The Quasigroup Variety Arising from a 2-Perfect 6-Cycle System of Order 13

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Abstract. In this paper we give a partial answer to a query of Lindner concerning the quasigroups arising from 2-perfect 6-cycle systems.

1. Introduction

The problem of finding Steiner triple systems on n points can be regarded as the problem of partitioning the complete graph on n vertices into edge-disjoint 3-cycles. An obvious generalisation of this is the problem of partitioning the complete graph into edge-disjoint cycles of length k. It is possible to use the resulting design to define a groupoid (as can be done for Steiner triple systems) but this is not necessarily a quasigroup. It is never possible to obtain a quasigroup when $k = 4$, but for larger k a quasigroup can be defined provided a certain additional condition is satisfied [2, 3, 5]. Systems satisfying this extra condition are known to exist for $k = 5, 6, 7$ and certain other odd values of k [3]. In the cases 5 and 7 the resulting quasigroups form a variety, but it is not known whether this is true in the case of $k = 6$. The quasigroups arising in the case of $k = 6$ satisfy certain conditions, one of which is:

$$(x \circ y = y \circ x) \rightarrow (x = y).$$

This is the only condition which is not a law and, in [5], Lindner asks if it can be replaced by a finite set of laws. In this paper we show that in the case of the 2-perfect 6-cycle system of order 13 of Lindner, Phelps, and Rodger [4], this quasi-identity can indeed be replaced by a law. We also note that the variety generated by the quasigroup arising from this 6-cycle system has a finite basis for its laws (as indeed do those arising from various other 2-perfect 6-cycle systems).

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2. Definitions and Preliminaries

Definition 2.1. A *6-cycle system* of order n is a decomposition of the complete graph, K_n, into a collection of edge-disjoint 6-cycles.

We use $(v_1, v_2, v_3, v_4, v_5, v_6)$ to represent the 6-cycle containing edges $v_1 v_2$, $v_2 v_3, \ldots, v_5 v_6, v_6 v_1$.

Definition 2.2. A 6-cycle system is said to be *2-perfect* if the set of 3-cycles formed by taking edges between vertices which are distance two apart in the 6-cycles is also an edge-disjoint decomposition of K_n.

So if $(v_1, v_2, v_3, v_4, v_5, v_6)$ is a 6-cycle of the system then (v_1, v_3, v_5) and (v_2, v_4, v_6) are in the set of 3-cycles formed by taking edges between points distance two apart in the 6-cycles. If this distance-2 graph is indeed a partition of the edges of K_n, then it is a Steiner triple system of order n (Lindner [5]).

It is known that necessary and sufficient conditions for the existence of a 2-perfect 6-cycle system of order n are that $n \equiv 1$ or 9 mod 12 and $n > 9$, [5]. Hence the smallest possible order of such a system is 13. The existence of a 2-perfect 6-cycle system of order 13 is shown by example in [4]. Here we concentrate on this particular 2-perfect 6-cycle system which we shall denote by C_{13}.

Being 2-perfect is a necessary and sufficient condition on a 6-cycle system in order that a quasigroup may be defined from it in the following manner [5]. Define the binary operation $\circ$ on the vertex set V of K_n by:

(i) $(\forall v \in V)(v \circ v = v)$;
(ii) $(\forall u, v \in V, u \neq v)(v \circ u = z$ and $u \circ v = w)$ if and only if (u, v, w, s, t, z) is a 6-cycle of the system, for some s and t.

Such a quasigroup satisfies the laws:

(i) $(\forall x \in V)(x \circ x = x)$;
(ii) $(\forall x, y \in V)((x \circ y) \circ y = x)$;
(iii) $(\forall x, y \in V)((x \circ y) \circ (y \circ [x \circ y]) = x \circ (y \circ x))$; and the quasi-identity
(iv) $(\forall x, y \in V)((x \circ y = y \circ x) \rightarrow (x = y))$ (antisymmetry).

Moreover, any quasigroup satisfying these conditions yields a 2-perfect 6-cycle system (Lindner [5]). We shall refer to the laws (i), (ii), and (iii) as *the standard laws.*

We also need a few definitions from the theory of universal algebra.

Definition 2.3. An algebra is *simple* if it has no proper non-trivial homomorphic image, (equivalently, no non-trivial congruences).

Definition 2.4. A variety of algebras is said to be *congruence permutable* if all algebras in the variety have permutable congruences.

Definition 2.5. An algebra is said to be *para-primal* if it is finite, generates a congruence permutable variety and is such that each of its non-trivial subalgebras is simple.

The quasigroup arising from the system C_{13} will be denoted by $\widehat{C}_{13}$.

3. The simplicity of $\widehat{C}_{13}$

In this section we prove the simplicity of the quasigroup $\widehat{C}_{13}$ using the technique described below. This technique depends on the facts that if one has a proper homomorphic image of any algebra, then there must be two distinct elements in the original algebra whose homomorphic images are equal, and that any homomorphic image must obey the *laws* that hold in the original quasigroup (although not necessarily, of course, the antisymmetry condition). So let us consider a 6-cycle in C_{13}.

Lemma 3.1. *If any two of the elements in the set* $\{x, y, x \circ y, y \circ (x \circ y), x \circ (y \circ x), y \circ x\}$, *other than the pairs* $\{x, y \circ (x \circ y)\}$, $\{y, x \circ (y \circ x)\}$ *and* $\{x \circ y, y \circ x\}$ *(that is, the diametrically opposite pairs), are equated, then all elements in the set become equal.*

Proof: The result follows from the fact that $x \circ x = x$ is a standard law.

(1) If $x = y$ then $x \circ y = x \circ x = x$, $y \circ (x \circ y) = x \circ (x \circ x) = x \circ x = x$, etc.
(2) If $x = x \circ y$, then $x \circ x = x \circ y$, so, since we have a quasigroup, $x = y$, and we are back in case (1).
(3) If $x = y \circ x$, then, as above, we have $x = y$.
(4) If $x = x \circ (y \circ x)$, then $x \circ x = x \circ (y \circ x)$ so $x = y \circ x$ and we are back in case (3).

Since any adjacent vertices can play the rôles of x and y, this covers all possible cases. ∎

The significance of the above result is two-fold:

(1) If it can be shown that equating any arbitrary pair of diametrically opposite vertices causes the whole quasigroup to collapse, then it has no proper non-trivial homomorphic images.
(2) If, in the course of our calculation, we have equality of two elements in a hexagon which are not diametrically opposite, then we know immediately that the whole hexagon collapses.

In the example given in [4] of a 2-perfect 6-cycle system of order 13, K_{13} has vertex set $\mathbf{Z}_{13}$ and the collection of 6-cycles is defined as follows: $C_{13} = \{(0,5,2,8,7,9) + i \mid 0 \leq i \leq 12\}$ where $(v_1, v_2, v_3, v_4, v_5, v_6) + i = (v_1 + i, v_2 + i, \ldots, v_6 + i)$ with each component reduced modulo 13.

Consider this system. From the way in which the cycles are generated, it is clear that it is sufficient to check the three pairs $\{0,8\}$, $\{5,7\}$ and $\{2,9\}$ from the first cycle, $(0,5,2,8,7,9)$, to determine whether or not it is simple.

$\{0,8\}$: These points are not diametrically opposite in the cycles

$$(8,0,10,3,2,4) \qquad \text{and} \qquad (6,11,8,1,0,2)$$

so these both collapse, forcing the collapse of the first cycle as well. We now have equality of all elements except 12, but any cycle containing 12, such as $(3,8,5,11,10,12)$, will collapse as well.

{5,7}: These points are not diametrically opposite in the cycles

$$(11,3,0,6,5,7) \qquad \text{and} \qquad (5,10,7,0,12,1)$$

so these both collapse, forcing the collapse of the first cycle as well. We now have equality of all elements except 4, but any cycle containing 4, such as $(2,7,4,10,9,11)$, will collapse as well.

{2,9}: These points are not diametrically opposite in the cycles

$$(2,7,4,10,9,11) \qquad \text{and} \qquad (7,12,9,2,1,3)$$

so these both collapse, forcing the collapse of the first cycle as well. We now have equality of all elements except 6, but again any cycle containing 6, such as $(6,11,8,1,0,2)$, will collapse as well.

Since $\widehat{C}_{13}$ has no non-trivial homomorphic images, all quasigroups in the variety generated by C_{13} must be antisymmetric, and so must satisfy laws forcing antisymmetry, although not necessarily finitely many. This is also true of the quasigroups arising from many other 2-perfect 6-cycle systems including C_{21}, C_{25} and C_{37} of [4].

4. The existence of a finite basis

In this section we show that, at least in the case of $\widehat{C}_{13}$, there must be a finite set of laws implying antisymmetry, since the variety generated by $\widehat{C}_{13}$ has a finite basis for its laws.

Lemma 4.1. *The variety generated by the quasigroup arising from a 2-perfect 6-cycle system has permutable congruences.*

Proof: It is known ([7]) that the existence of a Mal'cev polynomial $p(x,y,z)$ with the properties:

(1) $p(x,x,z) = z$
(2) $p(x,z,z) = x$

implies that congruences permute.

In the quasigroups arising from 2-perfect 6-cycle systems the Mal'cev polynomial can be taken to be $p(x,y,z) = y \circ [z\backslash x]$ where $z\backslash x$ is the element such that $z \circ [z\backslash x] = x$. From the definition, $p(x,z,z) = z \circ [z\backslash x] = x$. So it remains to show that $p(x,x,z) = x \circ [z\backslash x] = z$. Let $z\backslash x = a$; then $z \circ a = x$ and so $x \circ a = [z \circ a] \circ a = z$ by standard law (ii) as required. ∎

Theorem 4.2. *The variety generated by $\widehat{C}_{13}$ is finitely based.*

Proof: MacKenzie [6] proves that a congruence permutable variety generated by a finite set of para-primal algebras has a finite basis.

Here $\widehat{C}_{13}$ is a finite algebra which is para-primal since it has no proper non-trivial subalgebras, so its variety satisfies MacKenzie's conditions and is finitely based. ∎

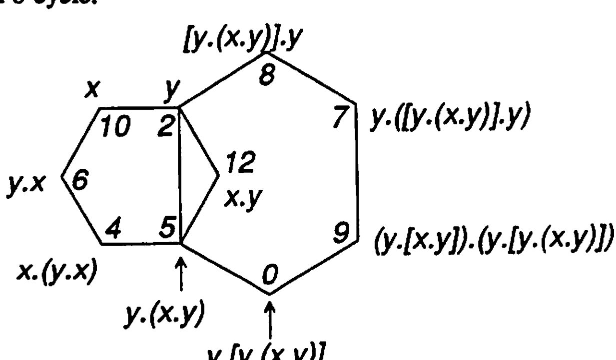
This is also true of the varieties generated by the quasigroups arising from C_{21}, C_{25} and C_{37} of [4].

5. The search for a law implying antisymmetry

The variety defined by the two-variable laws of any variety is also the variety of all algebras whose two-generator subalgebras belong to the original variety (see Neumann [5, p. 21]). Since a quasigroup, all of whose two-generator sub-quasigroups arise from 2-perfect 6-cycle systems, clearly itself arises from such a system (since the defining conditions are all two-variable), if there are laws in a quasigroup arising from a 2-perfect 6-cycle system which define antisymmetry, then they must be two-variable.

Having decided that the varieties generated by the quasigroups arising from certain 2-perfect 6-cycle systems must be finitely based, the authors began a search for two-variable laws in these quasigroups. The method used depends upon the fact that since each edge belongs to a unique 6-cycle, two vertices determine a unique 6-cycle in which they are adjacent.

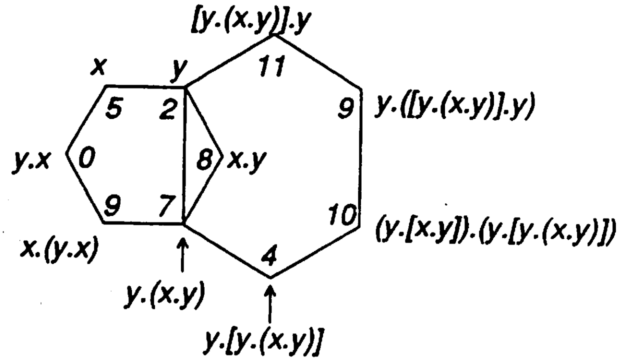
We start with any 6-cycle from C_{13} and we label the vertices $(x, y, x\circ y, y\circ[x\circ y], x\circ[y\circ x], y\circ x)$ as well as with the elements of $\mathbf{Z}_{13}$. Choose two non-adjacent vertices in this 6-cycle. This pair is adjacent in a unique 6-cycle which 'overlaps' the original 6-cycle.



Example 5.1

Here 5 and 2 are the non-adjacent pair chosen from the first 6-cycle, so the 'overlapping' 6-cycle is $(2,5,0,9,7,8)$. The labels in terms of x and y are derived in the obvious way from the labels on 2 and 5.

Since a 6-cycle system on 13 points has so few vertices it does not take many 6-cycles to obtain a repeated numerical label. Although one might hope that such a repetition would yield a law, this does not work in general.

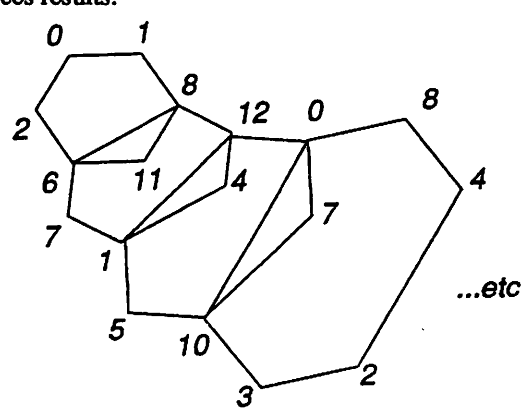


Example 5.2

Since the label 9 occurs in both 6-cycles one might hypothesise the law $x \circ (y \circ x) = y \circ ([y \circ (x \circ y)] \circ y)$ but from Example 5.1 it can be seen that this equation is not always satisfied.

The repetition of an entire 6-cycle is more promising; however, even this does not always produce a law.

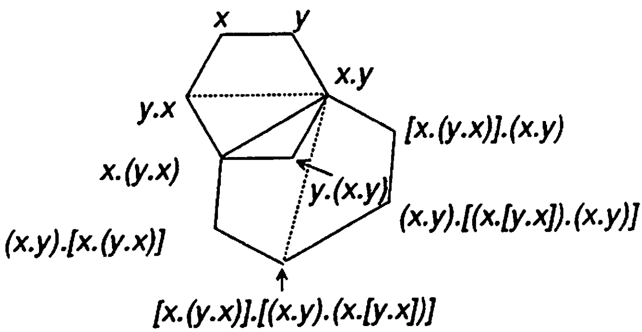
The cyclic nature of the system(s) under consideration suggests the use of a pattern in the selection of the 'overlapping' 6-cycle to be chosen at each stage. This often produces results.



Example 5.3

This pattern produces Law (a) below. In C_{13} the first 6-cycle always repeats on the tenth 6-cycle. (This pattern also produces laws in other systems).

The purpose of seeking these laws was to obtain laws implying antisymmetry, so we want the substitution $x \circ y = y \circ x$ to reduce the law to $x = y$. Consider the following diagram.



When $x \circ y = y \circ x$ the vertex of the 'overlapping' 6-cycle which has as its label $[x \circ (y \circ x)] \circ [(x \circ y) \circ (x \circ [y \circ x])]$ is also equal to $y \circ x$. This can be verified as follows:

Let $x \circ y = y \circ x = z$; then

$$\begin{aligned} [x \circ (y \circ x)] \circ [(x \circ y) \circ (x \circ [y \circ x])] &= [x \circ z] \circ [z \circ (x \circ z)] \\ &= x \circ [z \circ x] \qquad \text{(by standard law (iii))} \\ &= x \circ [(y \circ x) \circ x] \\ &= x \circ y \qquad \text{(by standard law (ii))}. \end{aligned}$$

The dotted lines in the diagram indicate which words in x and y become equal when the substitution $x \circ y = y \circ x$ is made.

The 'overlapping' of 6-cycles, in a way that makes it possible to equate diametrically opposite vertices in successive 6-cycles when $x \circ y = y \circ x$, produces laws which can be simplified when the substitution $x \circ y = y \circ x$ is made. Unfortunately, in the majority of cases, being able to simplify the law when $x \circ y = y \circ x$ does not reduce it to $x = y$, but to $x = x$ or a similar tautology.

Example 5.4 produces Law (b) below and reduces to $x = x$ when $x \circ y = y \circ x$.

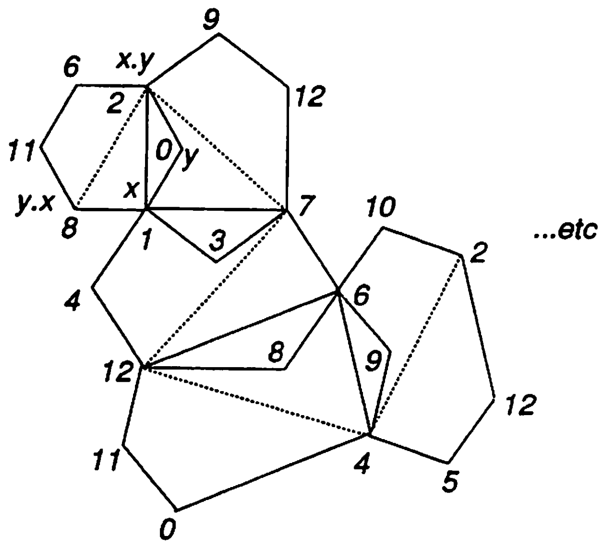
The search conducted using this method was fruitful, and many two-variable laws have been found for the quasigroup $\widehat{C}_{13}$ other than the standard laws mentioned earlier. The following laws were amongst those we found:

Law (a):

$$y = [[b(pb)]([bp][b(pb)])][[b(pb)](bp)],$$

where

$$b = [a(na)][(an)(a[na])]$$
$$p = [a(na)][an]$$
$$a = [z(mz)][(zm)(z[mz])]$$
$$n = [z(mz)][zm]$$
$$z = [x(yx)][(xy)(x[yx])]$$
$$m = [x(yx)][xy].$$



Example 5.4

Law (b):

$$x = g(g[[g(fg)]g]),$$

where

$$f = d[[d([d(cd)]d)]d],$$
$$g = d[d([d(cd)]d)],$$
$$c = b[[b([b(ab)]b)]b],$$
$$d = b[b([b(ab)]b)],$$
$$a = v[[v([v(zv)]v)]v],$$
$$b = v[v([v(zv)]v)],$$
$$z = w[[w([w(uw)]w)]w],$$
$$v = w[w([w(uw)]w)],$$
$$w = x[x[(xy)x]],$$
$$u = x[(x[(xy)x])x].$$

Law (c):

$$xy = \{[(yx)(xy)][(xy)(yx)]\}\{[(xy)(yx)][(yx)(xy)]\}.$$

6. The antisymmetry law

Theorem 6.1. *A law which implies antisymmetry in the variety generated by* $\widehat{C}_{13}$ *is:*

$$c = d([[y(xy)]a]d),$$

where

$$d = [[y(xy)]a][b([y(xy)]b)],$$
$$c = [b([y(xy)]b)][[y(xy)]a],$$
$$b = [y(xy)][(yx)[y(yx)]],$$
$$a = [(yx)[y(yx)]][y(xy)].$$

Proof: In order to show that this law implies antisymmetry, we make the substitution $x \circ y = y \circ x$. Then:

$$a = [(x \circ y) \circ (y \circ [x \circ y])] \circ [y \circ (x \circ y)]$$
$$= x \circ y \qquad \text{(by standard law (ii))};$$
$$b = [y \circ (x \circ y)] \circ [(x \circ y) \circ (y \circ [x \circ y])]$$
$$= [y \circ (x \circ y)] \circ [x \circ (y \circ x)]$$
$$= y \circ x \qquad \text{(by standard laws (ii) and (iii))};$$
$$c = [(y \circ x) \circ ([y \circ (x \circ y)] \circ [y \circ x])] \circ [(y \circ [x \circ y]) \circ (x \circ y)]$$
$$= y \circ x \qquad \text{(by standard law (ii))};$$
$$d = [(y \circ [x \circ y]) \circ (x \circ y)] \circ [(y \circ x) \circ ([y \circ (x \circ y)] \circ [y \circ x])]$$
$$= y \circ [(y \circ x) \circ y]$$
$$= y \circ x.$$

So the right hand side is

$$[y \circ x] \circ [([y \circ (x \circ y)] \circ [x \circ y]) \circ (y \circ x)]$$
$$= [y \circ x] \circ [y \circ (y \circ x)]$$
$$= x \circ (y \circ x).$$

Hence the law reduces to $x \circ y = x \circ (y \circ x)$, but since $x \circ x = x$ we have

$$x \circ y = x \circ (y \circ x) \quad \text{implies} \quad (x \circ y) \circ (x \circ y) = x \circ (x \circ y) \quad \text{and so} \quad x \circ y = x.$$

Hence $x \circ y = x \circ x$ and thus $y = x$. ∎

We have also found laws for the quasigroups arising from C_{25} and C_{37} of [4], and from the other cyclic 2-perfect 6-cycle system of order 13 (see Gower [1]), but as yet we do not have laws implying antisymmetry for these.

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