frequency 4.

Consequently, we really need a design in 32 blocks of six such that the blocks split into 4 partial-resolution classes, the 4 blocks of each class being disjoint. This can be achieved as follows.

Call the element $1_k, 2_k, \ldots, 8_k$, where k ranges from 1 to 4. We reserve eight positions in a block for 2 elements from each k-class. Clearly 2 of these positions must be empty. We also use R to designate symbols 1,2,3,4 and S to designate symbols 5,6,7,8. Then we may write down the blocks according to the following schema.

R	R	R	-
S	S	S	-
R	S	-	R
S	R	-	S
R	-	S	S
S	-	R	R
-	R	S	R
-	S	R	S

This array merely uses the fact that the elements that do not appear with one element of frequency 4 must appear with the other 3 elements of frequency 4. Now we are able to fill in the R and S positions since there are exactly 3 one-factors on 4 elements. We may thus replace the R and S letters by these 1-factors to give

1,2	1,2	1,2	-
3,4	3,4	3,4	-
5,6	5,6	5,6	-
7,8	7,8	7,8	-
1,3	5,7	-	1,2
2,4	6,8	-	3,4
5,7	1,3	-	5,6
6,8	2,4	-	7,8

1,4	-	5,7	5,7
2,3	-	6,8	6,8
5,8	-	1,3	1,3
6,7	-	2,4	2,4
-	1,4	5,8	1,4
-	2,3	6,7	2,3
-	5,8	1,4	5,8
-	6,7	2,3	6,7

Note that elements in column i (i running from 1 to 4) must be given the subscript i. This array establishes the result of

Lemma 9. D(2,7,36) = 16.

5. The Case v = 37.

We immediately find that a packing in 18 blocks would have negative weight; so we try 17 blocks and find that the maximal-weight packing has weight 2. It contains 8 elements of frequency 4, 29 of frequency 3. Using Balance Lemma 2 from [4], we see that there are also 2 possible packing of weight zero. The first of these would have 9 elements of frequency 4, 27 of frequency 3, one of frequency 2; the second would have one element of frequency 5, 6 of frequency 4, and 30 of frequency 3. We consider this second possibility.

Since the weight of each block is zero, the dual of this packing is a PBD on 17 elements with one block of length 5, 6 of length 4, and 30 of length 3. Each element occurs with frequency 7. Thus the 5 elements from the long block must occur 6 times each in the triples and we may delete them to leave a design on 12 elements that consists of 30 pairs, arranged in 5 sets of one-factors, as well as 4 quadruples (each element occurs twice in the quadruples). The quadruples are uniquely determined by this information (they are merely the dual of K_6 - e, where e is a 1-factor of K_6). We may write them as: (1,2,3,4), (5,6,7,8), (1,5,9,10), (2,6,11,12), (3,7,9,11), (4,8,10,12). It remains to be see whether the 30 missing pairs from elements 1,2,3, ...,11,12, can be arranged in five 1-factors. This may be done as follows:

1,6	2,10	3,5	4,7	9,12	8,11
1,7	2,8	3,12	4,5	6,9	10,11
1,8	2,9	3,6	4,11	5,12	7,10
1,11	2,5	3,10	4,6	7,12	8,9
1,12	2,7	3,8	4,9	5,11	6,10

We thus have

Lemma 10. D(2,7,37) = 17.

6. The Case v = 38.

For v = 38, the weight bound is 19 and a design in 19 blocks has weight zero. It must contain 19 elements of frequency 3 and 19 elements of frequency 4. Since each block has weight zero, it contains 4 elements of frequiency 4, 3 of frequency 3. These facts suggest a cyclic solution; one is easily obtained by cycling (mod 19) the initial block $(1_1,2_1,4_1,9_1,1_2,5_2,11_2)$. Thus we have

Lemma 11. D(2,7,38) = 19.

7. Remarks on the Cases 39 ≤ v ≤ 43.

Of course, the packing numbers in this range would be easy to obtain using the conic bound if only there existed a BIBD (43,43,7,7,1). We use the (r,λ) design (r = 7, λ = 1) given in [2] to supply some information about D(2,7,v) in the range 39 ≤ v < 43.

This design, due to McCarthy, contains 3 blocks of length 7, 9 of length 5, 20 of length 4, 9 of length 3, 2 of length 1. By dualizing this design, we find that D(2,7,43) ≥ 25. The fact (cf. [2]) that a (7,1) design can not have v > 28 can not be used to give a bound on the packing, since the dual of a packing is only a (7,1) design if the packing has weight zero. So we may state

Lemma 12. 25 ≤ D(2,7,43).

Delete the element x that occurs in both the blocks of length 1 in the McCarthy design; the result is a (7,1) design on 24 varieties in 41 blocks. Dualize this design to establish that D(2,7,41) ≥ 23.

Alternatively, we might note that the packing number D(2,4,23) = 40, and the packing contains 22 elements of frequency 7, one element of frequency 6. By adding a block that consists of this single element and then dualizing the design, we find that D(2,7,41) ≥ 23. Thus we have

Lemma 13. 23 ≤ D(2,7,41) ≤ D(2,7,42).

The result for 42 varieties follows from noting that D(2,4,24) = 42 and that the (2,4, 24) packing is a (7,1) design; so dualizing shows that D(2,7,42) ≥ 24. This gives us

Lemma 14. 24 ≤ D(2,7,42) ≤ D(2,7,43).

Now we take the PBD on 22 elements with 35 blocks of length 4, one block B of length 7 (cf [1]). Let the block of length 7 be abcdefg. All elements occur 7 times except the 7 elements of B, which occur 6 times each. Now select a block C = (a123) and replace B and C by (123), (abc), (ade), (afg), (bdf), (ceg). This produces a (7,1) design on 22 elements in 40 blocks. Alternatively, we might simply replace block B by abc, ade, bdf, ceg, fg. By dualizing either of these designs, we see that D(2,7,40) ≥ 22. On the other hand, the weight bound for 40 shows that D(2,7,40) ≤ 25. We state this result as

Lemma 15. $22 \leq D(2,7,40) \leq 25$.

For v = 39, the weight bound is 22. Any packing in 22 blocks of maximal weight would have weight 6, 37 elements of frequency 4, 2 elements of frequency 3. By using Balance Lemma 2 from [4], we see that the possibilities for packings are limited to those in the following table.

Frequency			6	5	4	3	2
Weight	6			37	2		
	4			38		1	
	4		1	35	3		
	2		1	36	1	1	
	2		2	33	4		
	0		2	34	2	1	
	0	1	0	34	3		
	0		2	34	1	1	
	0		3	31	5		

The second case may be rejected, by looking at the dual, since D(2,4,22) = 37 (cf. [1]). All other cases, save the first and the last, are easily rejected by noting that any element of frequency > 4 can not occur in a block without having at least one companion element of frequency < 4 (the weight of any block ≥ 0). But this produces a repeated pair, and so we have

Lemma 16. Any (2,7,39) packing in 22 blocks either consists of 37 element of frequency 4 and 2 elements of frequency 3, or it consists of 3 elements of frequency 5, 5 elements of frequency 3, and 31 elements of frequency 2; in the second case, the dual is a PBD.

Since we know that D(2,7,38) = 19, we have

Lemma 17. $19 \leq D(2,7,39) \leq 22$.

8. Conclusion.

We summarize the results of this paper in the following table of v versus D, where D = D(2,7,v).

v	D	v	D	v	D
7-12	1	29	8	36	16
13-17	2	30	9	37	17
18-21	3	31	9	38	19
22-24	4	32	10	39	19-22
25-26	5	33	11	40	22-25
27	6	34	12	41	23-D(42)
28	8	35	15	42	24-D(43)
				43	25≤ D(43)

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