# Some Extremal Problems on the Cycle Length Distribution of Graphs*

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**Abstract**

The cycle length distribution ($CLD$) of a graph of order $n$ is $(c_1, c_2, \ldots, c_n)$, where $c_i$ is the number of cycles of length $i$, for $i = 1, 2, \ldots, n$. For an integer sequence $(a_1, a_2, \ldots, a_n)$, we consider the problem of characterizing those graphs $G$ with minimum possible edge number and with $CLD(G) = (c_1, c_2, \ldots, c_n)$ such that $c_i \geq a_i$ for $i = 1, 2, \ldots, n$. The number of edges in such a graph is denoted by $g(a_1, a_2, \ldots, a_n)$. In this paper, we give the lower and upper bounds of $g(0, 0, k, \ldots, k)$ for $k = 2, 3, 4$.

**Key words:** pancyclic graphs; cycle length distribution; bound.

**AMS Subject Classifications:** 05C38; 05C35.

## 1 Introduction

Throughout this paper all graphs will be simple, undirected, finite, and connected. The cycle length distribution ($CLD$) of a graph of order $n$ is $(c_1, c_2, \ldots, c_n)$, where $c_i$ is the number of cycles of length $i$. For an integer sequence $(a_1, a_2, \ldots, a_n)$, we would like to characterize those graphs $G$ with maximum possible edge number and with $CLD(G) = (c_1, c_2, \ldots, c_n)$ such

*Partially supported by NSF of China (Nos.11401290, 61379021); NSF of Fujian (Nos.2015J01018, 2014J01020); Project of Fujian Education Department (No.JA14204); IRTSTFJ; Fujian provincial key laboratory of granular computing and its application.

that $c_i \leq a_i$ for $i = 1, 2, \ldots, n$. The number of edges in such a graph is denoted by $f(a_1, a_2, \ldots, a_n)$. For an integer sequence $(a_1, a_2, \ldots, a_n)$, we would like to characterize those graphs $G$ with minimum possible edge number and with $CLD(G) = (c_1, c_2, \ldots, c_n)$ such that $c_i \geq a_i$ for $i = 1, 2, \ldots, n$. The number of edges in such a graph is denoted by $g(a_1, a_2, \ldots, a_n)$. Shi [13] gave the definition of the cycle length distribution ($CLD$) of a graph and raised some related problems which have attracted the attention of researchers for a long time in the following:

**Problem 1.1** ( [13]) *Determine* $f(a_1, a_2, \ldots, a_n)$.

**Problem 1.2** ( [13]) *Determine* $g(a_1, a_2, \ldots, a_n)$.

In particular, we denote $f(1, 1, \ldots, 1)$ and $g(1, 1, \ldots, 1)$ by $f(n)$ and $g(n)$, respectively. Erdös raised the problem of determining $f(n)$ (see [1] Problem 11 ) in 1975. Partial results of Problem 1.1 are given in [2, 3, 6–12, 14, 15, 17]. Jia [5]has given the lower and upper bounds of $g(n)$ for $n$ sufficiently large. Recently, George *et al.* [4] determined $g(0, 0, 1, 1, \ldots, 1)$ for $3 \leq n \leq 22$.

In this paper, we generalize Jia's constructive technique of graphs with $g(n)$ to graphs with $g(0, 0, k, \ldots, k)$ for $k = 2, 3, 4$. The similar results with respect to $g(0, 0, k, \ldots, k)$ for $k = 2, 3, 4$ are obtained.

# 2 Bounds on $g(0, 0, k, \ldots, k)$ for $k = 2, 3, 4$.

Let $n$ and $k$ be positive integers and let $C_n$ be an $n$-cycle. We assume that all edges in $E(G) - E(C_n)$ are drawn inside the bounded region of $C_n$ and called by chords. Label the vertices of $C_n \subset G$ by $v_1, \ldots, v_n$ in cyclic order. Denote $G[H]$ by the edge-induced subgraph of $G$ whose edge set is $H$ and whose vertex set consists of all ends of edges of $H$.

**Lemma 2.1** ( [16]) *For a positive integer $r$,*

$$M(r) \leq 2^{r+1} - 1.$$

*where $r$ and $M(r)$ are denoted by the number of chords and distinct cycles of a Hamilton graph $G$, respectively.*

**Lemma 2.2**

$$g(0,0,k,\ldots,k) \geq n + log_2(kn - 2k + 1) - 1.$$

**Proof.** Let $G$ be a graph with $g(0,0,k,\ldots,k)$ edges such that $G$ has at least $k$ $t$-cycles for $3 \leq t \leq n$. Clearly, $G$ has $r = g(0,0,k,\ldots,k) - n$ chords. By Lemma 2.1, $G$ has at most $2^{r+1} - 1$ distinct cycles. Thus, we have

$$k(n-2) \leq 2^{r+1} - 1 = 2^{g(0,0,k,\ldots,k)-n+1} - 1.$$

Hence

$$g(0,0,k,\ldots,k) \geq n + log_2(kn - 2k + 1) - 1.$$

□

**Theorem 2.3** *When $n$ is large,*

$$g(0,0,2,\ldots,2) \leq n + \frac{3}{2}log_2 n + \frac{5}{2}.$$

**Proof.** We now prove this theorem by constructing a graph which has at least two $t$-cycles for $3 \leq t \leq n$. Define

$$t_i = 2^{i-1} + i, i = 1,2,\ldots,k, t_{k+1} = 1.$$

where $k$ is a positive integer such that

$$2^{k-1} + k < n \leq 2^k + k.$$

Then

$$k \leq log_2 n + 1.$$

Let $G_1^1$ be a graph which contains $C_n$ and the following edges(see Fig. 1(a)):

$$e_1 = v_1v_3, e_{i+1} = v_{t_i}v_{t_{i+1}}, i = 1,2,\ldots,k.$$

If $t_k = n$, then we don't need to add $e_{k+1}$. For convenience, we refer to $v_nv_1$ as $e_{k+1}$ in this case.

If $2^{k-1}+k<p\leq n$ and $q=n-t_k$, then $q\geq 1$. We have

$$p-q-(k+1)=\sum_{i=0}^{k-2}a_i2^i.$$

where $a_i=0$ or $1, i=0,1,\ldots,k-2$.

Define

$$E_2=\begin{cases} E_{21} & \text{if } a_0=0; \\ E_{22} & \text{if } a_0=1. \end{cases} \tag{2.1}$$

where $E_{21},E_{22}$ can choose either of the paths $v_1v_2v_4$ and $v_1v_3v_4$ and either of the paths $v_1v_2v_3v_4$ and $v_1v_3v_2v_4$, respectively.

For $1\leq i\leq k-2$, we define

$$E_{i+2}=\begin{cases} e_{i+2} & \text{if } a_i=0; \\ v_{t_{i+1}}v_{t_{i+1}+1}\cdots v_{t_{i+2}} & \text{if } a_i=1. \end{cases} \tag{2.2}$$

By (2.1) and (2.2), we have

$$|\varepsilon(E_2)|=2+a_0, |\varepsilon(E_{i+2})|=a_i2^i+1, i=1,2,\cdots,k-2.$$

Let $E_{k+1}$ be the path

$$E_{k+1}: v_{t_k}v_{t_k+1}\ldots v_{t_{k+1}}.$$

Then $|\varepsilon(E_{k+1})|=n-t_k+1=q+1$. Therefore, the paths

$$E_2,E_3,\ldots,E_k,E_{k+1}.$$

form a cycle of length

$$\begin{aligned}\sum_{j=2}^{k+1}|\varepsilon(E_j)| &= |\varepsilon(E_2)|+\sum_{i=1}^{k-2}|\varepsilon(E_{i+2})|+|\varepsilon(E_{k+1})| \\ &= 1+\sum_{i=0}^{k-2}(a_i2^i+1)+q+1 \\ &= 1+p-q-(k+1)+k-1+q+1 \\ &= p.\end{aligned}$$

Let $S$ be $p$-cycle. For $E_2=E_{21}$ or $E_{22}$, without loss of generality, assume that $S$ contain $E_{21}$. If $v_1v_2, v_2v_4\in E(S), v_1v_3, v_3v_4\notin E(S)$, then there exists a new cycle $S\prime$ such that $S\prime = S - v_1v_2 - v_2v_4 + v_1v_3 + v_3v_4$. If $v_1v_2, v_2v_4 \notin E(S), v_1v_3, v_3v_4 \in E(S)$, then there exists a new cycle $S\prime$ such that $S\prime = S - v_1v_3 - v_3v_4 + v_1v_2 + v_2v_4$. Thus $|E(S\prime)| = |E(S)| = p$. For either choice of $E_2$, there exist two distinct cycles with length $p$.

If $k+1 < p \leq 2^{k-1} + k$, then we have

$$p - k - 1 = \sum_{i=0}^{k-2} a_i 2^i.$$

Let $E_2, \ldots, E_k$ be as defined in (2.1) and (2.2). Therefore, the paths

$$E_2, E_3, \ldots, E_k, e_{k+1}.$$

form a cycle of length

$$\begin{aligned} 1 + \sum_{j=2}^{k} |\varepsilon(E_j)| &= |\varepsilon(E_2)| + \sum_{i=1}^{k-2} |\varepsilon(E_{i+2})| + 1 \\ &= 1 + \sum_{i=0}^{k-2} (a_i 2^i + 1) + 1 \\ &= 1 + p - k - 1 + k - 1 + 1 \\ &= p. \end{aligned}$$

For either choice of $E_2$, there exist two distinct cycles with length $p$.

Denoted the number of edges added to $C_n$ by $k_1$, then

$$k_1 \leq k + 1 \leq log_2 n + 2.$$

Consider $3 \leq p \leq k+1$. Let $G_1^2 = G_1^1[v_1v_2, v_2v_3, v_3v_4, e_1, e_2, \ldots, e_{k+1}]$. Then $|V(G_1^2)| = k+2$. Let $f_j = v_1 v_{t_{k-2j}}, j = 1, 2, \ldots, m$, where $m = \lfloor \frac{k-1}{2} \rfloor$. Clearly, $|V(G_1^2)| - 2m = k+2-2\lfloor \frac{k-1}{2} \rfloor$, $3 \leq k+2-2(\frac{k-1}{2}) \leq k+2-2\lfloor \frac{k-1}{2} \rfloor \leq k+2-2\frac{k-2}{2} = 4$. If $|V(G_1^2)| - 2m = 3$, then $f_m = v_1v_4$. If $|V(G_1^2)| - 2m = 4$, then $f_m = v_1v_7$.

Let $G_1^3$ be the graph with added edges $f_j$ for $1 \leq j \leq m$ to $G_1^2$(see Fig. 1(a)). If $p = k+1$, then the paths

$$E_2, e_3, e_4, \ldots, e_{k+1}.$$

form a cycle of length $k+1$. For either choice of $E_2$, there exist two distinct cycles of length $k+1$ if and only if $E_2 = E_{21}$.

If $k+2-2j \le p \le k+3-2j, j = 1,2,\ldots,m$ and $p \ne k+1$, then the paths

$$E_2, e_3, e_4, \ldots, e_{k-2j}, f_j.$$

form a cycle of length $p = k-2j+2+a_0, a_0 \in \{0,1\}$, where the value of $a_0$ dependents on the choices of $E_2$. For either choice of $E_2$, there exist two distinct cycles of length $p$.

If $3 \le p \le k+2-2m \le 4$ or 5, then we denote $H_1 = G_1^3[v_1v_2, v_2v_3, v_3v_4, e_1, e_2, v_1v_4]$ or $H_1 = G_1^3[v_1v_2, v_2v_3, v_3v_4, v_4v_5, e_1, e_2, e_3, v_1v_7]$ and easily verify that $H_1$ contains at least two cycles with length for $3 \le p \le k+1-2m$.

Denote the number of edges added to $G_1^2$ by $k_2$, then $k_2 = m = \lfloor \frac{k-1}{2} \rfloor \le \frac{k-1}{2} \le \frac{1}{2}log_2 n + \frac{1}{2}$.

Let $H_1^*$ be a graph with at least two $t$-cycles for every $3 \le i \le n$ by adding $k_1 + k_2$ edges to $C_n$. Thus

$$\varepsilon(H_1^*) = n + k_1 + k_2 \le n + log_2 n + 2 + \frac{1}{2}log_2 n + \frac{1}{2} = n + \frac{3}{2}log_2 n + \frac{5}{2}.$$

□

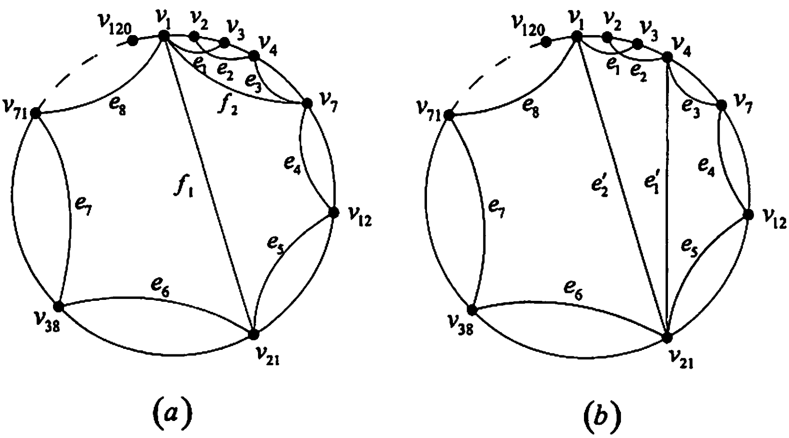


Figure 1: A construction of graphs having at least two $t$-cycles for $3 \le t \le 120$.

The following corollaries are immediate from Lemma 2.2 and Theorem 2.3.

**Corollary 2.4** *When n is large,*

$$n + log_2(n - \frac{3}{2}) \leq g(0,0,2,\ldots,2) \leq n + \frac{3}{2}log_2 n + \frac{5}{2}.$$

Recall that $|V(G_1^2)| = k+2$ and $k \leq log_2 n + 1$. By Corollary 2.4, if we add edges to $G_1^2$ by using the similar method as we construct $H_1^*$ by adding edges to $C_n$ such that $G_1^2$ be constructed a graph which has at least two $t$-cycles for $3 \leq t \leq k+2$(see Fig. 1(b)), then $k_2 \leq \frac{3}{2}log_2(k+2) + \frac{5}{2} \leq \frac{3}{2}log_2(log_2 n + 3) + \frac{5}{2}$. Thus $g(0,0,2,\ldots,2) \leq n + k_1 + k_2 \leq n + log_2 n + \frac{3}{2}log_2(log_2 n + 3) + \frac{9}{2}$. It is easy to see that

**Corollary 2.5** *For a sufficiently large integer n,*

$$g(0,0,2,\ldots,2) \leq n + log_2 n + \frac{3}{2}log_2 log_2 n + O(1).$$

**Remark 2.1** Let $k$ be the number of chords in a (2)-pancyclic graph of order $n$. Zamfirescu [18]proved that $k \leq \lfloor\frac{\sqrt{16n+1}-5}{2}\rfloor$. By Corollary 2.5, we improve it and have $k \leq log_2 n + \frac{3}{2}log_2 log_2 n + O(1)$ .

**Theorem 2.6** *When n is large,*

$$g(0,0,3,\ldots,3) \leq n + \frac{8}{7}log_2 n + 5.$$

**Proof.** In a similar manner as in the proof of Theorem 2.3, we construct a graph which has at least three $t$-cycles for $3 \leq t \leq n$. Define

$$t_{i+5} = 2^{i+2} + i + 3, i = 1,2,\ldots,k-4, t_{k+2} = 1.$$

where $k$ is a positive integer such that

$$2^{k-2} + k - 1 < n \leq 2^{k-1} + k - 1.$$

Then

$$k \leq log_2 n + 2.$$

Let $G_2^1$ be a graph which contains $C_n$ and the following edges(see Fig. 2) :

$$e_1 = v_1v_3, e_2 = v_2v_4, e_3 = v_3v_8, e_4 = v_4v_9, e_5 = v_9v_{12}.$$

$$e_{i+5} = v_{t_{i+5}}v_{t_{i+6}}, i = 1, 2, \ldots, k-4.$$

If $t_{k+1} = n$, then we don't need to add $e_{k+1}$. For convenience, we refer to $v_nv_1$ as $e_{k+1}$ in this case.

If $2^{k-2} + k - 1 < p \leq n$ and $q = n - t_{k+1}$, then $q \geq 1$. We have

$$p - q - (k+1) = \sum_{i=0}^{k-6} a_{i+3}2^{i+3} + 2a_1 + 4b_2 + b_1 - 1.$$

where $a_i = 0$ $or$ $1, i = 1$ or $3, 4, \ldots, k-3$ and $b_j = 0$ $or$ $1, j = 1, 2$.

Define

$$E_4 = \begin{cases} E_{41} & \text{if } b_1 = b_2 = 1; \\ E_{42} & \text{if } b_1 = 0, b_2 = 1; \\ E_{43} & \text{if } b_1 = 1, b_2 = 0; \\ E_{44} & \text{if } b_1 = b_2 = 0. \end{cases} \qquad (2.3)$$

where $E_{41}$ can choose any one of three paths $v_1v_2v_3v_4v_5v_6v_7v_8v_9$, $v_1v_3v_2v_4v_5v_6v_7v_8v_9$, $v_1v_2v_3v_8v_7v_6v_5v_4v_9$; $E_{42}$ can choose any one of three paths $v_1v_3v_4v_5v_6v_7v_8v_9$, $v_1v_2v_4v_5v_6v_7v_8v_9$, $v_1v_3v_8v_7v_6v_5v_4v_9$; $E_{43}$ can choose any one of three paths $v_1v_2v_3v_4v_9$, $v_1v_2v_3v_8v_9$,$v_1v_3v_2v_4v_9$; $E_{44}$ can choose any one of three paths $v_1v_3v_8v_9$, $v_1v_3v_4v_9$, $v_1v_2v_4v_9$.

$$E_5 = \begin{cases} e_5 & \text{if } a_1 = 0; \\ v_9v_{10}v_{11}v_{12} & \text{if } a_1 = 1. \end{cases} \qquad (2.4)$$

For $1 \leq i \leq k-5$, we define

$$E_{i+5} = \begin{cases} e_{i+5} & \text{if } a_{i+2} = 0; \\ v_{t_{i+5}}v_{t_{i+5}+1}\cdots v_{t_{i+6}} & \text{if } a_{i+2} = 1. \end{cases} \qquad (2.5)$$

By (2.3), (2.4) and (2.5), we have

$$|\varepsilon(E_4)| = 4b_2 + b_1 - 1 + 4 = 4b_2 + b_1 + 3, |\varepsilon(E_5)| = 2a_1 + 1;$$

$$|\varepsilon(E_{i+5})| = a_{i+2}2^{i+2} + 1, i = 1, 2, \cdots, k-5.$$

Let $E_{k+1}$ be the path

$$E_{k+1} : v_{t_{k+1}}v_{t_{k+1}+1}\ldots v_{t_{k+2}}.$$

Then $|\varepsilon(E_{k+1})| = n - t_{k+1} + 1 = q + 1$. Therefore, the paths

$$E_4, E_5, \ldots, E_k, E_{k+1}.$$

form a cycle of length

$$\begin{aligned}
\sum_{j=4}^{k+1} |\varepsilon(E_j)| &= \sum_{i=1}^{k-5} |\varepsilon(E_{i+5})| + |\varepsilon(E_{k+1})| + |\varepsilon(E_4)| + |\varepsilon(E_5| \\
&= \sum_{i=0}^{k-6} (a_{i+3}2^{i+3} + 1) + (q+1) + (4b_2 + b_1 + 3) + (2a_1 + 1) \\
&= (p - q - (k+1) - 2a_1 - 4b_2 - b_1 + 1 + k - 5) + (q+1) \\
&+ (4b_2 + b_1 + 3) + (2a_1 + 1) \\
&= p.
\end{aligned}$$

In a similar manner as we discuss $E_2$ in the proof of Theorem 2.2. For any choice of $E_4$, there exist three distinct cycles with length $p$.

If $k < p \leq 2^{k-2} + k - 1$, then we have

$$p - k - 1 = \sum_{i=0}^{k-6} a_{i+3}2^{i+3} + 2a_1 + 4b_2 + b_1 - 1.$$

Let $E_4, \ldots, E_k$ be as defined in (2.3), (2.4) and (2.5). Therefore, the paths

$$E_4, E_5, \ldots, E_k, e_{k+1}.$$

form a cycle of length

$$\begin{aligned}
1 + \sum_{j=4}^{k} |\varepsilon(E_j)| &= \sum_{i=1}^{k-5} |\varepsilon(E_{i+5})| + |\varepsilon(E_4)| + |\varepsilon(E_5| + 1 \\
&= \sum_{i=0}^{k-6} (a_{i+3}2^{i+3} + 1) + (4b_2 + b_1 + 3) + (2a_1 + 1) + 1 \\
&= (p - k - 1 - 2a_1 - 4b_2 - b_1 + 1 + k - 5) \\
&+ (4b_2 + b_1 + 3) + (2a_1 + 1) + 1 \\
&= p.
\end{aligned}$$

For any choice of $E_4$, there exist three distinct cycles with length $p$.

Denote the number of edges added to $C_n$ by $k_1$, then

$$k_1 \leq k + 1 \leq log_2 n + 3.$$

Consider $3 \leq p \leq k+1$, let $G_2^2 = G_2^1[v_1v_2, v_2v_3, \ldots, v_8v_9, e_1, e_2, \ldots, e_{k+1}]$, then $|V(G_2^2)| = k + 5$. Let

$$f_j = v_1 v_{t_{k+1-7j}}, j = 1, 2, \ldots, m.$$

where $m = \lfloor \frac{k-4}{7} \rfloor$. If $12 = t_6 < t_{k+1-7m} < t_{13} = 1035$, then $f_{m+1} = v_1v_{12}, f_{m+2} = v_1v_9$. If $9 \leq t_{k+1-7m} \leq t_6 = 12$, then $f_m = v_1v_{12}, f_{m+1} = v_1v_9$.

Let $G_2^3$ be the graph with added edges $f_j$ for $1 \leq j \leq m + 1$(or $1 \leq j \leq m + 2$) to $G_2^2$(see Fig. 2). If $p = k + 1$, then the paths

$$E_4, E_5, e_6, e_7, \ldots, e_{k+1}.$$

form a cycle of length $k+1$. For any choice of $E_4$, there exist three distinct cycles of length $k + 1$ if and only if $E_4 = E_{44}, E_5 = e_5$.

If $k + 2 - 7j \leq p \leq k + 8 - 7j, j = 1, 2, \ldots, m$ and $p \neq k + 1$, then the paths

$$E_4, E_5, e_6, e_7, \ldots, e_{k-7j}, f_j.$$

form a cycle of length $p = k + 1 + 4b_2 + b_1 + 2a_1 - 7j, a_1, b_1, b_2 \in \{0, 1\}$, where the value of $a_1, b_1$ and $b_2$ dependent on the choices of $E_5$ and $E_4$, respectively. For any choice of $E_4$, there exist three distinct cycles of length $p$.

If $3 \leq p \leq k+1-7m \leq 12$, then we denote $H_2 = G_2^3[v_1v_2, v_2v_3, \ldots, v_{11}v_{12}, v_1v_9, v_1v_{12}, e_1, e_2, e_3, e_4, e_5]$ and easily verify that $H_2$ satisfies at least three cycles with length $3 \leq p \leq k + 1 - 7m$.

Denote the number of edges added to $G_2^2$ by $k_2$, then $k_2 \leq m + 2 = \lfloor \frac{k-4}{7} \rfloor + 2 \leq \frac{log_2 n}{7} + 2$.

Let $H_2^*$ be a graph with at least three $t$-cycles for every $3 \leq i \leq n$ by adding $k_1 + k_2$ edges to $C_n$. Thus

$$\varepsilon(H_2^*) = n + k_1 + k_2 \leq n + log_2 n + 3 + \frac{log_2 n}{7} + 2 = n + \frac{8 log_2 n}{7} + 5.$$

□

The following corollaries are immediate from Lemma 2.2 and Theorem 2.6.

**Corollary 2.7** *When $n$ is large,*

$$n + log_2(3n-5) - 1 \le g(0,0,3,\dots,3) \le n + \frac{8}{7}log_2 n + 5.$$

Applying the above construction to $G_2^2$ by using the similar method as we construct $H_2^*$, we shall obtain the following upper bound.

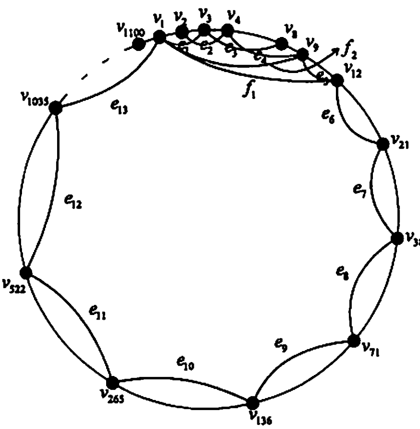


Figure 2: A construction of graphs having at least three $t$-cycles for $3 \le t \le 1100$.

**Corollary 2.8** *For a sufficiently large integer $n$,*

$$g(0,0,3,\dots,3) \le n + log_2 n + \frac{8}{7}log_2 log_2 n + O(1).$$

**Theorem 2.9** *When $n$ is large,*

$$g(0,0,4,\dots,4) \le n + \frac{5}{4}log_2 n + 5.$$

**Proof.** In a similar manner as in the proof of Theorem 2.3, we construct a graph which has at least four $t$-cycles for $3 \le t \le n$. Define

$$t_{i+4} = 2^{i+1} + i + 3, i = 1, 2, \dots, k-3, t_{k+2} = 1.$$

where $k$ is a positive integer such that

$$2^{k-2}+k<n\leq 2^{k-1}+k.$$

Then

$$k\leq log_2 n+2.$$

Let $G_3^1$ be a graph which contains $C_n$ and the following edges(see Fig. 3)
:

$$e_1=v_1v_3, e_2=v_2v_4, e_3=v_4v_7, e_4=v_5v_8.$$

$$e_{i+4}=v_{t_{i+4}}v_{t_{i+5}}, i=1,2,\ldots,k-3.$$

If $t_{k+1}=n$, then we don't need to add $e_{k+1}$. For convenience, we refer to $v_nv_1$ as $e_{k+1}$ in this case.

If $2^{k-2}+k<p\leq n$ and $q=n-t_{k+1}$, then $q\geq 1$. We have

$$p-q-(k+1)=\sum_{i=0}^{k-3}a_i2^i.$$

where $a_i=0$ *or* $1, i=0,1,\ldots,k-3$.

Define

$$E_3=\begin{cases} E_{31} & \text{if } a_0=0;\\ E_{32} & \text{if } a_0=1.\end{cases} \tag{2.6}$$

where $E_{31}, E_{32}$ can choose either of the paths $v_1v_2v_4$ and $v_1v_3v_4$ and either of the paths $v_1v_2v_3v_4$ and $v_1v_3v_2v_4$, respectively.
and

$$E_4=\begin{cases} E_{41} & \text{if } a_1=0;\\ E_{42} & \text{if } a_1=1.\end{cases} \tag{2.7}$$

where $E_{41}, E_{42}$ can choose either of the paths $v_4v_7v_8$ and $v_4v_5v_8$ and either of the paths $v_4v_5v_6v_7v_8$ and $v_4v_7v_6v_5v_8$, respectively.

For $1\leq i\leq k-4$, we define

$$E_{i+4}=\begin{cases} e_{i+4} & \text{if } a_{i+1}=0;\\ v_{t_{i+4}}v_{t_{i+4}+1}\cdots v_{t_{i+5}} & \text{if } a_{i+1}=1.\end{cases} \tag{2.8}$$

By (2.6), (2.7)and (2.8), we have

$$|\varepsilon(E_3)|=2+a_0, |\varepsilon(E_4)|=2+2a_1;$$

$$|\varepsilon(E_{i+4})| = 1 + a_{i+1}2^{i+1}, i = 1, 2, \cdots, k-4.$$

Let $E_{k+1}$ be the path

$$E_{k+1} : v_{t_{k+1}} v_{t_{k+1}+1} \cdots v_{t_{k+2}}.$$

Then $|\varepsilon(E_{k+1})| = n - t_{k+1} + 1 = q + 1$. Therefore, the paths

$$E_3, E_4, \ldots, E_k, E_{k+1}.$$

form a cycle of length

$$\begin{aligned}
\sum_{j=3}^{k+1} |\varepsilon(E_j)| &= \sum_{i=1}^{k-4} |\varepsilon(E_{i+4})| + |\varepsilon(E_3)| + |\varepsilon(E_4)| + |\varepsilon(E_{k+1})| \\
&= \sum_{i=0}^{k-3} (a_i 2^i + 1) + 2 + q + 1 \\
&= p - q - (k+1) + k - 2 + 2 + q + 1 \\
&= p.
\end{aligned}$$

In a similar manner as we discuss $E_2$ in the proof of Theorem 2.2. For either choice of $E_3$ and $E_4$, there exist four different cycles with length $p$.

If $k + 1 < p \leq 2^{k-2} + k$, then we have

$$p - k - 1 = \sum_{i=0}^{k-3} a_i 2^i.$$

Let$E_3, \ldots, E_k$ be as defined in (2.6), (2.7)and (2.8). Therefore, the paths

$$E_3, E_4, \ldots, E_k, e_{k+1}.$$

form a cycle of length

$$\begin{aligned}
1 + \sum_{j=3}^{k} |\varepsilon(E_j)| &= \sum_{i=1}^{k-4} |\varepsilon(E_{i+4})| + |\varepsilon(E_3)| + |\varepsilon(E_4)| + 1 \\
&= \sum_{i=0}^{k-3} (a_i 2^i + 1) + 2 + 1 \\
&= 1 + p - k - 1 + k - 2 + 2 \\
&= p.
\end{aligned}$$

For either choice of $E_3$ and $E_4$, there exist four different cycles with length $p$.

Denote the number of edges added to $C_n$ by $k_1$, then

$$k_1 \leq k+1 \leq log_2 n + 3.$$

Consider $3 \leq p \leq k+1$, let $G_3^2 = G_1[v_1v_2, v_2v_3, v_3v_4, v_5v_6, v_7v_8, v_1v_3, v_2 v_4, v_5v_8, v_4v_7, e_5, e_6, \ldots, e_{k+1}]$, then $|V(G_3^2)| = k+4$. Let

$$f_j = v_1 v_{t_{k+1-4j}}, j = 1, 2, \ldots, m.$$

where $m = \lfloor \frac{k-4}{4} \rfloor$. If $8 = t_5 < t_{k+1-4m} \leq t_8 = 39$, then $f_{m+1} = v_1v_8, f_{m+2} = v_1v_4$. If $4 \leq t_{k+1-7m} \leq t_5 = 8$, then $f_m = v_1v_8, f_{m+1} = v_1v_4$.

Let $G_3^3$ be the graph with added edges $f_i$ for $1 \leq i \leq m+1$(or $1 \leq i \leq m+2$) to $G_3^2$(see Fig. 3). If $p = k+1$, then the paths

$$E_3, E_4, e_5, e_6, \ldots, e_{k+1}.$$

form a cycle of length $k+1$. For either choice of $E_3$ and $E_4$, there exist four distinct cycles of length $k+1$ if and only if $E_3 = E_{31}, E_4 = E_{41}$.

If $k+2-4j \leq p \leq k+5-4j, j = 1, 2, \ldots, m$ and $p \neq k+1$, then the paths

$$E_3, E_4, e_5, e_6, \ldots, e_{k-4j+1}, f_j.$$

form a cycle of length $p = k - 4j + 2 + a_0 + 2a_1, a_0, a_1 \in \{0, 1\}$, where the value of $a_0, a_1$ dependent on the choices of $E_3$ and $E_4$, respectively. For either choice of $E_3$ and $E_4$, there exist four distinct cycles of length $p$.

If $3 \leq p \leq k+1-4m \leq 8$, then we denote $H_3 = G_3^3[v_1v_2, v_2v_3, \ldots, v_7v_8, v_1v_4, v_1v_8, e_1, e_2, e_3, e_4]$ and easily verify that $H_3$ satisfy at least four cycles with length $3 \leq p \leq k+1-4m$.

Denote the number of edges added to $G_3^2$ by $k_2$, then $k_2 = m+2 = \lfloor \frac{k-4}{4} \rfloor + 2 \leq \frac{k}{4} + 1 \leq \frac{1}{4} log_2 n + 2$.

Let $H_3^*$ be a graph with at least four $t$-cycles for every $3 \leq i \leq n$ by adding $k_1 + k_2$ edges to $C_n$. Thus

$$\varepsilon(H_3^*) = n + k_1 + k_2 \leq n + log_2 n + 3 + \frac{1}{4} log_2 n + 2 = n + \frac{5}{4} log_2 n + 5.$$

□

The following corollaries are immediate from Lemma 2.2 and Theorem 2.9.

**Corollary 2.10** *When n is large,*

$$n + log_2(n - \frac{7}{4}) + 1 \leq g(0, 0, 4, \ldots, 4) \leq n + \frac{5}{4}log_2 n + 5.$$

Applying the above construction to $G_3^2$ by using the similar method as we construct $H_3^*$, we shall obtain the following upper bound.

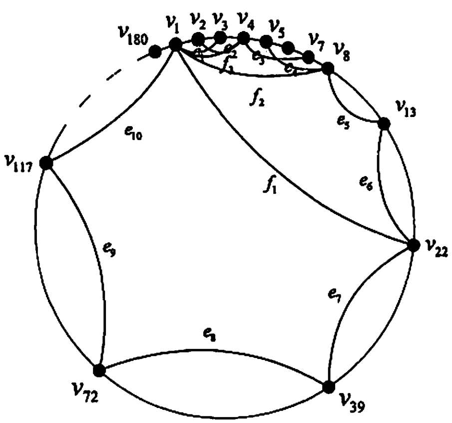


Figure 3: A construction of graphs having at least four $t$-cycles for $3 \leq t \leq 180$.

**Corollary 2.11** *For a sufficiently large integer n,*

$$g(0, 0, 4, \ldots, 4) \leq n + log_2 n + \frac{5}{4}log_2 log_2 n + O(1).$$

# 3 Conclusion

In this paper, we studied the extremal functions $g(0, 0, k, \ldots, k)$ for $k = 2, 3, 4$ and gave the partial results of Problem 1.2. The Problems 1.1

and 1.2, which are so called the cycle-size packing and covering problems and have been studied extensively, remain open. Now we present some of the many corresponding(unanswered) problems and conjectures which have been developed in the following:

**Problem 3.1** ( [10]) *Determine the maximum number of edges in a hamiltonian graph on n vertices with no repeated cycle lengths.*

**Conjecture 3.2** ( [8])

$$\lim_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} = \sqrt{2.4}.$$

Let $f_2(n)$ be the maximum number of edges in a 2-connected graph on $n$ vertices in which no two cycles have the same length.

**Conjecture 3.3** ( [2]) $lim \frac{f_2(n)-n}{\sqrt{n}} = 1.$

**Conjecture 3.4** ( [5]) $g(n) = n + log_2 n + O(1)$ *as* $n \longrightarrow \infty$.

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