Two-Fold Orbitals

J. Lauri, R. Mizzi
Department of Mathematics
University of Malta
Malta
josef.lauri@um.edu.mt
russell.mizzi@um.edu.mt

R. Scapellato
Dipartimento di Matematica
Politecnico di Milano
Milano
Italy
raffaele.scapellato@polimi.it

Abstract

Two-fold automorphisms (or "TF-isomorphisms") of graphs are a generalisation of automorphisms. Suppose α, β are two permutations of $V = V(G)$ such that for any pair (u, v), u, $v \in V$, (u, v) is an arc of G if and only if $(\alpha(u), \beta(v))$ is an arc of G. Such a pair of permutations is called a two-fold automorphism of G. These pairs form a group that is called two-fold automorphism group. Clearly, it contains all the pairs (α, α) where α is an automorphism of G. The two-fold automorphism group of G can be larger than $\mathrm{Aut}(G)$ since it may contain pairs (α, β) with $\alpha \neq \beta$. It is known that when this happens, $\mathrm{Aut}(G) \times \mathbb{Z}_2$ is strictly contained in $\mathrm{Aut}(G \times K_2)$. In the literature, when this inclusion is strict, the graph G is called *unstable*.

Now let $\Gamma \leq S_V \times S_V$. A two-fold orbital (or "TF-orbital") of Γ is an orbit of the action $(\alpha, \beta) : (u, v) \mapsto (\alpha(u), \beta(v))$ for $(\alpha, \beta) \in \Gamma$ and $u, v \in V$. Clearly, Γ is a subgroup of the TF-automorphism group of any of its TF-orbitals. We give a short proof of a characterization of TF-orbitals which are disconnected graph and prove that a similar characterization of TF-orbitals which are digraphs might not be possible. We shall also show that the TF-rank of Γ, that is the number of its TF-orbitals, can be equal to 1 and we shall obtain necessary and sufficient conditions on Γ for this to happen.

JCMCC **100** (2017), pp. 133-154

Then, we shall show that, unlike the case of a coherent configuration made up of orbitals of a permutation group, the usual definition of structure constants does not, in general, hold for TF-orbitals. We shall give some simple conditions on Γ which guarantee that its TF-orbitals do admit the definition of the structure constants and we shall show that, in general, TF-orbitals allow "structure constants" based on walks of length 3 rather than length 2. We shall then use this last result to obtain, for a rank 3 strongly regular graph G, a necessary condition for it to be unstable, by using the fact that the two-fold automorphism group of G must have TF-rank 2.

1 General Introduction and Notation

A *mixed graph* is a pair $G = (V(G), A(G))$ where $V(G)$ is a finite set and $A(G)$ is a set of ordered pairs of elements of $V(G)$. The elements of $V(G)$ are called *vertices* and the elements of $A(G)$ are called *arcs*. When referring to an arc (u, v), we say that u is *adjacent to* v and v is *adjacent from* u. Sometimes we use $u \longrightarrow v$ to represent an arc $(u, v) \in V(G)$. The vertex u is the *start-vertex* and v is the *end-vertex* of a given arc (u, v). An arc of the form (u, u) is called a *loop*. A mixed graph cannot contain multiple arcs, that is, it cannot contain the arc (u, v) more than once, but it can contain loops, that is, arcs of the form (u, u). A mixed graph G is called *bipartite* if there is a partition of $V(G)$ into two sets X and Y, which we call *colour classes*, such that for each arc (u, v) of G the set $\{u, v\}$ intersects both X and Y. Note that a bipartite mixed graph cannot have loops because the corresponding vertex would have to be in $X \cap Y$, which is empty. A set S of arcs is *self-paired* if, whenever $(u, v) \in S$, (v, u) is also in S. If $S = \{(u, v), (v, u)\}$, then we consider S to be the unordered pair $\{u, v\}$; this unordered pair is called an *edge*. Therefore, a mixed graph can contain edges, non-self paired arcs and loops. Let S_V denote the symmetric group of all permutations on V. An element of $S_V \times S_V$ will be called a *two-fold permutation* or TF-*permutation* and each subgroup Γ of $S_V \times S_V$ will be said to be a *two-fold pemutation group* or TF-*permutation group*. A TF-*orbital* of Γ is an orbit of the action of Γ on $V \times V$. If (u, v) is an arc, denote by $\Gamma(u, v)$ the TF-orbital of Γ containing (u, v). Note that a TF-orbital is, in general, a mixed graph.

A *graph* is a mixed graph without loops whose arc-set is self-paired. The edge-set of a graph is denoted by $E(G)$. A *digraph* is a mixed graph with no loops in which no set of arcs is self-paired.

A bipartite digraph with a bipartition $V(G) = X \cup Y$ is said to be *strongly bipartite* if each arc of G is incident from a vertex in X to a vertex in Y.

Given a mixed graph G and a vertex $v \in V(G)$, we define the *in-neighbourhood* $N_{in}(v)$ by $N_{in}(v) = \{x \in V(G) | (x, v) \in A(G)\}$. Similarly, we define the *out-neighbourhood* $N_{out}(v)$ by $N_{out}(v) = \{x \in V(G) | (v, x) \in A(G)\}$. The *in-degree* $\rho_{in}(v)$ of a vertex v is defined by $\rho_{in}(v) = |N_{in}(v)|$ and the *out-degree* $\rho_{out}(v)$ of a vertex v is defined by $\rho_{out}(v) = |N_{out}(v)|$. When G is a graph, these notions reduce to the usual neighbourhood $N(v) = N_{in}(v) = N_{out}(v)$ and degree $\rho(v) = \rho_{in}(v) = \rho_{out}(v)$.

Let G be a graph and let $v \in V(G)$. Let $N(v)$ be the neighbourhood of v. We say that G is *vertex-determining* if $N(x) \neq N(y)$ for any two distinct vertices x and y of G [12].

A *two-fold isomorphism* or TF-*isomorphism* from G to H is a pair of bijections α, β: $V(G) \to V(H)$ such that (u, v) is an arc of G if and only if $(\alpha(u), \beta(v))$ is an arc of H. When such a pair exists, we say that G and H are TF-*isomorphic* and write $G \cong^{TF} H$. The TF-isomorphism is denoted by (α, β). The inverse $(\alpha^{-1}, \beta^{-1})$ of (α, β) is a TF-isomorphism from H to G.

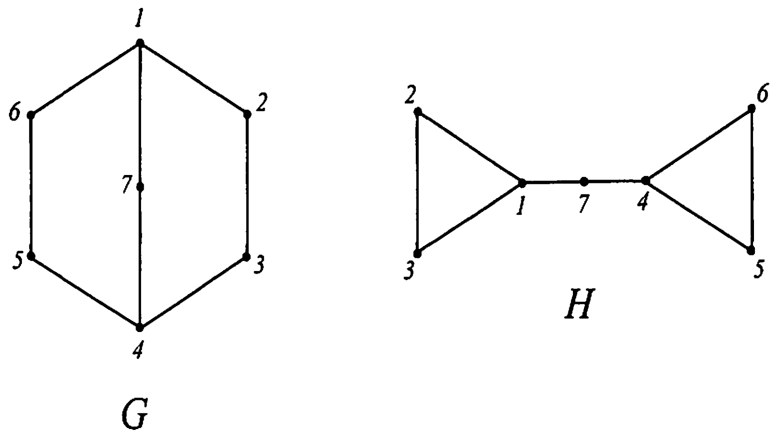

Figure 1: Two non-isomorphic but TF-isomorphic graphs.

The two graphs G and H in Figure 1, which have the same vertex set $V(G) = V(H)$, are non-isomorphic and yet TF-isomorphic. In fact (α, β), where $\alpha = (2\ 5)$ and $\beta = (1\ 4)(3\ 6)$, is a TF-isomorphism from G to H.

This concept was first studied by Zelinka [19, 20] in the the context of isotopy of digraphs. Some graph properties are preserved by a TF-isomorphism. Such is the case with the degree sequence, as illustrated by Figure 1. In [8] we showed that alternating trails or **A**-trails, which we shall define in full below, are invariant under TF-isomorphisms. For instance, the alternating trail $5 \longrightarrow 6 \longleftarrow 1 \longrightarrow 2$ in G is mapped by (α, β) to the similarly alternating trail $2 \longrightarrow 3 \longleftarrow 1 \longrightarrow 2$ which we shall later be calling "semi-closed".

When $G = H$, (α, β) is said to be a TF-*automorphism* and it is called non-trivial if $\alpha \neq \beta$. The set of all TF-automorphisms of G with multiplication defined by $(\alpha, \beta)(\gamma, \delta) = (\alpha\gamma, \beta\delta)$ is a subgroup of $S_{V(G)} \times S_{V(G)}$ and it is called the *two-fold automorphism group* of G and is denoted by $\mathrm{Aut}^{\mathrm{TF}}(G)$. Note that if we identify an automorphism α with the TF-automorphism (α, α), then $\mathrm{Aut}(G) \subseteq \mathrm{Aut}^{\mathrm{TF}}(G)$. Hence $\mathrm{Aut}(G) = \mathrm{Aut}^{\mathrm{TF}}(G)$ whenever all TF-automorphisms are trivial. It is possible for an asymmetric graph G, that is a graph with $|\mathrm{Aut}(G)| = 1$, to have non-trivial TF-automorphisms. This was one of our main results in [8]. For any graph G, the automorphism group of the direct product $G \times K_2$ contains all permutations arising from lifting elements of $\mathrm{Aut}(G)$, but may have further elements: in general, $\mathrm{Aut}(G) \times \mathbb{Z}_2 \subseteq \mathrm{Aut}(G \times K_2)$, but the inclusion might be strict. Whenever $\mathrm{Aut}(G) \times \mathbb{Z}_2 \subset \mathrm{Aut}(G \times K_2)$, the graph is called *unstable*. In [9] we showed that G is unstable if and only if it has a non-trivial TF-automorphism and in [10] we highlight some of the advantages of using TF-automorphisms to study unstable graphs.

The *canonical double cover* of a mixed graph G, denoted by $\mathbf{CDC}(G)$, (also called its *duplex* especially in computational chemistry literature, for example, [14]) is the mixed graph whose vertex set is $V(G) \times \{0, 1\}$ and in which there is an arc from (u, i) to (v, j) if and only if $i \neq j$ and there is an arc from u to v in G. The canonical double cover of G is often described as the direct product $G \times K_2$ [3, 2], and is sometimes also called the *bipartite double cover* of G. In [7] we proved the following result:

Theorem 1.1. *Let G, H be graphs. Then $G \cong^{\mathrm{TF}} H$ if and only if* $\mathbf{CDC}(G) \cong \mathbf{CDC}(H)$. □

Let G be a mixed graph. The *incidence double cover* of G, denoted by $\mathbf{IDC}(G)$ is a bipartite graph with vertex set $V(\mathbf{IDC}(G)) = V(G) \times \{0, 1\}$ and edge set $E(\mathbf{IDC}(G)) = \{(u, 0), (v, 1)\} \mid (u, v) \in A(G)\}$. The reader may refer to [5] for more information regarding the incidence double cover of graphs and its relevance to the study of semi-symmetric graphs. For graphs, the canonical double cover is identical to the incidence double cover. In [9], we proved the following extension of Theorem 1.1.

Theorem 1.2. *Let G, H be mixed graphs. Then $G \cong^{TF} H$ if and only if* $\mathbf{IDC}(G) \cong \mathbf{IDC}(H)$. □

The *alternating double cover* or **ADC** of G, denoted by $\mathbf{ADC}(G)$, is the direct product of G and the digraph P with $V(P) = \{0, 1\}$ and $A(P) = \{(0,1)\}$. This means that the vertex set of $\mathbf{ADC}(G)$ is $V(G) \times V(P)$ and $((u,0),(v,1)) \in A(\mathbf{ADC}(G))$ if and only if $(u,v) \in A(G)$. Clearly, $\mathbf{ADC}(G)$ is strongly bipartite, having vertices of the form $(u,0)$ as sources and vertices of the form $(u,1)$ as sinks. In fact, $\mathbf{ADC}(G)$ is obtained from $\mathbf{IDC}(G)$ by changing any edge $\{(u,0),(v,1)\}$ to an arc $((u,0),(v,1))$. The following is a straightforward consequence of Theorem 1.2.

Corollary 1.3. *Let G, H be mixed graphs. Then $G \cong^{TF} H$ if and only if* $\mathbf{ADC}(G) \cong \mathbf{ADC}(H)$. □

A set P of arcs (possibly containing also loops) of a mixed graph G is called a *trail* of *length* k if its elements can be ordered in a sequence a_1, $a_2, \ldots$, a_k such that each a_i has a common vertex with a_{i+1} for all $i = 1, \ldots, k-1$. If u is the vertex of a_1, that is not in a_2 and v is the vertex of a_k which is not in a_{k-1}, then we say that P *joins* u and v; u is called the *first vertex* of P and v is called the *last vertex* with respect to the sequence a_1, a_2, $\ldots$, a_k. If, whenever $a_i = (x,y)$, either $a_{i+1} = (x,z)$ or $a_{i+1} = (z,y)$ for some new vertex z, P is called an *alternating trail* or **A**-*trail.* If the first vertex u and the start-vertex v of an **A**-trail P are different, then P is said to be *open.* If they are equal then we have to distinguish between two cases. When the number of arcs is even, P is called *closed* while when the number of arcs is odd, P is called *semi-closed.* In a closed **A**-trail, for two arcs in the sequence that meet at a vertex x, either both arcs are incident from x or both arcs are incident to x. In a semi-closed **A**-trail, there is exactly one vertex x such that for the two arcs meeting at x, one arc is incident to x and the other is incident from x.

In [8], we proved the following result which we shall use later on.

Proposition 1.4. *Let G and G' be mixed graphs and P be an* **A**-*trail in G. Let (α, β) be any* TF-*isomorphism from G to G'. Then there exists an* **A**-*trail P' in G' such that (α,β) restricted to P maps P to P'. Moreover, P is closed if and only if P' is closed.*

Proposition 1.4 implies that alternating trails are invariant under the action of a TF-isomorphism.

Any other graph theoretical terms which we use are standard and can be found in any graph theory textbook such as [1]. For information on automorphism groups, the reader is referred to [11]. For an introduction to coherent configurations [6] and [18].

2 Disconnected Two-Fold Orbital Graphs and Digraphs

Let $\Gamma \leq S_V \times S_V$ and G be a TF-orbital of Γ. If G is a graph, we say that G is a *two-fold orbital graph* or a **TOG**. If G is a digraph, we say that G is a *two-fold orbital digraph* or a **TOD**. In [7], Theorem 4.3 provides a characterisation of disconnected **TOG**s. In this section we give a much shorter proof of this result and then investigate whether the result can be extended to cases when the TF-orbital yields a digraph or a graph.

Let G be any mixed graph. Consider the relation R on the set $A(G)$ defined by: xRy if and only if x and y are the first and last arcs of an **A**-trail of G. Clearly R is an equivalence relation. As shown in [9], Corollary 5.2, if G is a connected bipartite graph, then R has two equivalence classes, while if G is a connected non-bipartite graph, then G has only one equivalence class.

Therefore, if R is defined on the arcs of a disconnected graph, then the number of equivalence classes will be equal to the sum of the number of non-bipartite components and twice the number of bipartite components of G.

Proposition 2.1. *Let H and K be two connected components of a disconnected graph G such that $(\alpha, \beta) \in Aut(G)$ maps an arc (u, v) of H to an arc of K. Then there exists a bijection from one equivalence class induced by R on the arcs of H to one equivalence class induced by R on K.*

Proof. Define $\phi_{\alpha,\beta} : A(G) \to A(G)$ as follows: $\phi_{\alpha,\beta}(a) = b$ if and only if given that $a = (u, v)$ then $b = (\alpha(u), \beta(v))$. Let $a = (u, v)$ and $b = (x, y)$ be arcs of H such that aRb. It is easy to check that since **A**-trails are invariant under the action of a TF-isomorphism, as shown in Proposition

1.4, $\phi_{\alpha,\beta}(a) R \phi_{\alpha,\beta}(b)$. The same argument applies to $\phi_{\alpha^{-1},\beta^{-1}}$. Hence $\phi_{\alpha,\beta}$ is a bijection as stated. □

Corollary 2.2. *Let H and K be two connected components of a graph G such that $(\alpha, \beta) \in \mathrm{Aut}(G)$ maps an arc (u, v) of H to an arc of K. Then one of the following conditions holds:*

(a) H and K are non-bipartite and TF-isomorphic;

(b) One of them, say H, is non-bipartite and K is bipartite and isomorphic to $\mathbf{CDC}(H)$;

(c) H and K are bipartite and isomorphic.

Proof. Assume first that H is non-bipartite, then the equivalence class containing arc (u, v) consists of all the arcs of H. This must therefore be mapped by $\phi_{\alpha,\beta}$ to an equivalence class of containing arcs of K amongst which we find $(\alpha(u), \beta(v))$.

If also K is non-bipartite, the equivalence class containing the image of (u, v) under (α, β) consists of all the arcs of K. Clearly $\mathbf{ADC}(H)$ and $\mathbf{ADC}(K)$ must then be isomorphic and hence, by Corollary 1.3, H and K are TF-isomorphic.

If K is bipartite, R has two equivalence classes. Letting U and V be the colour classes of K, then one equivalence class of R consists of arcs running from U to V and the other consists of arcs running from V to U. These two classes of R form two strongly bipartite digraphs which we denote by $\overrightarrow{K}$ and $\overleftarrow{K}$ such that the arcs of $\overrightarrow{K}$ run from U to V and $(u, v) \in A(\overrightarrow{K})$ if and only if $(v, u) \in A(\overleftarrow{K})$. In this case, $\phi_{\alpha,\beta}$ takes all the arcs of H to one of these sets of arcs, that is, to either $\overrightarrow{K}$ or $\overleftarrow{K}$. Let us assume that the arcs of H are mapped into the arcs of $\overrightarrow{K}$. Note that for any edge $\{x, y\}$ of H, the arcs (x, y) and (y, x) must be mapped to distinct arcs say (x_0, y_1) and (y_0, x_1) both running from U to V. The vertex set of $\overrightarrow{K}$ must be twice the size of the vertex set of H. Hence $|V(K)| = 2|V(H)|$. The digraph $\overrightarrow{K}$ covers the arcs of H once. Note that K can be obtained from $\overrightarrow{K}$ by substituting each of its arcs with an edge. Let $\overleftarrow{K}$ be obtained from $\overrightarrow{K}$ by replacing every arc (u, v) with the arc (v, u). Now $\phi_{\alpha,\beta}(A(H)) = A(\overrightarrow{K})$ and $\phi_{\beta,\alpha}(A(H)) = A(\overleftarrow{K})$. But both $\phi_{\alpha,\beta}$ and $\phi_{\beta,\alpha}$ are bijections. Therefore, define $\phi : K \to \mathbf{CDC}(H)$ such that ϕ sends an arc $(\alpha(u), \beta(v))$ of K to $((u, 0), (v, 1))$ of $\mathbf{CDC}(H)$ and the corresponding arc $(\beta(v), \alpha(u))$ of

K to $((u,1),(v,0))$. Therefore, the edge $\{u,v\}$ of H is covered by the two edges $\{(u,0),(v,1)\}$ and $\{(u,1),(v,0)\}$. This gives that K is isomorphic to $\mathbf{CDC}(H)$, that is,(b) holds.

We are left with the case where both H and K are bipartite: then an equivalence class of one is mapped by $\phi_{\alpha,\beta}$ to an equivalence class of the other. The digraphs induced by these equivalent classes may be obtained by directing the arcs from one (vertex) colour class to the other. The TF-automorphism (α,β) of G restricted to such a digraph containing (u,v) say $\vec{H}$ is TF-isomorphism from $\vec{H}$ to the digraph $\vec{K}$ induced by the equivalence class of R containing the image of (u,v). Since H and K are both bipartite they must be isomorphic. Therefore (c) holds. □

We are now in a position to give a short proof of the main result in [7].

Theorem 2.3. *Let G be a* **TOG** *with no isolated vertices and let its components be $G_1,\ldots,G_k$ and:*

$$|V(G_1)| \geq |V(G_2)| \geq \ldots \geq |V(G_k)|.$$

Then each G_i $i \in \{1, \ldots, k\}$ is still a **TOG**. *Moreover:*

(i) if $|V(G_1| = |V(G_k)|$, then G_1, G_2, $\ldots, G_k$ are pairwise TF-isomorphic:

(ii) otherwise, there exists a unique index $r \in \{1, \ldots, k-1\}$ such that

$$G_1 \cong G_2 \cong \ldots G_r \ncong G_{r+1} \cong^{\mathbf{TF}} \ldots \cong^{\mathbf{TF}} G_k$$

where $G_1 \cong$ $\mathbf{CDC}(G_k)$.

Proof. First note that since G is a **TOG**, given any components say G_p, G_q then there exists $(\alpha,\beta) \in \mathrm{Aut}(G)$ which maps an arc (u,v) of G_p to an arc of G_q. Therefore, we can use the result of Corollary 2.2 to any pair of components. If $|V(G_p)| = |V(G_q)|$ then either (a) or (c) of Corollary 2.2 holds, that is, the two components are either non-bipartite and TF-isomorphic or bipartite and isomorphic, since TF-isomorphisms and isomorphisms preserve the order of a graph. On the other hand if, say, $|V(G_p)| > |V(G_q)|$ then (b) holds and so $G_p = \mathbf{CDC}(G_q)$. Furthermore, the set of elements of $\mathbf{\Gamma} = \mathrm{Aut}(G)$ which map arcs of any component G_i to arcs of G_i, irrespective of whether G_i is bipartite or non-bipartite, restricts to a group $\mathbf{\Gamma_i}$ of TF-automorphisms of G_i and therefore G_i is a **TOG** on $\mathbf{\Gamma_i}$. □

The fact that disconnected **TOG**s have connected components of only two TF-isomorphism types, need not be true for **TOD**s. Our next result proves this, but first we need to introduce a class of examples.

Let $m = 2k > 2$ be an integer. A residue class modulo Z_m can be said to be "even" or "odd" according to whether it belongs to the subgroup of index 2 or not (this notion would make no sense if m were odd). Define a digraph $G = G(m)$ by: $V(G) = Z_m \times Z_m$ and $A(G)$ consists of the ordered pairs of the following four kinds: $((x, y), (x, y+1))$ x even, $((x, y), (x, y-1))$ x odd, $((x, y), (x+1, y))$ y even, $((x, y), (x-1, y))$ y odd. In other words, two vertices form an arc when they share a coordinate and the other coordinate differs by 1, where the higher of the two coordinate values is the first vertex if the equal coordinate is odd and the second vertex if the equal coordinate is even. Note that G is a digraph because $m > 2$, otherwise it would be C_4. It is clear that G is connected: its underlying graph is the cartesian product of two m-cycles.

Geometrically, G can be seen as a torus obtained by taking a $m \times m$ grid and identifying each side with its opposite. For this reason we refer to a digraph constructed this way as an m-torus digraph. Figure 2 shows the m-torus digraph for $m = 6$. The squares are of two kinds: let us call *black* those that induce a closed A-trail and *white* those that induce a directed cycle.

Let us first determine $\mathrm{Aut}(G_m)$. Call an arc 'horizontal' when its end-vertices share the first coordinate and 'vertical' when they share the second one. Let Δ be the subgroup of $\mathrm{Aut}(G_m)$ consisting of the maps of the following four kinds:

$$
\begin{aligned}
\text{type } 00 : (x, y) &\mapsto (x+i, y+j) \; i,\ j \text{ even} \\
\text{type } 01 : (x, y) &\mapsto (x+i, -y+j) \; i \text{ odd},\ j \text{ even} \\
\text{type } 10 : (x, y) &\mapsto (-x+i, y+j) \; i \text{ even},\ j \text{ odd} \\
\text{type } 11 : (x, y) &\mapsto (x+i, y+j) \; i,\ j \text{ odd.}
\end{aligned}
$$

The set of all automorphisms of the above types is closed under composition and is therefore a subgroup of $\mathrm{Aut}(G_m)$. Also the mapping from Δ to $\mathbb{Z}_2 \times \mathbb{Z}_2$ defined by sending an automorphism of type $\varepsilon\varepsilon'$ to $(\varepsilon, \varepsilon')$ is a homomorphism. Each element of Δ takes a horizontal arc into a horizontal arc and a vertical arc into a vertical arc. Since there are k even and k odd elements in $\mathbb{Z}_m$, Δ has order $4k^2$. Now consider the map $\psi : (x, y) \mapsto (y, x)$ which commutes with all the elements of Δ. We adjoin to Δ, this map ψ.

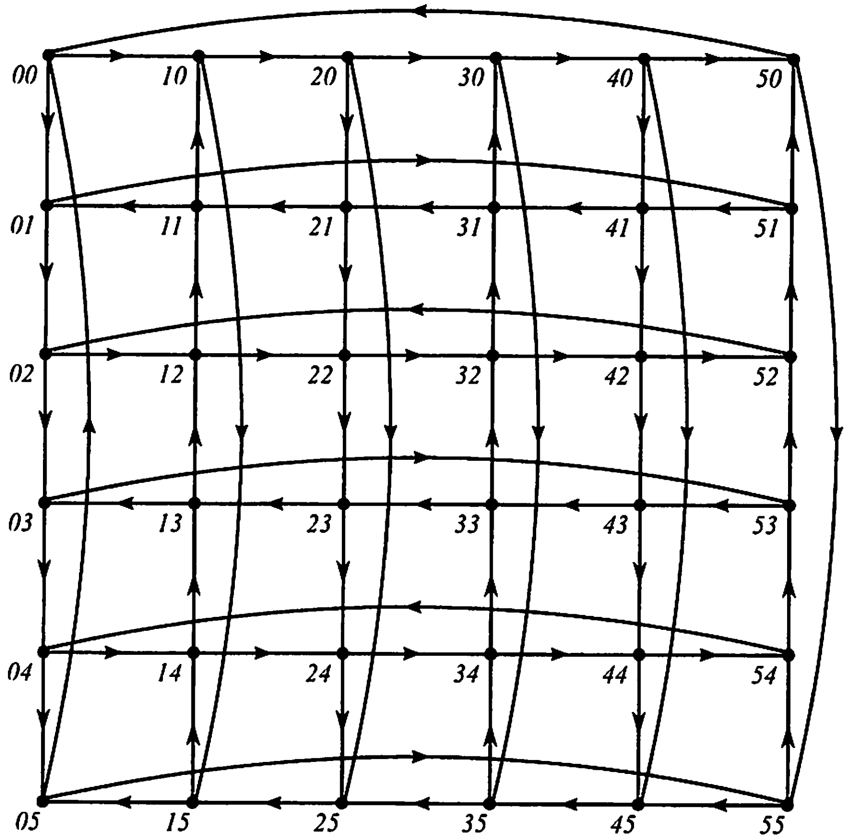

Figure 2: The m-torus digraph for $m = 6$.

Hence $\langle \Delta, \psi \rangle$ is the direct product of Δ and $\langle \psi \rangle$ and so has order $8k^2$. So we observe that the order of Δ equals the number of vertices and that of $\langle \Delta, \psi \rangle$ equals the number of arcs of G_m.

Let us now consider an automorphism ϕ of $\mathrm{Aut}(G_m)$ that fixes and arc e. Note that e belongs to a unique closed A-trail of order 4 and to a unique directed 4-cycle. Both trails must be fixed by ϕ and clearly they are fixed arc-wise. This causes further A-trails and directed cycles to be fixed arc-wise by ϕ and, since G_m is connected, it follows that $\phi = \mathrm{id}$. Therefore $\mathrm{Aut}(G_m)$ is semiregular on arcs. Since $\langle \Delta, \psi \rangle$ has as many elements as there are arcs, this forces it to be a regular group, hence $\langle \Delta, \psi \rangle = \mathrm{Aut}(G_m)$. Note that Δ is transitive, hence regular on the vertex set of G_m. In the full automorphism group, the stabilizer of a vertex v has order 2 and the non-trivial element swaps the arcs incident with v.

Now, for the same values of m, define the digraphs $H = H(m)$ as follows: $V(H) = Z_m \times Z_m \times Z_2$, $A(H_m)$ consists of the ordered pairs of the following kinds: $((x, y, 0), (x, y+1, 1))$ x even, $((x, y, 0), (x, y-1, 1))$ x odd, $((x, y, 0), (x+1, y, 1))$ y even, $((x, y, 0), (x-1, y, 1))$ y odd, $((x, y, 0), (x+1, -y, 1))$ and $((x, y, 1), (x+1, -y, 0))$. Note that this definition follows the same track of that of $G(m)$, with the fundamental difference that the third coordinate distinguishes between sources and sinks. Unlike $G(m)$, the digraph $H(m)$ is disconnected: it splits into $(m^2)/2$ components of order 4, each isomorphic to a black square. The maps obtained from those in Δ by preserving the third coordinate are automorphisms of $H(m)$. Of course, it has further automorphisms.

Let $K(m)$ be the union of $G(m)$ and $H(m)$. Define two permutations of $V(K)$: α swaps (x, y) with $(x, y, 0)$ and fixes $(x, y, 1)$ for all x, y; β swaps (x, y) with $(x, y, 1)$ and fixes $(x, y, 0)$ for all x, y. Then (α, β) is a TF-automorphism of $K(m)$. In fact, due to the definition of $H(m)$, the two-fold permutation (α, β) takes arcs of G_m to arcs of H and conversely. Together with the aforementioned maps, this TF-automorphism generates a group Γ' of TF-automorphisms of $K(m)$. It can be shown that $K(m) = \Gamma'(u, v)$ for any arc (u, v) of $K(m)$, therefore $K(m)$ is a **TOD**.

We are now set to state and prove the announced result.

Proposition 2.4. *Given any positive integer $k \geq 2$, there exists a disconnected* **TOD** *with at least k mutually non-*TF*-isomorphic components.*

Proof. Let $K(m)$ be obtained as above. It is true that $K(m)$ contains connected components of only two types. However, one can take any union of graphs $K(m_1)$, $K(m_2)$, $K(m_3)$, .. and, as above, for any m_i, m_j, there always exists some TF-isomorphism from G_{m_i} to a number of black squares in H_{m_j} with vertices (a, b), $(a, b+1)$, $(a+1, b+1)$, $(a+1, b)$, where both a and b are even. This gives the required **TOD**. □

3 TF-rank equal to 1

The rank of a permutation group Γ is defined to be the number of orbitals of Γ. The study of permutation groups of rank 3 is an important meeting point of permutation group theory and graph theory. For instance, the reader may refer to [13].

Let $\mathbf{\Gamma} \leq S_V \times S_V$ be a two-fold permutation group acting on the set

$V \times V$ and let

$$\Sigma = \{\beta^{-1}\alpha : (\alpha, \beta) \in \mathbf{\Gamma}\}.$$

We say that $\mathbf{\Gamma}$ is Σ-*transitive* on V if for any $u, v \in V$, there exists $\beta^{-1}\alpha \in \Sigma$ such that $\beta^{-1}\alpha(u) = v$, that is $\alpha(u) = \beta(v)$. We say that $\mathbf{\Gamma}$ is TF-*transitive* on V if, for all $u, v \in V$, there exists $(\alpha, \beta) \in \mathbf{\Gamma}$ such that $\alpha(u) = v$ *and* $\beta(u) = v$.

We know, in general, that the number of orbitals of a permutation group (Γ, V) is at least 2 and this happens only when (Γ, V) is 2-transitive.

However, unlike the usual rank, the TF-rank can be equal to 1. This is possible because TF-permutations can take arcs to loops. The following result characterizes the actions whose TF-rank is equal to 1.

Theorem 3.1. *Let* $\mathbf{\Gamma} \subseteq S_V \times S_V$ *be a two-fold permutation group. Then,* $(\mathbf{\Gamma}, V \times V)$ *has* TF-*rank equal to* 1 *if and only if* $\mathbf{\Gamma}$ *is both* Σ-*transitive and* TF-*transitive.*

Proof. Suppose that $(\mathbf{\Gamma}, V \times V)$ has TF-rank equal to 1. Therefore, all the arcs and loops are members of the same TF-orbital. Consider any arc (u, v) and the loop (w, w). There exists $(\alpha, \beta) \in \mathbf{\Gamma}$ such that $(\alpha, \beta)(u, v) = (w, w)$ which implies that $\alpha(u) = w\beta(v)$. This holds for any pair of elements u, v of V, since every pair of vertices are joined by two arcs running in opposite directions and there is a loop at each vertex. Hence TF-rank equal to 1 implies that $\mathbf{\Gamma}$ is Σ-transitive. Now, for any pair of vertices u, $v \in V$, consider the loops (u, u) and (v, v). There exists $(\alpha, \beta) \in \mathbf{\Gamma}$ such that $(\alpha, \beta)(u, u) = (v, v)$ by virtue of the fact that all loops are members of the same TF-orbital. This implies that $\alpha(u) = v$ and $\beta(u) = v$. Therefore, $\mathbf{\Gamma}$ is TF-transitive.

Conversely, suppose that $\mathbf{\Gamma}$ is Σ-transitive and TF-transitive on V. Since Γ is Σ-transitive on V, for any (u, v), there exists $\beta^{-1}\alpha \in \Sigma$ such that $\beta^{-1}\alpha(u) = v$ so $\alpha(u) = \beta(v)$. Therefore, any arc has a loop in its TF-orbital. If $\mathbf{\Gamma}$ is TF-transitive, then any given pair of loops are, by definition, in the same TF-orbital since, given u, $v \in V$, there is (α, β) such that $\alpha(u) = v$ and $\beta(u) = v$, therefore $(\alpha, \beta)(u, u) = (v, v)$. Hence, all the loops are in the same TF-orbital. Therefore, all the arcs are also in the same TF-orbital since any arc has some loop in its orbital. Hence $\mathbf{\Gamma}$ has TF-rank equal to 1. □

A trivial example for which the TF-rank is 1 is $S_V \times S_V$ itself. Now let us construct a simple, yet non-trivial example. Let $|V| = p$, with p prime. Consider $\mathbf{\Gamma}'$ generated by (α, α^{-1}) where $\alpha = (12 \ldots p)$. The presence of the TF-permutation (α, α^{-1}) ensures that all the TF-orbitals are self-paired, that is, they are graphs. The TF-orbitals for $p = 5$ are illustrated in Figure 3.

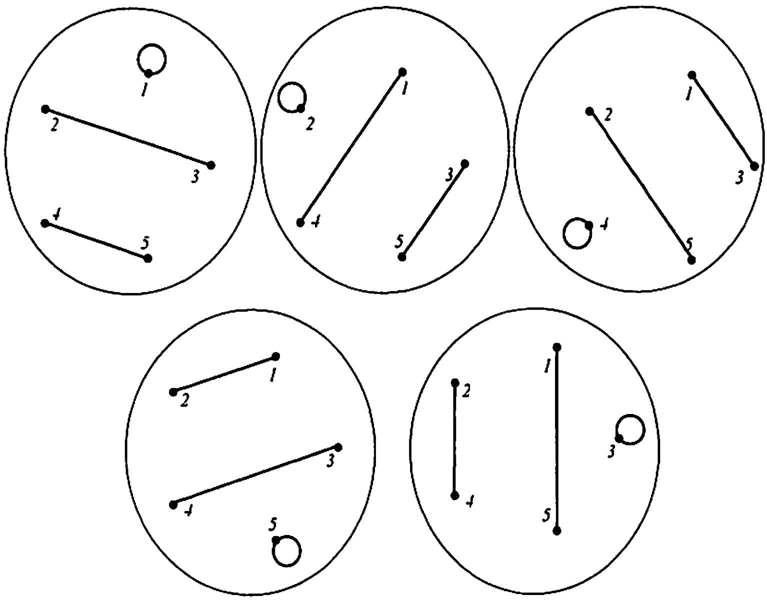

Figure 3: The TF-orbitals generated by (α, α^{-1}) where $\alpha = (1\ 2\ 3\ 4\ 5)$.

In order to get TF-rank equal to 1, we need to gather all the loops in one orbital. This can be done by letting $\mathbf{\Gamma} = \langle(\alpha, \alpha), (\alpha, \alpha^{-1})\rangle$. Since all the loops are in the same orbital and every edge has a loop in its orbital under (α, α^{-1}), then all the edges and all the loops are in the same TF-orbital so that the TF-rank is equal to 1.

4 Structure constants and TF-Orbitals

Let $\mathbf{\Gamma} \leq \mathcal{S}_V \times \mathcal{S}_V$ and let $R_1, R_2, \ldots, R_r$ be the TF-orbitals of $(\mathbf{\Gamma}, V \times V)$ considered as mixed graphs. These TF-orbitals partition the arcs of $K^{\circ}_{|V|}$ which is the complete graph on V with loops. It is clear that if $\mathbf{\Gamma}$ contains a non trivial (α, β), then some loop is in the same TF-orbital as some arc because $\alpha \neq \beta$ implies that there exists some vertex x such that $\alpha(x) \neq \beta(x)$, that is the loop (x, x) is in the same TF-orbital as the arc $(\alpha(x), \beta(x))$.

When considering the orbitals of a permutation group (Γ, V) as par-

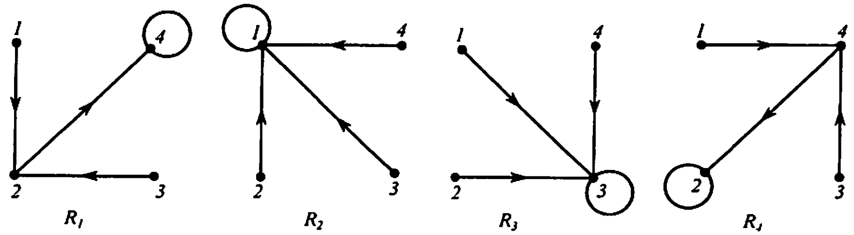

Figure 4: The TF-orbitals for $\mathbf{\Gamma} = \langle(\alpha, \beta)\rangle$ where $\alpha = (1\ 2\ 3\ 4)$ and $\beta = (2\ 4)$.

titioning the edges of $K^{\circ}_{|V|}$, we obtain a coherent configuration and the system of orbitals admits the *structure constants* p^k_{ij}: Given arc (a, b) in R_k then the number of vertices x such that (a, x) is in R_i and (x, b) is in R_j equals p^k_{ij} and is independent of the choice of (a, b) in R_k.

However, the system of TF-orbitals of $(\mathbf{\Gamma}, V \times V)$ does not, in general, admit the structure constants. The following counterexample is enough to show this. Let $\mathbf{\Gamma} = \langle((1\ 2\ 3\ 4), (2\ 4))\rangle = \{((1\ 2\ 3\ 4), (2, 4)), ((1\ 3)(2\ 4), \text{id})), ((4\ 3\ 2\ 1), (2\ 4)), (\text{id}, \text{id})\}$. The TF-orbitals are shown in Figure 4. Consider the arc $(1, 2)$ in class R_1. There is a path in R_4; the arc $(1, 4)$ followed by the arc $(4, 2)$. This would make $p^1_{44} = 1$. But now consider the arc $(2, 4)$ also in class R_1. There is no similar directed path from 4 to 2 with both arcs in R_4. Therefore, p^1_{44} cannot be defined.

On the other hand, there exist systems of TF-orbitals that admit structure constants. For example, let $\mathbf{\Gamma} = \langle((1\ 2\ 3\ 4), (1\ 2)(3\ 4))\rangle = \{((1\ 2\ 3\ 4), (1\ 2)(3\ 4)), ((1\ 3)(2\ 4), \text{id})), ((4\ 3\ 2\ 1), (1\ 2)(3\ 4)), (\text{id}, \text{id})\}$. The TF-orbitals are shown in Figure 5. It can easily be checked from this figure that these TF-orbitals admit structure constants. Here we give two conditions on $\mathbf{\Gamma}$ which are sufficient for this to happen.

A two-fold permutation group $\mathbf{\Gamma}$ is said to satisfy *Property K* if, for any $x, y \in V$ and any $(\alpha, \beta) \in \mathbf{\Gamma}$, the arcs (x, y) and $(\beta(x), \beta(y))$ are in the same TF-orbital.

Theorem 4.1. *Suppose $\mathbf{\Gamma} \leq S_V \times S_V$ has Property K. Then, given any arc (a, b) in the TF-orbital R_k, the number of vertices x such that (a, x) is in R_i and (x, b) is in R_j is independent of the choice of (a, b) in R_k. Therefore the TF-orbitals admit the definition of structure constants p^k_{ij}.*

Proof. Let (a, b) and (a', b') be two arcs in the TF-orbital R_k. Therefore there is an (α, β) in $\mathbf{\Gamma}$ mapping (a, b) into (a', b'). Let x be such that (a, x) is in R_i and (x, b) in R_j. Then (α, β) maps (a, x) into (a', x'), which is therefore also in R_i. Now, Property K gives that (x, b) and $(\beta(x), \beta(b)) = (x', b')$ are both in the same TF-orbital, therefore both in R_j. Therefore β is a bijection from

$$W = \{x : (a, x) \in R_i, (x, b) \in R_j\}$$

to

$$W' = \{x' : (a', x') \in R_i, (x', b') \in R_j\},$$

which gives the required result. □

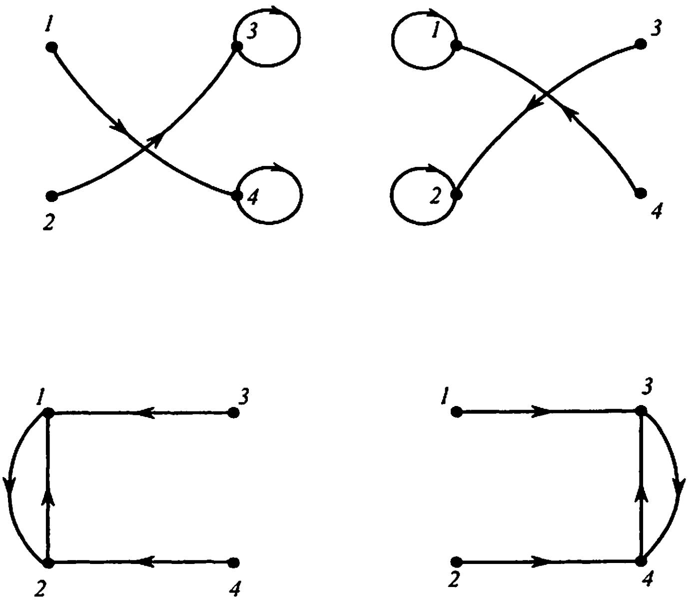

Figure 5: The system of TF-orbitals for $\mathbf{\Gamma} = \langle((1\ 2\ 3\ 4), (1\ 2)(3\ 4))\rangle$.

Now, let us say that a two-fold permutation group $\mathbf{\Gamma}$ is said to satisfy *Property M* if, for any (α, β) in $\mathbf{\Gamma}$, (β, β) is also in $\mathbf{\Gamma}$. Clearly, Property M implies Property K, but the converse does not hold in general. For example, let $\mathbf{\Gamma}$ be the set of all pairs (γ, γ^{-1}), $\gamma \in \mathcal{S}_V$. The TF-orbitals of $\mathbf{\Gamma}$ are only two, R_o consisting of loops and R_1 consisting of arcs of the form (u, v) where $u \neq v$. Every permutation β takes arcs to arcs and loops to loops, hence Property K is satisfied. However, most permutations differ from their inverse, so Property M is not satisfied.

The fact that Property M implies Property K makes it easier to obtain two-fold permutation groups fulfilling Property K. For example, let Γ_1 be a permutation group acting on V and let Δ_1 consist of all two-fold permutations (β, β) with $\beta \in \Gamma_1$. Fix $\alpha_0 \in S_V \setminus \Gamma_1$ and $\beta_0 \in \Gamma_1$ and let $\mathbf{\Gamma} = \langle \Delta_1, (\alpha_0, \beta_0) \rangle$. Clearly, $\mathbf{\Gamma}$ satisfies Property M, hence Property K.

As another example, let Γ_1 be a permutation group and Γ_2 be a subgroup of Γ_1. Consider the direct product $\Gamma_1 \times \Gamma_2$. For any element (α, β) of $\Gamma_1 \times \Gamma_2$, β is also in Γ_1, therefore (β, β) is also in $\Gamma_1 \times \Gamma_2$, which therefore satisfies Property M, hence Property K.

The system of TF-orbitals of such two-fold permutation groups has the property that the space generated by linear combinations of their adjacency matrices is closed under matrix multiplication and therefore forms an algebra. However, they do not form a coherent configuration because this space of matrices does not contain the identity and is not closed under taking of transpose. If the TF-orbitals happen to be all undirected graphs then the matrices are all symmetric and therefore we obtain a structure which satisfies the axioms of a coherent configuration except that it does not contain the identity. Klin (personal communication and [4]) has shown that such structures can arise naturally in other contexts.

Although TF-orbitals do not in general admit structure constants, we can prove that an extension of the structure constants to directed alternating walks of length 3 can, in general, be defined.

Theorem 4.2. *Let $\mathbf{\Gamma} \leq S_V \times S_V$ and let R_1, R_2, $\ldots$, R_r be* TF*-orbitals of $\mathbf{\Gamma}$. Let i, j, k and s be any elements of $\{1, 2, \ldots, r\}$. Let (a, b) be an arc in R_s. Then the number of arcs (y, x), such that $(y, x) \in R_j$, $(a, x) \in R_i$ and $(y, b) \in R_k$ is independent of the choice of arc (a, b) in R_s.*

Proof. Figure 6 illustrates the arcs in the statement. An arc $(a,' b')$ also in R_s is also shown. For any such arc $(a,' b')$ in R_s, there is an (α, β) such that $(\alpha, \beta)(a, b) = (a', b')$.

Let the sets W and W' be defined as follows:

$$\begin{aligned} W &= \{(y, x) \in R_j \ : \ (a, x) \in R_i \text{ and } (y, b) \in R_k\} \\ W' &= \{(y', x') \in R_j \ : \ (a', x') \in R_i \text{ and } (y', b') \in R_k\} \end{aligned}$$

Then clearly (α, β) induces a bijection from W to W', thus $|W| = |W'|$. But (a, b) and (a', b') are two arbitrary arcs in R_s, therefore the result fol-

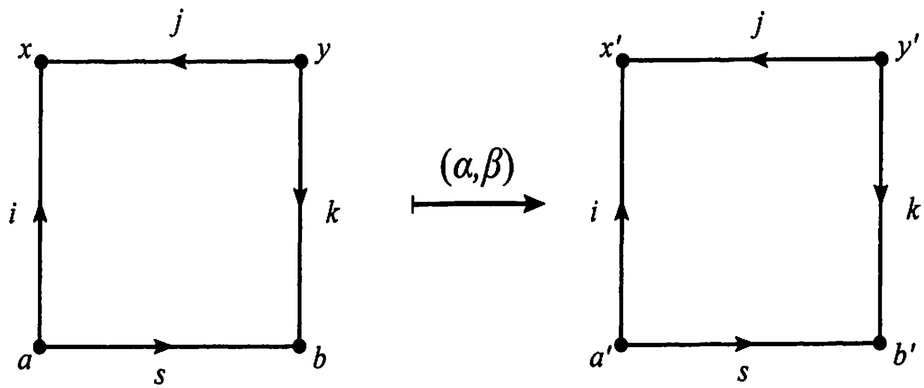

Figure 6: Definition of p^s_{ijk} for TF-orbitals.

lows. □

If (a,b) is an arc in R_s, we denote the number of arcs (y,x), such that $(y,x) \in R_j$, $(a,x) \in R_i$ and $(y,b) \in R_k$ by p^s_{ijk}.

Theorem 4.2 can be expressed in terms of matrices by:

$$A_i A_j^T A_k = \Sigma_{t=1}^r p^s_{ijk} A_s$$

where A_x is the adjacency matrix of the orbital R_x and A_x^T is its transpose.

5 Unstable rank 3 strongly regular graphs

There are two cases of instability which are considered to be trivial. First of all, if G is bipartite, then $G \times K_2$ consists of two copies of G and therefore, unless Aut(G) is trivial, G is unstable. Secondly, if G has two vertices u, v which share the same neighbourhood set, then G admits the TF-automorphism (α, id) where $\alpha = (u\ v)$ and therefore G is unstable. Hence, when we consider unstable graphs we tacitly exclude such graphs.

Let us now direct our attention to unstable strongly regular graphs of rank 3 with the usual parameters n, k, λ, μ in order to illustrate a simple application of Theorem 4.2. Let G be such a rank 3 strongly regular graph. We shall assume that $k > \mu$ so that G is vertex-determining. It should be noticed that the excluded case only consists of trivial graphs (disconnected unions of complete graphs or complete multi-bipartite graphs). Therefore the orbitals of $\mathrm{Aut}(G)$ on $V(G) \times V(G)$ are $G_\circ$ where $V(G_\circ) = V(G)$ and $A(G_\circ) = \{(u, u) : u \in V(G)\}$, G and $\overline{G}$. It is well known that the adjacency matrix A of G satisfies:

$$A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J$$

where J is the all-1's matrix. Now in [17], Proposition 2 it is proved that $\lambda = \mu$ if G is unstable. We give a different proof by applying TF-orbitals.

Theorem 5.1. *Let G be a rank 3 unstable strongly regular graph with parameters n, k, λ, μ. Let p be the number of walks of length 3 joining any pair of adjacent vertices in G and let q be the number of walks of length 3 joining any pair of non-adjacent vertices in G. Then*

$$\begin{aligned} \lambda &= \mu \\ q &= \mu k \\ p &= \mu k + k - \mu. \end{aligned}$$

Proof. Since the graph G is unstable, $\mathrm{Aut}^{\mathrm{TF}}(G)$ contains non-trivial TF-automorphisms. Consider the TF-orbitals of $\mathrm{Aut}^{\mathrm{TF}}(G)$ on $V(G) \times V(G)$. Of course, the three orbitals $G_\circ$, G and $\overline{G}$ cannot be split into more TF-orbitals. We also know that since the TF-permutations (whether trivial or not) are automorphisms of G, the orbital G remains unchanged. But since some loop must be in the same TF-orbital as some arc, we conclude that all loops join $\overline{G}$ to form one TF-orbital $\overline{G}^\circ$ (where $\overline{G}^\circ$ is the complement of G with a loop at every vertex). Therefore we have that the TF-rank of $\mathrm{Aut}^{\mathrm{TF}}(G)$ is 2 and the TF-orbitals are G and $\overline{G}^\circ$.

Now, the adjacency matrix of G is A, but that of $\overline{G}^\circ$ is $J - A$ since $\overline{G}^\circ$ has a loop at every vertex. Let $G = R_1$ and let $\overline{G}^\circ = R_2$ and let us use Theorem 4.2. By applying the existence of structure constants to R_1 and R_2, we get

$$A^3 = p^1_{111}A + p^2_{111}(J - A)$$

which we can write as

$$A^3 = pA + q(J - A) \qquad \cdots (1)$$

where $p = p^1_{111}$ and $q = p^2_{111}$.

Let $\theta = \lambda - \mu$. Therefore (1) becomes:

$$A^2 = kI + \theta A + \mu(J - I) \qquad \ldots(2)$$

Substituting for A^2 again in (2), gives

$$(k - \mu - p + q + \theta^2)A = (q - \mu k - \theta\mu)J + I(\theta(\mu - k)).$$

But the right-hand side is a constant matrix except for the diagonal and this is equal to A times a constant. This can only happen if:

$$\begin{aligned} k - \mu - p + q + \theta^2 &= 0 \\ q - \mu k - \theta\mu &= 0 \\ \theta(\mu - k) &= 0 \end{aligned}$$

therefore $\theta = 0$ since $k \neq \mu$. It follows that:

$$\begin{aligned} \lambda &= \mu \\ q - \mu k &= 0 \\ \text{and} \qquad k - \mu - p + q &= 0 \end{aligned}$$

□

Note that the fact that $q = \mu k$ is not surprising. In G, every vertex is non-adjacent to itself (while every vertex is adjacent to itself in the other TF-orbital $\overline{G}^{\circ}$). The number of walks of length 3 joining u to itself is equal to twice the number of triangles containing u. But the degree of u is k, every edge lies on $\lambda = \mu$ triangles and therefore $\lambda k = \mu k$ counts every triangle containing u twice and hence is the same as q.

Equation $p = \mu k + k - \mu$ can also be obtained by a direct counting argument. Let u and v be adjacent. There are $(k-1)(\mu-1)$ walks from u to v without repeated edges. The remaining walks have either u or v repeated. The former are of the form (u, x), (x, u), (u, v), where x is any neighbour of u (including v itself), hence there are k of them. There are also k of the latter walks. Note that (u, v), (v, u), (u, v) appears twice. The overall number of walks from u to v is thus $(k-1)(m-1+2)-1 = km+k-m$.

An unstable strongly regular graph of rank 3 seems to be very unstable, in this sense. Since every loop is in the other TF-orbital $\overline{G}^{\circ}$, it follows that, for every pair of non-adjacent vertices a, b in G, the edge $\{a, b\}$ is in $\overline{G}^{\circ}$

and therefore in the same TF-orbital as the loop (c, c). Therefore, for every vertex c, there exists (α, β) in $\text{Aut}^{\text{TF}}(G)$ such that $\alpha(a) = c = \beta(b)$. This is similar to TF-transitivity, but only for pairs of vertices not adjacent in G.

An example of a well-known strongly regular graph which is unstable with $\lambda = \mu$ (although it does not have rank 3) is the Shrikhande graph S illustrated in Figure 7 which has the same parameters as the cartesian product $K_4 \Box K_4$ (shown to be unstable in [17]), namely $(16, 6, 2, 2)$.

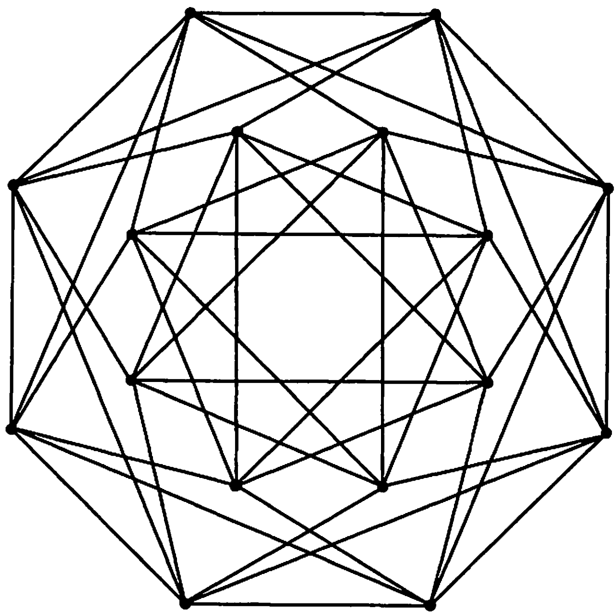

Figure 7: The Shrikhande graph.

Using the computational package Sage [16] we find that $|\text{Aut}(S)| = 192$ but $|\text{Aut}(S \times K_2)| = 23040$, confirming the instability of S. However $\lambda = \mu$ does not force instability. For example, the last graph in the list of 3845 strongly regular graphs with parameters $(n, k, \lambda, \mu) = (35, 18, 9, 9)$ in [15] is the graph G whose vertices are the lines of $PG(3, 2)$ with adjacency when they have one point in common. It can be constructed as the graph whose vertex set is consists of all 3-subsets of $\{1, 2, \ldots, 7\}$ and two of them are adjacent when they have exactly one element in common. Again using Sage, it can be shown that $|\text{Aut}(G)| = 40320$ and $|\text{Aut}(G \times K_2)| = 80640$, therefore G is stable.

We finally note that if G is strongly regular, then at least one of G

and $\overline{G}$ is stable because the parameters λ and μ cannot be equal for both graphs.

References

[1] A. Bondy and U.S.R. Murty. *Graph Theory (Graduate Texts in Mathematics)*. Springer, 2008.

[2] R. Hammack, W. Imrich, and S. Klavžar. *Handbook of Product Graphs, Second Edition*. Discrete Mathematics and Its Applications. Taylor & Francis, 2011.

[3] W. Imrich and S. Klavzar. *Product Graphs: Structure and Recognition*. Wiley, 2000.

[4] M. H. Klin. Lectures on Coherent Configurations 2014. http://www.savbb.sk/ karabas/edu/sschool14/.

[5] M. H. Klin, J. Lauri, and M. Ziv-Av. Links between two semisymmetric graphs on 112 vertices via association schemes. *J. Symb. Comput.*, 47(10):1175–1191, 2012.

[6] M. H. Klin, C. Rücker, G. Rücker, and G. Tinhofer. Algebraic Combinatorics in Mathematical Chemistry. Methods and Algorithms. I. Permutation Groups and Coherent (Cellular) Algebras., 1997.

[7] J. Lauri, R. Mizzi, and R. Scapellato. Two-fold orbital digraphs and other constructions. *International J. of Pure and Applied Math.*, 1:63–93, 2004.

[8] J. Lauri, R. Mizzi, and R. Scapellato. Two-fold automorphisms of graphs. *Australasian J. Combinatorics.*, 49:165–176, 2011.

[9] J. Lauri, R. Mizzi, and R. Scapellato. A generalisation of isomorphisms with applications. *Preprint*, 2014.

[10] J. Lauri, R. Mizzi, and R. Scapellato. Unstable graphs: A fresh outlook via TF-automorphisms. *Ars Mathematica Contemporanea*, 8(1):115–131, 2015.

[11] J. Lauri and R. Scapellato. *Topics in Graph Automorphisms and Reconstruction*. Cambridge University Press, 2003.

[12] D. Marušič, R. Scapellato, and N. Zagaglia Salvi. A characterization of particular symmetric (0, 1)-matrices. *Linear Algebra Appl.*, 119:153–162, 1989.

[13] G. Pearce and C. E. Praeger. Rank 3 Transitive Decompositions of Complete Multipartite Graphs. *Graphs and Combinatorics*, 29(3):669–680, 2013.

[14] M. Randić. Symmetry properties of chemical graphs. V. Internal rotation in $XY_3XY_2XY_3$. *J. Comput. Chem.*, 4(1):73–83, 1983.

[15] T Spence. Strongly Regular Graphs on at most 64 vertices. `http://www.maths.gla.ac.uk/ es/srgraphs.php`.

[16] W. A. Stein et al. *Sage Mathematics Software (Version 6.3)*. The Sage Development Team, 2014. `http://www.sagemath.org`.

[17] D. B. Surowski. Stability of arc-transitive graphs. *J. Graph Theory*, 38(2):95–110, 2001.

[18] J.H. van Lint and R.M. Wilson. *A Course in Combinatorics*. Cambridge University Press, 2001.

[19] B. Zelinka. The group of autotopies of a digraph. *Czechoslovak Mathematical Journal*, 21(4):619–624, 1971.

[20] B. Zelinka. Isotopy of digraphs. *Czechoslovak Mathematical Journal*, 22(3):353–360, 1972.