# A note on connected domination critical graphs

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**Abstract**

Let $\gamma_c(G)$ denote the connected domination number of the graph $G$. A graph $G$ is said to be connected domination edge critical, or simply $\gamma_c$-critical, if $\gamma_c(G+e) < \gamma_c(G)$ for each edge $e \in E(\overline{G})$. We answer a question posed by Zhao and Cao concerning $\gamma_c$-critical graphs with maximum diameter.

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## 1 Introduction

A set $S \subseteq V(G)$ of a graph $G$ is a *dominating set* if every vertex not in $S$ is adjacent to a vertex in $S$. The *domination number* $\gamma(G)$ is the minimum cardinality of all dominating sets. A *connected dominating set* $S$ in a graph $G$ is a subset $S$ of $V(G)$ such that the induced subgraph $\langle S \rangle$ is connected and $S$ dominates $G$. Every connected graph $G$ has a connected dominating set, since $S = V(G)$ is such a set. The *connected domination number* $\gamma_c(G)$ is the minimum cardinality of all connected dominating sets of $G$. A dominating set of $G$ of cardinality $\gamma(G)$ is called a $\gamma(G)$*-set*, while a connected dominating set of $G$ of cardinality $\gamma_c(G)$ is called a $\gamma_c(G)$*-set*.

The *open neighborhood* of $v$ is the set $N(v) = \{u \in V \mid uv \in E\}$. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v \in S} N(v)$. The domination-related concepts not defined here can be found in [?].

A graph $G$ is *connected domination edge critical*, or just *$\gamma_c$-critical*, if $\gamma_c(G+e) < \gamma_c(G)$ for any edge $e \in E(\overline{G}) \neq \emptyset$. For non-adjacent vertices $u$ and $v$ in $G$, if $S$ is a connected dominating set of $G+uv$, we will denote $S$ by $S_{uv}$.

It is also shown in [?], and we restate it here for emphasis, that the addition of an edge to a graph can change the connected domination number by at most two.

**Theorem 1** [?] *For any edge $e \in E(\overline{G})$,*

$$\gamma_c(G) - 2 \leq \gamma_c(G+e) \leq \gamma_c(G).$$

Chen, Sun, and Ma [?] characterized the $\gamma_c$-critical graphs $G$ with $\gamma_c(G) = 2$. They showed that no tree with order $n \geq 3$ is $\gamma_c$-critical. We give the following stronger result for $\gamma_c$-critical graphs $G$ with $\gamma_c(G) \geq 3$.

**Theorem 2** *If $G$ is a $\gamma_c$-critical graph and $\gamma_c(G) \geq 3$, then $G$ has at most one endvertex.*

**Proof.** Suppose $G$ is a $\gamma_c$-critical graph with two endvertices $u$ and $v$. If $u$ and $v$ have a common support vertex $w$, then it is easily seen that $G$ is not $\gamma_c$-critical, by considering $G+uv$, where $\gamma_c(G) = \gamma_c(G+uv)$. Let $u'$ and $v'$ be the support vertices of $u$ and $v$, respectively. Note that $u'$ and $v'$ is in every $\gamma_c$-set $S$ of $G$, and there is a $u'-v'$ path in $G$. Consider $S_{uv}$ and assume that $u, v \in S_{uv}$. Assume, without loss of generality, that $u' \in S_{uv}$. Suppose that $v' \in S_{uv}$. Since $|S_{uv}| < S$, we have $|S_{uv} \setminus \{u,v\}| \leq |S| - 3$, and $S' = S_{uv} \setminus \{u,v\}$ dominates $G$ but is not connected in $G$, otherwise $\gamma_c(G) < |S|$. Note that since $S'$ is not connected, it has exactly two components $C_1$ and $C_2$, with $u' \in C_1$ and $v' \in C_2$. Since $V(G) \setminus \{u,v\}$ is connected, there is a vertex $x \in N(C_1)$, and a vertex $y \in N(C_2)$ such that $xy \in E(G)$. Then $S' \cup \{x,y\}$ is connected and dominates $G$, and we have $|S' \cup \{x,y\}| = |S_{uv}| < |S|$, a contradiction. Hence $v' \notin S_{uv}$, and $v \in S_{uv}$ only to dominate $v'$. Note that $N(v') \setminus \{v\} \neq \emptyset$, and $(N(v') \setminus \{v\}) \cap S_{uv} = \emptyset$. Let $x \in N(v')$, and $x \neq v$. Then $S_{uv} \cup \{x\} \setminus \{v\}$ is connected and dominates $G$. But $|S_{uv} \cup \{x\} \setminus \{v\}| = |S_{uv}| < |S|$, a contradiction. Hence, not both of $u$ and $v$ are in $S_{uv}$. Therefore assume, without loss of generality, that $u \in S_{uv}$ and $v \notin S_{uv}$. Then $u \in S_{uv}$ dominates $v$, and $v' \notin S_{uv}$, and there exists $x \in N(v')$ such that $x \in S_{uv}$. Then $S_{uv} \setminus \{u\} \cup \{x\}$ is connected and dominates $G$. But $|S_{uv} \setminus \{u\} \cup \{x\}| = |S_{uv}| < |S|$, a contradiction. □

## 2 $\gamma_c$-critical graphs with maximum diameter

Zhao and Cao [?] gave the following diameter result.

**Theorem 3** [?] *If $G$ is a $k_c$-critical graph, then the diameter of $G$ is at most $k$ and this bound is sharp.*

As an example, they construct a class of $k_c$-critical graphs $G_{k-2}$ with diameter $k$ as follows. $V(G_{k-2}) = \{a_i, b_j, c_j : 0 \leq i \leq k-2, 1 \leq j \leq 2\}$ and $E(G_{k-2}) = \{a_i a_{i+1}, a_{k-2}b_1, a_{k-2}c_1, b_1 b_2, b_1 c_1, b_2 c_2, c_1 c_2 : 0 \leq i \leq k-3\}$. (See Figure ??.)

They pose the following question: Does every $k_c$-critical graph with diameter $k$ have the graph in Figure ?? as an induced subgraph?

We answer their question by providing a class of graphs with the necessary properties that does not contain the graph in Figure ?? as an induced subgraph. We construct a $k_c$-critical graph $G_k$ with $diam(G_k) = k$. Let $V(G_k) = \{v_i, x, y | \ 0 \leq i \leq k\}$, and $E(G_k) = \{v_i v_{i+1}, x v_{k-3}, x v_{k-2}, xy, y v_k | \ 0 \leq \ i \ \leq k-1\}$. (See Figure ??.)

**Proposition 4** *The graph $G_k$ is $k_c$-critical with diameter $k$, for $k \geq 4$.*

**Proof.** Since $v_i \in D$, $1 \leq i \leq k-3$, for every connected dominating set $D$, and at least three of the remaining vertices, $v_{k-2}$, $v_{k-1}$, $v_k$, $x$, and $y$, are required to form a connected dominating set, $\gamma_c(G_k) \geq (k-3)+3 = k$. The set $C = \{v_i | \ 1 \leq i \leq k\}$ is a connected dominating set of $G_k$. Thus, $\gamma_c(G_k) \leq k$, and so $\gamma_c(G_k) = k$.

We now show that $G_k$ is $k_c$-critical. For non-adjacent vertices $u$ and $v$, let $S_{uv}$ be a connected dominating set of $G_k + uv$. Consider $S_{v_0 v_i}$, for $2 \leq i \leq k$. Here $S_{v_0 v_i} = \{v_2, \ldots, v_k\}$ is a connected dominating set with $|S_{v_0 v_i}| < k$. For $S_{v_0 x}$, and $S_{v_0 y}$, $\{v_2, \ldots, v_{k-3}, x, y, v_k\}$ is a connected dominating set with cardinality less than $k$. Now consider $S_{v_i v_j}$, with $1 \leq i \leq k-2$, $3 \leq j \leq k$, and $i \leq j-2$. Here, $S_{v_i v_j} = \{v_l | \ 1 \leq l \leq k, l \neq j-1\}$ is a connected dominating set with $|S_{v_i v_j}| < k$. By the symmetry of $G_k$, relabel $x$ and $y$ as $v_{k-2}$ and $v_{k-1}$, respectively, and then use the same argument as above to show $|S_{v_i x}| < k$ and $|S_{v_i y}| < k$, whenever $1 \leq i \leq k-3$. For $S_{v_{k-2} y}$, $\{v_i | \ 1 \leq i \leq k-2\} \cup \{y\}$ is a connected dominating set with cardinality less than $k$. For $S_{x v_{k-1}}$, $\{v_i | \ 1 \leq i \leq k-3\} \cup \{x, v_{k-1}\}$ is a connected dominating set of cardinality less than $k$. Finally, for $S_{v_{k-1} y}$, $\{v_i | \ 1 \leq i \leq k-1\}$ is a connected dominating set of cardinality less than $k$.

Hence, $G_k$ is $k_c$-critical, and by our construction, $\text{diam}(G_k) = k$. □

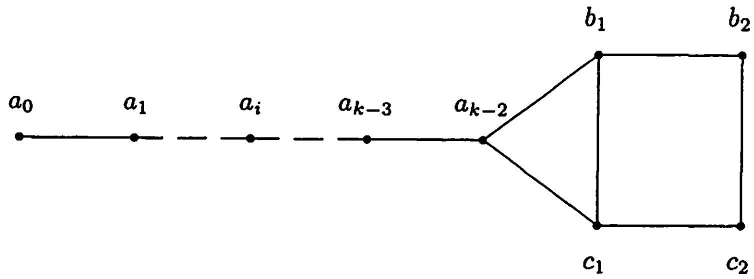


Figure 1: $k_c$-critical graph $G_{k-2}$ with diameter $k$

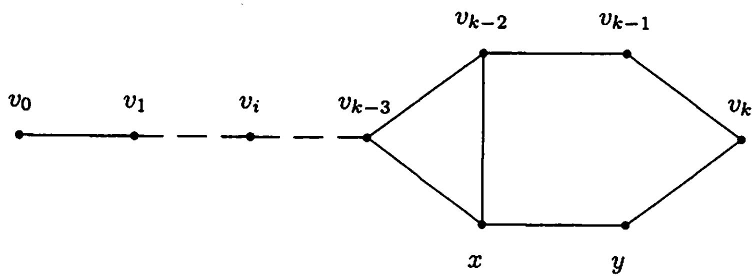


Figure 2: $k_c$-critical graph $G_k$ with diameter $k$

**Theorem 5** *If $G$ is a $k_c$-critical graph with diameter $k \geq 4$ and minimum order, then $G$ is the graph in Figure ?? or the graph in Figure ??.*

**Proof.** Let $G$ be a $k_c$-critical graph with diameter $k \geq 4$ and with minimum order. Suppose that $|G| < k + 2$. Let $P_k = v_0, v_1, \ldots, v_k$ be a diametrical path of $G$. If $G = P_k$, then $\gamma_c(G) < k$, a contradiction, hence $|G| \geq k + 2$. Suppose then that $|G| = k + 2$, and let $y$ be the additional vertex not on $P_k$. If $y$ is adjacent to an interior vertex of $P_k$, then $\gamma_c(G) < k$. Assume, without loss of generality, that $y$ is adjacent to $v_k$. Since $\text{diam}(G) = k \geq 4$, $y$ is not adjacent to $v_0$. But then $G = P_{k+1}$, and hence not $k_c$-critical. Thus $|G| \geq k + 3$.

Assume that $|G| = k + 3$, with two additional vertices $x$ and $y$ not on $P_k$. If both $x$ and $y$ are adjacent to interior vertices of $P_k$, then $\gamma_c(G) < k$, a contradiction. Assume, without loss of generality, that $y$ is not adjacent to any interior vertex of $P_k$. We consider two cases. Either $y$ is adjacent to one of $v_0$ or $v_k$, or $y$ is not adjacent to any vertex of $P_k$.

Suppose first the latter case, that $y$ is not adjacent to any vertex of $P_k$. Then $y$ is adjacent to $x$. Because $G$ has at most one endvertex, both $v_0$ and $v_k$ have degree at least 2. As a consequence, we must have $x$ adjacent to both $v_0$ and $v_k$, contradicting our assumption that $\text{dist}(v_0, v_k) = k \geq 4$.

Assume then, without loss of generality, that $y$ is adjacent to $v_k$. Since $y$ is not adjacent to any other vertex of $P_k$ and since $\text{dist}(v_0, y) \leq k$, $y$ is adjacent to $x$. Thus, there exists a path, containing $x$, from $v_0$ to $y$ of length at most $k$. Hence, $x$ is adjacent to some vertex of $P_k$. If $x$ is adjacent to $v_i$ on $P_k$, $i \leq k-4$, then $\text{dist}(v_0, v_k) < k$, a contradiction. This implies that $x$ is adjacent to some of the vertices $v_j$ on $P_k$, where $k-3 \leq j \leq k$.

Suppose $x$ is adjacent to $v_k$. If $x$ is not adjacent to an interior vertex of $P_k$, then $\text{diam}(G) > k$, a contradiction. If $x$ is adjacent to $v_{k-3}$, then $\text{dist}(v_0, v_k) < k$, also a contradiction. If $x$ is adjacent to $v_{k-2}$, then $\{v_1, \ldots, v_{k-2}, x\}$ is a connected dominating set with cardinality less than $k$, a contradiction. Suppose that $x$ is adjacent to $v_{k-1}$ and consider $G + v_0y$, which requires at least $k$ connected vertices to be dominated. Hence, we have $\gamma_c(G + v_0y) = \gamma_c(G)$, which contradicts the assumption that $G$ is $\gamma_c$-critical. Consequently, $x$ is not adjacent to $v_k$. Also, $x$ is not adjacent to all three vertices $v_{k-3}$, $v_{k-2}$, and $v_{k-1}$, since then $\{v_1, \ldots, v_{k-3}, y, x\}$ is a connected dominating set of cardinality less than $k$.

Now consider the cases where $x$ is adjacent to exactly one of the vertices $v_{k-3}$, $v_{k-2}$, or $v_{k-1}$. Assume $x$ is adjacent to $v_{k-1}$. Then $\gamma_c(G + v_0y) \geq k$, implying that $G$ is not $\gamma_c$-critical. If $x$ is adjacent to $v_{k-2}$, then $\gamma_c(G + xv_{k-1}) = k$ (See Figure ??), implying that $G$ is not $\gamma_c$-critical. If $x$ is adjacent to $v_{k-3}$, then $\gamma_c(G + xv_{k-2}) = k$ (See Figure ??), implying that $G$ is not $\gamma_c$-critical.

It follows that $x$ is adjacent to exactly two of the vertices $v_{k-3}$, $v_{k-2}$, and $v_{k-2}$. If $x$ is adjacent to $v_{k-3}$ and $v_{k-1}$, then $\{v_1, \ldots, v_{k-3}, x, y\}$ is a connected dominating set of cardinality less than $k$. If $x$ is adjacent to $v_{k-3}$ and $v_{k-2}$, then $G$ is the graph in Figure ??. If $x$ is adjacent to $v_{k-2}$ and $v_{k-1}$, then $G$ is the graph in Figure ??. □