Vertex-equalized Edge-colorings

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Abstract

We define a new fairness notion on edge-colorings, requiring that the number of vertices in the subgraphs induced by the edges of each color are within one of each other. Given a (not necessarily proper) k-edge-coloring of a graph G, for each color $i \in \mathbb{Z}_k$ let $G[i]$ denote the (not necessarily spanning) subgraph of G induced by the edges colored i. Let $\nu_i(G) = |V(G[i])|$. Formally, a k-edge-coloring of a graph G is said to be *vertex-equalized* if for each pair of colors $i, j \in \mathbb{Z}_k$, $|\nu_i(G) - \nu_j(G)| \leq 1$. In this paper a characterization is found for connected graphs that have vertex-equalized k-edge-colorings for each $k \in \{2,3\}$ (see Corollary 4.1 and Corollary 4.2).

1 Introduction

Fairness notions in edge-colorings of graphs have been extensively studied. In what follows, a graph G is called *even* if all vertices of G have even degree. Given a (not necessarily proper) k-edge-coloring of a graph G, for each color $i \in \mathbb{Z}_k$ let $G(i)$ denote the spanning subgraph of G with edge set equal to the edges colored i, and let $G[i]$ be the (not necessarily spanning) subgraph induced by the edges colored i. Then a k-edge-coloring of G is called an *even k-edge-coloring* if for each color $i \in \mathbb{Z}_k$, $G(i)$ is an even graph. A k-edge-coloring of G is said to be *equitable* if for each vertex $v \in V(G)$ and for each pair of colors $i, j \in \mathbb{Z}_k$, $|deg_{G(i)}(v) - deg_{G(j)}(v)| \in \{0, 1\}$. Moreover, a k-edge-coloring of G is said to be *evenly-equitable* if

(i) for each color $i \in \mathbb{Z}_k$, $G(i)$ is an even graph, and

(ii) for each vertex $v \in V(G)$ and for each pair of colors $i, j \in \mathbb{Z}_k$, $|deg_{G(i)}(v) - deg_{G(j)}(v)| \in \{0, 2\}$.

A k-edge-coloring of G is said to be *balanced* if the edges between each pair of vertices are shared as fairly as possible among the k colors. A k-edge-coloring of G is called *equalized* if G contains $\lfloor |E(G)|/k \rfloor$ or $\lceil |E(G)|/k \rceil$ edges of each of the k colors.

In 1970's de Werra studied these special types of edge-colorings for bipartite graphs. Due to his work in [1, 2, 3, 4] it is known that for each $k \in \mathbb{N}$ every bipartite graph has a k-edge-coloring that is balanced, equitable and equalized at the same time. Several other results exist for more general graphs. In particular, Hilton proved in [6] that each even graph has an evenly-equitable k-edge-coloring for each $k \in \mathbb{N}$, thereby completely settling this problem. The existence of equitable k-edge-colorings is much more problematic, and very unlikely to be completely solved. For example, settling the existence of equitable Δ-edge-colorings is equivalent to classifying the Class 1 graphs (see [9, 10] for example). One general result on this topic was found by Hilton and de Werra [8] who proved that if $k \geq 2$ and G is a simple graph such that no vertex in G has degree equal to a multiple of k, then G has an equitable k-edge-coloring. More recently, Zhang and Liu [11] extended this result by proving that for each $k \geq 2$, if the subgraph of G induced by the vertices with degree divisible by k is a forest, then G has an equitable k-edge-coloring, thereby verifying a conjecture made by Hilton in [7].

In this paper we consider a new fairness notion, requiring that the number of vertices in the subgraphs induced by the edges of each color are within one of each other. Given a k-edge-coloring of a graph G, for each color $i \in \mathbb{Z}_k$ let $G[i]$ denote the (not necessarily spanning) subgraph of G induced by the edges colored i. Let $\nu_i(G) = |V(G[i])|$. Formally, a k-edge-coloring of a graph G is said to be *vertex-equalized* if for each pair of colors $i, j \in \mathbb{Z}_k$, $|\nu_i(G) - \nu_j(G)| \leq 1$. In this paper a characterization is found for connected graphs that have vertex-equalized k-edge-colorings for each $k \in \{2, 3\}$ (see Theorems 2 and 3).

If H is edge-colored with colors in $\mathbb{Z}_k$ then define $m(H)$ to be a color $c \in \mathbb{Z}_k$ for which $\nu_c(H) \leq \nu_{c'}(H)$ for all $c' \in \mathbb{Z}_k$. Throughout the paper S_i denotes the star with i edges and $i+1$ vertices (so S_i is the same as the complete bipartite graph $K_{1,i}$).

The following lemma will be very useful in proving the main results.

Lemma 1. *Each non-empty connected graph has a spanning subgraph that is a union of vertex-disjoint non-empty stars.*

Proof. Let G be a non-empty connected graph, and T be a spanning tree of G. Let H be formed from T by greedily removing the middle edge in any path of length 3 until no 3-path remains. Then clearly each component is a star and $\delta(H) \geq 1$ since removing a middle edge never creates a vertex of degree 0. □

2 Vertex-equalized 2-edge-colorings

Theorem 2. *Suppose G is a connected simple graph. Then G has a vertex-equalized 2-edge-coloring if and only if $G \neq K_2$.*

Proof. It is clear that K_2 has no vertex-equalized 2-edge-colorings. To prove sufficiency, assume that $G \neq K_2$. If G is empty, then the result is trivial; otherwise by Lemma 1, G has a spanning subgraph H consisting of vertex-disjoint non-empty stars. Form a non-decreasing ordering $(G_1, G_2, ..., G_s)$ of the components in H with respect to the number of edges in each component. Then form an ordering $(e'_1, e'_2, ..., e'_t)$ of the edges of H where if $e'_i \in G_k$, $e'_j \in G_l$ and $i < j$, then $k \leq l$. Alternately color these edges with 0 and 1. Suppose that in H the number of stars with exactly one edge is even. This procedure clearly yields a vertex-equalized 2-edge-coloring of H. If in H the number of stars with a single edge is odd, then $G_1 \cong K_2$, its edge e' is colored 0, and $\nu_0(H) \in \{\nu_1(H)+1, \nu_1(H)+2\}$. Also, since $G \neq K_2$, $s \geq 2$ (that is, G_2 exists). G is connected, so there must be an edge $e \neq e'$ incident with a vertex in G_1. Color e with 1. This gives a vertex-equalized 2-edge-coloring of $H+e$.

$$\text{Let } H_0 = \begin{cases} H & \text{if the number of stars in } H \text{ with a single edge is even} \\ H+e & \text{if the number of stars in } H \text{ with a single edge is odd.} \end{cases}$$

Now the vertex-equalized 2-edge-coloring of H_0 can be completed to a vertex-equalized 2-edge-coloring of G as follows. Let $E(G) \setminus E(H_0) = \bigcup_{i=1}^{p} e_i$ where $e_i = \{x_i, y_i\}$. For each k where $1 \leq k \leq p$, let $H_k = H_{k-1} + e_k$. Then for $1 \leq i \leq p$, if for some $c \in \{0, 1\}$ both x_i and y_i in H_{i-1} are incident with c then color e_i with c; otherwise color e_i with $m(H_{i-1})$. This gives a vertex-equalized 2-edge-coloring of G. □

3 Vertex-equalized 3-edge-colorings

Theorem 3. *Suppose G is a connected simple graph. Then G has a vertex-equalized 3-edge-coloring if and only if $G \neq K_2, S_2$.*

Proof. It is clear that K_2 and S_2 have no vertex-equalized 3-edge-colorings. To prove sufficiency, assume that $G \neq K_2, S_2$. If G is empty, then the result is trivial; otherwise by Lemma 1, G has a spanning subgraph H consisting of vertex-disjoint non-empty stars. We begin by coloring the edges in H together with at most two edges in $G - E(H)$, considering five cases in turn. In H let $a \in \mathbb{N}$ be the number of S_1's, and $b \in \mathbb{N}$ be the number of S_2's. Let $m = min\{a, b\}$. Properly edge-color the $3m$ edges in m of the S_1's and m of the S_2's with m edges of each color. For each $i \in \mathbb{Z}_3$, color with i all edges in $\lfloor (a-m)/3 \rfloor$ of the uncolored S_1's and all edges in $\lfloor (b-m)/3 \rfloor$ of the uncolored S_2's. The components that are left uncolored in H are all S_i's with $i \geq 3$, along with exactly one of the following

(i) one K_2 and no S_2's,

(ii) two K_2's and no S_2's,

(iii) one S_2 and no K_2's

(iv) two S_2's and no K_2's, or

(v) no other components.

Let L be the subgraph of H consisting of the uncolored components. Form a non-decreasing ordering $(L_1, L_2, ..., L_s)$ of the components in L with respect to the number of edges in each component. Then form an ordering $(e'_1, e'_2, ..., e'_t)$ of the edges of L where if $e'_i \in L_k$ and $e'_j \in L_l$ with $i < j$ then $k \leq l$.

Suppose we are in case (i); so $L_1 \cong K_2$, its edge being e'_1. If $s \geq 2$ then $L_2 \cong S_i$ where $i \geq 3$, in which case a vertex-equalized 3-edge-coloring of L can be produced by coloring e'_1 with 0, e'_2 with 1, e'_3 with 1, e'_4 with 2, and for $5 \leq k \leq t$ coloring e'_k with k (modulo 3). So now we can assume $s = 1$; so in H there is no component isomorphic to S_i where $i \geq 3$. If in H there is a component isomorphic to S_2, then $m \geq 1$ and so H contains 3 components $L_1, H' \cong S_2$ and $H'' \cong K_2$, such that currently in H' one edge is colored 1 and the other edge is colored 2, and in H'' the only edge is colored 0. Produce a vertex-equalized 3-edge-coloring of L by coloring e'_1 with 0, recoloring the edge in H'' with 1, and recoloring both edges in H' with 2. Finally suppose that in H there is no component isomorphic to S_i where $i \geq 2$; so $s = 1$ and $m = 0$. Then since $G \neq K_2$, in H there exist four components L_1, H', H'', H''', each isomorphic to K_2, such that currently the edge in H' is colored 0, the edge in H'' is colored 1, and the edge in H''' is colored 2. Since G is connected, there are at least two edges $e, e' \neq e'_1$ in G incident with a vertex in $V(L_1 \cup H')$. Color e'_1 with 0, e with 1, and e' with 2. This 3-edge-coloring of $L + \{e, e'\}$ is vertex-equalized.

In case (ii) $L_1, L_2 \cong K_2$, and $E(L_1) = \{e'_1\}$, $E(L_2) = \{e'_2\}$. Color e'_1 with 0, and e'_2 with 1. Since G is connected, there must be an edge $e \notin E(L_1 \cup L_2)$ incident with at least one vertex in $L_1 \cup L_2$. Color e with 2. For $3 \leq k \leq t$, color e'_k with $k-1$ (modulo 3). (In fact, thinking recursively, as $e'_3, \ldots, e'_t$ are colored in turn, the resulting partial edge-coloring of G is vertex-equalized.)

In case (iii) $L_1 \cong S_2$, and $E(L_1) = \{e'_1, e'_2\}$. Color e'_1 with 0, and e'_2 with 1. Since G is connected and $G \neq S_2$, there must be an edge $e \notin E(L_1)$ incident with at least one vertex in L_1. Color e with 2. For $3 \leq k \leq t$, color e'_k with $k - 1$ (modulo 3) to produce a vertex-equalized 3-edge-coloring of $L + e$.

In case (iv) $L_1, L_2 \cong S_2$, and $E(L_1) = \{e'_1, e'_2\}$, $E(L_2) = \{e'_3, e'_4\}$. Color e'_1 and e'_2 with 0, e'_3 with 1, and e'_4 with 2. For $5 \leq k \leq t$, color e'_k with $k - 1$ (modulo 3).

In case (v) for $1 \leq k \leq t$, color e'_k with $k - 1$ (modulo 3).

It is important to note that in each of the above cases a vertex-equalized 3-edge-coloring of a spanning subgraph H_0 of G has been found. Now the vertex-equalized 3-edge-coloring of H_0 can be completed to a vertex-equalized 3-edge-coloring of G. Let $E(G) \setminus E(H_0) = \{e_i \mid 1 \leq i \leq p\}$ where $e_i = \{x_i, y_i\}$. For each i where $1 \leq i \leq p$, let $H_i = H_{i-1} + e_i$ and recursively (inductively) color the remaining uncolored edges to produce a vertex-equalized 3-edge-coloring of G as follows. For $1 \leq i \leq p$, assuming that H_{i-1} has a vertex-equalized 3-edge-coloring in which $\nu_0(H_{i-1}) \geq \nu_1(H_{i-1}) \geq \nu_2(H_{i-1})$ (rename colors if necessary), one of the following statements holds:

(i) $\nu_0(H_{i-1}) = \nu_1(H_{i-1}) = \nu_2(H_{i-1})$,

(ii) $\nu_0(H_{i-1}) = \nu_1(H_{i-1}) = \nu_2(H_{i-1}) + 1$,

(iii) $\nu_0(H_{i-1}) = \nu_1(H_{i-1}) + 1 = \nu_2(H_{i-1}) + 1$.

In case (i) color e_i with c where c is any color occurring on an edge in H_{i-1} incident with x_i. In case (ii) color e_i with 2. In case (iii): color e_i with 1 if there is an edge colored 1 in H_{i-1} incident with x_i or y_i; otherwise color e_i with 2 if there is an edge colored 2 in H_{i-1} incident with x_i or y_i; and if e_i is still uncolored then color it with 0 (note that in this case each of x_i and y_i must be incident with edges colored 0 in H_{i-1}).

□

4 Further Remarks

Companion results for Theorem 2 and Theorem 3 follow easily for the case when G is connected, but not necessarily simple. In this section, it is assumed that edges join two distinct vertices; so loops are not described as edges.

Theorem 4. *Suppose G is a connected graph (possibly with loops and multiple edges) such that the underlying simple graph G_u has a vertex-equalized k-edge-coloring. Then G has a vertex-equalized k-edge-coloring.*

Proof. For each multiple edge $e = \{u, v\}$ in G, color e with $c \in \mathbb{Z}_k$ if $\{u, v\}$ in G_u is colored c. For each loop l at a vertex w, color l with $c \in \mathbb{Z}_k$ if c is the color of an edge in G_u that is incident with w. □

Corollary 4.1. *Suppose G is a connected graph (possibly with loops and multiple edges). Then G has a vertex-equalized 2-edge-coloring if and only if $G \neq K_2$.*

Proof. Clearly K_2 has no vertex-equalized 2-edge-coloring. To prove sufficiency let G be connected and $G \neq K_2$. Then in view of Theorems 2 and 4 we can assume that $G_u = K_2$. If G has any loops then color all loops with 0, and all edges with 1. If G has no loops, then color one edge with 0, and the remaining edges with 1. □

In what follows, $2K_2$ denotes a pair of vertices with a pair of parallel edges joining them.

Corollary 4.2. *Suppose G is a connected graph (possibly with loops and multiple edges). Then G has a vertex-equalized 3-edge-coloring if and only if $G \neq S_2, K_2, 2K_2$.*

Proof. Clearly S_2, K_2 and $2K_2$ have no vertex-equalized 3-edge-coloring. To prove sufficiency let G be connected and $G \neq S_2, K_2, 2K_2$. Then in view of Theorems 3 and 4 we can assume that $G_u = K_2$ or $G_u = S_2$. Suppose $G_u = K_2$. Then there are at least 3 edges in G. Color one such edge with 0, one with 1, and color all the other edges and loops in G with

2. Suppose $G_u = S_2$. Let $\{x, y\}$ and $\{y, z\}$ be the edges in G_u. If G has a loop, then color all loops in G with 0, all edges that join x to y with 1, and all edges that join y to z with 2. If G has no loops, then G has at least three edges. Color one edge with 0, one edge with 1, and the remaining edges with 2. □

Note that a generalization of Corollary 4.1 and Corollary 4.2 for disconnected graphs does not seem to be easy to obtain. For example, to settle the case with two colors (see Corollary 4.1) such a result would require the classification of all graphs G for which all vertex-equalized 2-edge-colorings satisfy $\nu_0(G) = \nu_1(G)$, since the graph consisting of two components G and K_2 would have no vertex-equalized 2-edge-coloring.

Also note that extending Theorem 2 and Theorem 3 to edge-colorings with four or more colors would require a different approach. This is because the idea of taking a spanning subgraph of a graph G, finding a vertex-equalized k-edge-coloring of this subgraph and then completing this coloring to a vertex-equalized k-edge-coloring of G by coloring a single edge at a time rarely works if $k \geq 4$. On the other hand, for a graph G that has many edges it is not difficult to see that one can take a vertex-equalized 3-edge-coloring of G and then recolor some of the edges in G with a new color to get a vertex-equalized 4-edge-coloring of G. Another approach for dense simple graphs would be to somehow find k edge-disjoint spanning subgraphs (for example, use Dirac's Theorem [5] k times to find k hamiltonian cycles in a graph on n vertices with $\delta \geq 2(k-1) + n/2$, coloring the edges in the i^{th} such subgraph with color i and all the other edges with any color to obtain a vertex-equalized k-edge-coloring in which $\nu_i = n$ for $1 \leq i \leq k$). Nevertheless, new ideas will be needed to settle the problem in general.

Finally the authors would like to note that an interesting related problem is to find the spectrum of $\nu_c(G)$ among all vertex-equalized k-edge-colorings of a graph G; that is, find $N(G) = \{\nu_c(G) \mid c \in \mathbb{Z}_k,\ G$ has a vertex-equalized k-edge-coloring with colors in $\mathbb{Z}_k\}$.

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