Extraordinary Subsets

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Abstract

For $n \geq 1$, we let a_n count the number of nonempty subsets S of $\{1, 2, 3, \ldots, n\} = [n]$, where the size of S equals the minimal element of S. Such a subset is called an *extraordinary* subset of $[n]$ and we find that $a_n = F_n$, the nth Fibonacci number. Then, for $n \geq k \geq 1$, we let $a(n, k)$ count the number of times the integer k appears among these a_n extraordinary subsets of n. Here we have $a(n, k) = a(n-1, k) + a(n-2, k-1)$, for $n \geq 3$ and $n > k \geq 2$. Formulas and properties for $t_n = \sum_{k=1}^{n} a(n, k)$ and $s_n = \sum_{k=1}^{n} k\, a(n, k)$ are given for $n \geq 1$. Finally, for fixed $n \geq 1$, we find that the sequence $a(n, k)$ is unimodal and examine the maximum element for the sequence. In this context the Catalan numbers make an entrance.

1. Determining a_n

For $n \geq 1$, let a_n count the number of subsets S of $\{1, 2, 3, \ldots, n\} = [n]$, where the minimal element in S equals the size of S. Such a subset is called an *extraordinary* subset of $[n]$. We find, for example, that $a_5 = 5$ — for the extraordinary subsets $\{1\}$, $\{2, 3\}$, $\{2, 4\}$, $\{2, 5\}$, and $\{3, 4, 5\}$ of $[5]$.

Both [1] and [2] have only $\{1\}$ as an extraordinary subset, so $a_1 = a_2 = 1$. For $n \geq 3$, consider the extraordinary subsets S of $[n]$ according to whether or not $n \in S$. (i) If $n \notin S$, then any of the a_{n-1} extraordinary subsets of $[n-1]$ prove to be extraordinary subsets of $[n]$. (ii) To obtain the extraordinary subsets of $[n]$ that contain n, start with each extraordinary subset S of $[n-2]$, increase each element by 1, then add in the element n. Since the situations in (i) and (ii) exhaust all possibilities and have nothing in common, it follows that

$$a_n = a_{n-1} + a_{n-2}, \quad a_1 = 1, \quad a_2 = 1, \text{ so}$$

$$a_n = F_n, \text{ the } n\text{th Fibonacci number.}$$

(This example appears as Exercise 50 on Pp. 263-264 of [1].)

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2. How Many Times Does an Element Appear among the a_n Extraordinary Subsets?

For $n \geq k \geq 1$, let $a(n,k)$ count the number of times that the element k appears among the a_n extraordinary subsets of $[n]$. The following table provides the values for $1 \leq k \leq n \leq 12$.

$n \setminus k$	1	2	3	4	5	6	7	8	9	10	11	12
1	1											
2	1											
3	1	1	1									
4	1	2	1	1								
5	1	3	2	2	2							
6	1	4	4	3	3	3						
7	1	5	7	5	5	5	5					
8	1	6	11	9	8	8	8	8				
9	1	7	16	16	13	13	13	13	13			
10	1	8	22	27	22	21	21	21	21	21		
11	1	9	29	43	38	34	34	34	34	34	34	
12	1	10	37	65	65	56	55	55	55	55	55	55

Table 1

For $n \geq 1$, $a(n,1) = 1$ because $\{1\}$ is the only extraordinary subset of $[n]$ of size 1. When $n \geq 3$ and $n > k \geq 2$, Table 1 suggests that

$$a(n,k) = a(n-1,k) + a(n-2,k-1).$$

This follows in general and is established by an argument similar to that given in Section 1. If S is an extraordinary subset of $[n]$, where $k \in S$ and $n \notin S$, then S is an extraordinary subset of $[n-1]$ and the number of times this happens is $a(n-1,k)$. Otherwise, consider each of the $a(n-2,k-1)$ extraordinary subsets of $[n-2]$ which contain $k-1$. For each such subset, increase each element by 1 and then add in the element n. This then provides the remaining extraordinary subsets of $[n]$ which contain k. As before, these two situations exhaust all possibilities and have nothing in common, so $a(n,1) = 1$, for $n \geq 1$, $a(2,2) = 0$, and

$$a(n,k) = a(n-1,k) + a(n-2,k-1), \; n \geq 3, \; n > k \geq 2.$$

Searching for patterns in Table 1, our first result reads as follows:

Theorem 1: For $n \geq 2$, $a(n,n) = F_{n-2}$.

Proof: Here we are counting the extraordinary subsets of $[n]$ that contain n. There is 1 extraordinary subset of size 2 — namely, $\{2,n\}$; $\binom{n-4}{1}$ such subsets of size 3 — the subsets $\{3,x,n\}$, where $3 < x < n$; $\binom{n-5}{2}$ such subsets of size 4 — the subsets $\{4,y,z,n\}$, where $4 < y < z < n$; ..., and,
(i) for n even, $\binom{n-(\frac{n}{2}+1)}{\frac{n}{2}-2} = \frac{n}{2}-1$ such subsets of size $\frac{n}{2}$ — those containing $\frac{n}{2}$ and n, and $\frac{n}{2}-2$ of the elements $\frac{n}{2}+1,\ldots,n-1$.
(ii) for n odd, $\binom{n-\frac{n-3}{2}}{\frac{n+3}{2}} = 1$ such subset of size $\frac{n+1}{2}$ — namely, the subset $\{\frac{n+1}{2}, \frac{n+3}{2}, \ldots, n\}$.

Consequently, since $1 = \binom{n-3}{0}$, it follows that

$$a(n,n) = \begin{cases} \binom{n-3}{0} + \binom{n-4}{1} + \binom{n-5}{2} + \cdots + \binom{\frac{n}{2}-1}{\frac{n}{2}-2} = F_{n-2}, & n \text{ even} \\ \binom{n-3}{0} + \binom{n-4}{1} + \binom{n-5}{2} + \cdots + \binom{\frac{n+3}{2}}{\frac{n+3}{2}} = F_{n-2}, & n \text{ odd.} \end{cases}$$

This follows from Lucas's Formula (1876), which appears as Theorem 12.4 on Pp. 155-156 of [3]. It will prove useful in the next section.

As we continue to examine the results in Table 1 we observe the following.

$$\begin{aligned} a(10,7) &= 21 = a(9,7) + a(8,6) = 13 + 8 = (8+5) + (5+3) \\ &= 8 + 2(5) + 3 = \tbinom{2}{0}a(8,7) + \tbinom{2}{1}a(7,6) + \tbinom{2}{2}a(6,5) \\ &= \tbinom{3}{0}a(7,7) + \tbinom{3}{1}a(6,6) + \tbinom{3}{2}a(5,5) + \tbinom{3}{3}a(4,4) \end{aligned}$$

This example now leads us to our next result.
Theorem 2: For $0 \leq r \leq n-k$,

$$a(n,k) = \sum_{i=0}^{r} \binom{r}{i} a(n-r-i, k-i).$$

Proof: The proof follows by induction. We see that the result is true for $r = 0$, and follows for $r = 1$ from above. Assuming the result true for $a(n-1,k)$ and $a(n-2,k-1)$, for $0 \leq r-1 \leq n-k-1$, we now find that

$$\begin{aligned} a(n,k) =& a(n-1,k) + a(n-2,k-1) \\ =& \sum_{i=0}^{r-1} \binom{r-1}{i} a((n-1)-(r-1)-i, k-i) \\ &+ \sum_{i=0}^{r-1} \binom{r-1}{i} a((n-2)-(r-1)-i, (k-1)-i) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{r-1} \binom{r-1}{i} a(n-r-i, k-i) \\
&+ \sum_{i=0}^{r-1} \binom{r-1}{i} a(n-r-i-1, (k-1)-i) \\
&= \binom{r-1}{0} a(n-r, k) + \sum_{i=1}^{r-1} \binom{r-1}{i} a(n-r-i, k-i) \\
&+ \sum_{i=1}^{r-1} \binom{r-1}{i-1} a(n-r-(i-1)-1, (k-1)-(i-1)) \\
&+ \binom{r-1}{r-1} a(n-r-(r-1)-1, (k-1)-(r-1)) \\
&= \binom{r-1}{0} a(n-r, k) + \sum_{i=1}^{r-1} \left(\binom{r-1}{i} + \binom{r-1}{i-1} \right) a(n-r-i, k-i) \\
&+ \binom{r-1}{r-1} a(n-2r, k-r) \\
&= \binom{r}{0} a(n-r, k) + \sum_{i=1}^{r-1} \binom{r}{i} a(n-r-i, k-i) + \binom{r}{r} a(n-2r, k-r) \\
&= \sum_{i=0}^{r} \binom{r}{i} a(n-r-i, k-i).
\end{aligned}
$$

3. The Total Number of Elements in the a_n Extraordinary Subsets of $[n]$

For $n \geq 1$, let t_n count the total number of elements (repeats are counted) that occur among the a_n extraordinary subsets of $[n]$. Hence, $t_n = \sum_{k=1}^{n} a(n, k)$. So $t_1 = 1$, $t_2 = 1$, and, from the results in Section **2**, it follows that for $n \geq 3$,

$$
\begin{aligned}
t_n &= \sum_{k=1}^{n} a(n, k) \\
&= a(n, 1) + \sum_{k=2}^{n-1} [a(n-1, k) + a(n-2, k-1)] + a(n, n) \\
&= a(n-1, 1) + \sum_{k=2}^{n-1} a(n-1, k) + \sum_{k=2}^{n-1} a(n-2, k-1) + a(n, n)
\end{aligned}
$$

$$= \left(a(n-1,1) + \sum_{k=2}^{n-1} a(n-1,k) \right) + \sum_{k=1}^{n-2} a(n-2,k) + a(n,n)$$

$$= \sum_{k=1}^{n-1} a(n-1,k) + \sum_{k=1}^{n-2} a(n-2,k) + a(n,n)$$

$$= t_{n-1} + t_{n-2} + a(n,n) = t_{n-1} + t_{n-2} + F_{n-2}.$$

The solution for this nonhomogeneous recurrence relation has the form $t_n^{(h)} + t_n^{(p)}$, where $t_n^{(h)}$ denotes the homogeneous part of the solution and $t_n^{(p)}$ the particular part. Following the techniques given in Chapter 7 of [1] and Chapter 10 of [2], we find that the forms for $t_n^{(h)}$ and $t_n^{(p)}$ are

$$t_n^{(h)} = c_1\alpha^n + c_2\beta^n \quad \text{and} \quad t_n^{(p)} = An\alpha^n + Bn\beta^n,$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, the golden ratio, and $\beta = \frac{1-\sqrt{5}}{2}$. Substituting $An\alpha^n + Bn\beta^n$ for t_n in the given recurrence relation we have

$$An\alpha^n + Bn\beta^n = A(n-1)\alpha^{n-1} + B(n-1)\beta^{n-1}$$
$$+A(n-2)\alpha^{n-2} + B(n-2)\beta^{n-2} + \frac{1}{\sqrt{5}}(\alpha^{n-2} + \beta^{n-2}),$$

from which it follows that $An\alpha^n = A(n-1)\alpha^{n-1} + A(n-2)\alpha^{n-2} + \frac{1}{\sqrt{5}}\alpha^{n-2}$ and $A = \frac{-1+\sqrt{5}}{10}$. A similar calculation leads to $B = \frac{-1-\sqrt{5}}{10}$. So

$$t_n = c_1\alpha^n + c_2\beta^n + \left(\frac{-1+\sqrt{5}}{10}\right) n\alpha^a + \left(\frac{-1-\sqrt{5}}{10}\right) n\beta^n, \text{ and from}$$

$$1 = t_1 = c_1\alpha + c_2\beta + \left(\frac{-1+\sqrt{5}}{10}\right)\alpha + \left(\frac{-1-\sqrt{5}}{10}\right)\beta, \text{ and}$$

$$1 = t_2 = c_1\alpha^2 + c_2\beta^2 + \left(\frac{-1+\sqrt{5}}{10}\right)(2)\alpha^2 + \left(\frac{-1-\sqrt{5}}{10}\right)(2)\beta^2, \text{ we have}$$

$$c_1 = \frac{3\sqrt{5}}{25} \quad \text{and} \quad c_2 = -\frac{3\sqrt{5}}{25}.$$

Therefore we have

Theorem 3:
$$t_n = \frac{3\sqrt{5}}{25}\alpha^n - \frac{3\sqrt{5}}{25}\beta^n - \frac{1}{10}n(\alpha^n + \beta^n) + \frac{\sqrt{5}}{10}n(\alpha^n - \beta^n)$$
$$= \frac{3}{5}F_n - \frac{1}{10}nL_n + \frac{1}{2}nF_n, \quad n \geq 1.$$

(Here L_n denotes the nth Lucas number. The Lucas numbers are defined recursively by $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$, $n \geq 2$.)

4. Properties of t_n

Using Theorem 3 we find the values of t_n, for $1 \leq n \leq 20$, to be as follows:

n	t_n	n	t_n	n	t_n	n	t_n
1	1	6	18	11	324	16	4957
2	1	7	33	12	564	17	8462
3	3	8	59	13	977	18	14406
4	5	9	105	14	1685	19	24465
5	10	10	185	15	2895	20	41455

Table 2

The sequence $t_1, t_2, t_3, \ldots$ appears as sequence $A010049$ in [4], where it is referred to as the sequence of Second-Order Fibonacci numbers. Further, t_n counts the total number of summands (parts) in the compositions of $n+1$, where 1 is not allowed as a summand.

The entries in Table 2 now lead to the following seven results.

Theorem 4: For $n \geq 1$, t_{3n} is divisible by 3.

Proof: First we observe that, for $n \geq 1$, $L_n = F_{n+1} + F_{n-1}$. (This is Corollary 55 on P. 80 of [3].) From this it follows that $L_{3n} = F_{3n+1} + F_{3n-1} = F_{3n+1} + (F_{3n+1} - F_{3n}) = 2F_{3n+1} - F_{3n}$. This result, along with Theorem 3, implies that

$$\begin{aligned} t_{3n} &= 3\left(\frac{1}{5}F_{3n} - \frac{1}{10}nL_{3n} + \frac{1}{2}nF_{3n}\right) \\ &= 3\left(\frac{1}{5}F_{3n} - \frac{n}{10}(2F_{3n+1} - F_{3n}) + \frac{1}{2}nF_{3n}\right) \\ &= 3\left(\frac{1}{5}\right)(F_{3n} - nF_{3n+1} + 3nF_{3n}) = 3\left(\frac{1}{5}\right)((2n+1)F_{3n} - nF_{3n-1}). \end{aligned}$$

Consequently, to prove this theorem we need to establish that $((2n+1)F_{3n} - nF_{3n-1})$ is a multiple of 5, for $n \geq 1$. We proceed by induction, observing that for $n = 1, 3F_3 - F_2 = 3(2) - 1 = 5$. So we assume the result true for an arbitrary (but fixed) $n\,(\geq 1)$ — that is, we assume that $((2n+1)F_{3n} - nF_{3n-1})$ is a multiple of 5. For the case of $n+1$, we have

$$\begin{aligned}
&(2(n+1)+1)F_{3(n+1)} - (n+1)F_{3(n+1)-1} \\
&= (2n+3)(3F_{3n} + 2F_{3n-1}) - (n+1)(2F_{3n} + F_{3n-1}) \\
&= (4n+7)F_{3n} + (3n+5)F_{3n-1} \\
&= (2(2n+1)+5)F_{3n} + (5n+5)F_{3n-1} - 2nF_{3n-1} \\
&= (5F_{3n} + (5n+5)F_{3n-1}) + 2((2n+1)F_{3n} - nF_{3n-1}),
\end{aligned}$$

which is a multiple of 5 since $((2n+1)F_{3n} - nF_{3n-1})$ is a multiple of 5 by the induction hypothesis. The result now follows for $n \geq 1$.

Theorem 5: For $n \geq 1$, t_{5n} is divisible by 5.

Proof: Once again we shall use induction. Since $t_5 = 10$, the result is true for $n = 1$. Assuming the result true for some arbitrary (but fixed) $n\,(\geq 1)$, we have t_{5n} divisible by 5. That is,

$$\begin{aligned}
t_{5n} &= \frac{3}{5}F_{5n} - \frac{1}{10}(5n)L_{5n} + \frac{1}{2}(5n)F_{5n} \\
&= \left(\frac{3}{5} + \frac{5n}{2}\right)F_{5n} - \frac{1}{2}n(2F_{5n+1} - F_{5n}) = \left(\frac{3}{5} + 3n\right)F_{5n} - nF_{5n+1}
\end{aligned}$$

is divisible by 5. Continuing now with $n+1$, we have

$$\begin{aligned}
t_{5(n+1)} &= \frac{3}{5}F_{5(n+1)} - \frac{1}{10}(5)(n+1)L_{5(n+1)} + \frac{1}{2}(5)(n+1)F_{5(n+1)} \\
&= \frac{3}{5}F_{5n+5} - \frac{1}{2}(n+1)(2F_{5n+6} - F_{5n+5}) + \frac{5}{2}(n+1)F_{5n+5} \\
&= \left(\frac{3}{5} + 3(n+1)\right)F_{5n+5} - (n+1)F_{5n+6} \\
&= \left(\frac{3}{5} + 3(n+1)\right)(5F_{5n+1} + 3F_{5n}) - (n+1)(8F_{5n+1} + 5F_{5n}) \\
&= (7n+10)F_{5n+1} + \left(\frac{29}{5} + 4n\right)F_{5n} \\
&= (10n+10)F_{5n+1} - 3nF_{5n+1} + \left(\frac{9}{5} + 9n\right)F_{5n} - (5n-4)F_{5n} \\
&= (10n+10)F_{5n+1} + 3\left(\left(\frac{3}{5} + 3n\right)F_{5n} - nF_{5n+1}\right) - (5n-4)F_{5n},
\end{aligned}$$

where the second summand is divisible by 5 from the induction hypothesis. Then by Theorem 16.1 on P. 196 of [3], for $m \geq 1$ and $n \geq 1, F_m$ divides

F_{mn}. Consequently, F_5 (= 5) divides F_{5n} and the summand $(5n-4)F_{5n}$ is divisible by 5. So the result for $n+1$ now follows.

Theorem 6: For $n \geq 0$, t_{5n+4} is divisible by 5.

Proof: If $n = 0$, then $t_{5n+4} = t_4 = 5$. This establishes the initial case. Assuming the result true for some arbitrary (but fixed) $n\,(\geq 0)$, from Theorem 3 the following is divisible by 5.

$$\begin{aligned}
t_{5n+4} &= \frac{3}{5}F_{5n+4} - \frac{1}{10}(5n+4)L_{5n+4} + \frac{1}{2}(5n+4)F_{5n+4} \\
&= \frac{3}{5}F_{5n+4} - \frac{1}{10}(5n+4)(2F_{5n+5} - F_{5n+4}) + \frac{1}{2}(5n+4)F_{5n+4} \\
&= (3n+3)F_{5n+4} - \left(n + \frac{4}{5}\right)F_{5n+5}.
\end{aligned}$$

Replacing n with $n+1$ we have

$$\begin{aligned}
t_{5(n+1)+4} &= t_{5n+9} = (3(n+1)+3)F_{5(n+1)+4} - \left((n+1) + \frac{4}{5}\right)F_{5(n+1)+5} \\
&= (3n+6)F_{5n+9} - \left(n + \frac{9}{5}\right)(F_{5n+9} + F_{5n+8}) \\
&= \left(2n + \frac{21}{5}\right)F_{5n+9} - \left(n + \frac{9}{5}\right)F_{5n+8} \\
&= \left(2n + \frac{21}{5}\right)(5F_{5n+5} + 3F_{5n+4}) - \left(n + \frac{9}{5}\right)(3F_{5n+5} + 2F_{5n+4}) \\
&= \left(\frac{78}{5} + 7n\right)F_{5n+5} + (4n+9)F_{5n+4} \\
&= \left(\left(\frac{78}{5} + 7n\right) + \left(3n + \frac{12}{5}\right) - 3\left(n + \frac{4}{5}\right)\right)F_{5n+5} \\
&\quad + (4n+9)F_{5n+4} \\
&= (10n+18)F_{5n+5} + 3\left((3n+3)F_{5n+4} - \left(n + \frac{4}{5}\right)F_{5n+5}\right) \\
&\quad - 5nF_{5n+4}.
\end{aligned}$$

Since F_5 (= 5) divides F_{5n+5} and the second summand is divisible by 5 due to the induction hypothesis, it follows that t_{5n+4} is divisible by 5 for $n \geq 0$.

Theorem 7: For $n \geq 1$, t_{6n} is even.

Proof: Since $t_n = \frac{3}{5}F_n - \frac{1}{10}nL_n + \frac{1}{2}nF_n$, it follows that

$$\begin{aligned}
t_{6n} &= \frac{3}{5}F_{6n} - \frac{1}{10}(6n)L_{6n} + \frac{1}{2}(6n)F_{6n} \\
&= \frac{3}{5}F_{6n} - \frac{3n}{5}(2F_{6n+1} - F_{6n}) + 3nF_{6n} \\
&= \frac{1}{5}[F_{6n}(18n+3) - 6nF_{6n+1}], \text{ so} \\
5t_{6n} &= F_{6n}(18n+3) - 6nF_{6n+1}.
\end{aligned}$$

Then, as $F_3\,(=2)$ divides F_{3k} for $k \geq 0$, we have $F_{6n} = F_{3(2n)}$ divisible by 2. So $5t_{6n}$ is even, and as $gcd(2,5) = 1$, it follows that 2 divides t_{6n}, for $n \geq 1$.

Theorem 8: For $n \geq 0$, t_{6n+5} is even.

Proof: From $t_n = \frac{3}{5}F_n - \frac{1}{10}nL_n + \frac{1}{2}nF_n$, it follows that

$$\begin{aligned}
t_{6n+5} &= \frac{3}{5}F_{6n+5} - \frac{1}{10}(6n+5)L_{6n+5} + \frac{1}{2}(6n+5)F_{6n+5} \\
&= \frac{3}{5}F_{6n+5} - \frac{1}{10}(6n+5)(2F_{6n+6} - F_{6n+5}) + \frac{1}{2}(6n+5)F_{6n+5} \\
&= \frac{1}{5}((18n+18)F_{6n+5} - (6n+5)F_{6n+6}), \text{ so} \\
5t_{6n+5} &= (18n+18)F_{6n+5} - (6n+5)F_{6n+6}.
\end{aligned}$$

Since $F_3\,(=2)$ divides F_{6k} for $k \geq 0$, we see that $5t_{6n+5}$ is even. Then, as the $gcd(2,5) = 1$, we have 2 divides t_{6n+5} for $n \geq 0$.

Theorem 9: For $n \geq 1$, if t_n is even, then $n \equiv 0$ or $5 \pmod 6$.

Proof: From the recursive definition of the Fibonacci numbers we know that F_n is even if and only if $n \equiv 0 \pmod 3$ (and so F_n is odd if and only if $n \equiv 1$ or $2 \pmod 3$). Since

$$\begin{aligned}
t_n &= \frac{3}{5}F_n - \frac{1}{10}nL_n + \frac{1}{2}nF_n \\
&= \frac{3}{5}F_n - \frac{1}{10}n(2F_{n+1} - F_n) + \frac{1}{2}nF_n \\
&= \left(\frac{3}{5} + \frac{3n}{5}\right)F_n - \frac{1}{5}nF_{n+1}, \text{ we have} \\
5t_n &= 3(n+1)F_n - nF_{n+1}.
\end{aligned}$$

With t_n even, it follows that $3(n+1)F_n - nF_{n+1}$ is even.

(1) If $n \equiv 1 \pmod 6$, then $n \equiv 1 \pmod 3$ and $n+1 \equiv 2 \pmod 3$, so both F_n and F_{n+1} are odd. Also, since $n \equiv 1 \pmod 6$, n is odd, so

$(n+1)$ is even. Then as nF_{n+1} is odd and $3(n+1)F_n$ is even, it follows that $3(n+1)F_n - nF_{n+1}$ is odd — contradicting $5t_n$ being even.

(2) If $n \equiv 2 \pmod 6$, then $n \equiv 2 \pmod 3$, so F_n is odd and F_{n+1} is even. Therefore, with n even, once again we have $3(n+1)F_n - nF_{n+1}$ odd — contradicting $5t_n$ being even.

Similar arguments for $n \equiv 3 \pmod 6$ and $n \equiv 4 \pmod 6$ likewise contradict that $5t_n$ is even.

Hence it follows that $n \equiv 0$ or $5 \pmod 6$.

From Theorems 7, 8, and 9 we now conclude this section with the following.

Theorem 10: For $n \geq 1$, t_n is even if and only if $n \equiv 0$ or $5 \pmod 6$.

5. The Sum of the Elements in the a_n Extraordinary Subsets of $[n]$

For $n \geq 1$, let s_n denote the sum of all the elements (repeats are counted) that occur among the a_n extraordinary subsets of $[n]$. So for a fixed value of n, $s_n = \sum_{k=1}^{n} k\,a(n,k)$. We find, for instance, that $s_1 = 1$, $s_2 = 1$, and $s_3 = 6$. To obtain a formula for s_n we consider the following for $n \geq 3$:

$$\begin{aligned} s_n &= \sum_{k=1}^{n} k\,a(n,k) = \sum_{k=1}^{n} k\,(a(n-1,k) + a(n-2,k-1)) + nF_{n-2} \\ &= \sum_{k=1}^{n} k\,a(n-1,k) + \sum_{k=1}^{n} ((k-1)+1)\,a(n-2,k-1) + nF_{n-2}. \end{aligned}$$

Since $a(n-1,n) = a(n-2,n-1) = a(n-2,0) = 0$, we find that

$$\begin{aligned} s_n &= \sum_{k=1}^{n-1} k\,a(n-1,k) + \sum_{k=2}^{n-1} (k-1)\,a(n-2,k-1) \\ &+ \sum_{k=1}^{n-1} a(n-2,k-1) + nFn-2 \\ &= \sum_{k=1}^{n-1} k\,a(n-1,k) + \sum_{k=1}^{n-2} k\,a(n-2,k) + \sum_{k=1}^{n-1} a(n-2,k-1) + nF_{n-2} \\ &= s_{n-1} + s_{n-2} + t_{n-2} + nF_{n-2} \\ &= s_{n-1} + s_{n-2} + \frac{3}{5}F_{n-2} - \frac{1}{10}(n-2)L_{n-2} + \frac{1}{2}(n-2)F_{n-2} + nF_{n-2} \\ &= s_{n-1} + s_{n-2} - \frac{1}{10}(n-2)L_{n-2} + \frac{1}{10}(15n-4)F_{n-2}, \quad n \geq 3. \end{aligned}$$

The form of the solution here is

$$s_n = c_1\alpha^n + c_2\beta^n + An\alpha^n + Bn\beta^n + Cn^2\alpha^n + Dn^2\beta^n,$$

where the particular part is $An\alpha^n + Bn\beta^n + Cn^2\alpha^n + Dn^2\beta^n$.
Upon substituting $An\alpha^n + Cn^2\alpha^n$ into the recurrence relation

$$\begin{aligned} s_n &= s_{n-1} + s_{n-2} - \frac{1}{10}(n-2)\alpha^{n-2} + \frac{1}{10}(15n-4)\frac{\alpha^{n-2}}{\sqrt{5}} \\ &= s_{n-1} + s_{n-2} + \left(\frac{\sqrt{5}-2}{5\sqrt{5}}\right)\alpha^{n-2} + \left(\frac{(3\sqrt{5}-1)n}{10}\right)\alpha^{n-2}, \end{aligned}$$

we learn that

$$\begin{aligned} &An\alpha^n + Cn^2\alpha^n \\ &= A(n-1)\alpha^{n-1} + C(n-1)^2\alpha^{n-1} + A(n-2)\alpha^{n-2} + C(n-2)^2\alpha^{n-2} \\ &= \left(\frac{\sqrt{5}-2}{5\sqrt{5}}\right)\alpha^{n-2} + \left(\frac{(3\sqrt{5}-1)n}{10}\right)\alpha^{n-2}, \quad \text{so} \\ &An\alpha^2 + Cn^2\alpha^2 = A(n-1)\alpha + C(n-1)^2\alpha + A(n-2) + C(n-2)^2 \\ &+ \left(\frac{\sqrt{5}-2}{5\sqrt{5}}\right) + \left(\frac{(3\sqrt{5}-1)}{10}\right)n, \text{ from which it follows that} \end{aligned}$$

$$A = -\frac{7}{50} + \frac{3\sqrt{5}}{25} \text{ and } C = -\frac{1}{10} + \frac{2\sqrt{5}}{25}.$$

Similar calculations give us

$$B = -\frac{7}{50} - \frac{3\sqrt{5}}{25} \text{ and } D = -\frac{1}{10} - \frac{2\sqrt{5}}{25}.$$

$$\begin{aligned} \text{So } s_n &= c_1\alpha^n + c_2\beta^n + \left(-\frac{7}{50} + \frac{3\sqrt{5}}{25}\right)n\alpha^n + \left(-\frac{7}{50} - \frac{3\sqrt{5}}{25}\right)n\beta^n \\ &+ \left(-\frac{1}{10} + \frac{2\sqrt{5}}{25}\right)n^2\alpha^n + \left(-\frac{1}{10} - \frac{2\sqrt{5}}{25}\right)n^2\beta^n. \end{aligned}$$

From $s_1 = 1$ and $s_2 = 1$, it follows that $c_1 = \frac{6\sqrt{5}}{125}$ and $c_2 = -\frac{6\sqrt{5}}{125}$, and this now provides the following.

Theorem 11: For $n \geq 1$,

$$s_n = \frac{6}{25}F_n + \frac{3}{5}nF_n - \frac{7}{50}nL_n + \frac{2}{5}n^2F_n - \frac{1}{10}n^2L_n.$$

6. Properties of s_n

Using Theorem 11, we find the values of s_n, for $1 \le n \le 20$, to be as follows:

n	s_n	n	s_n	n	s_n	n	s_n
1	1	6	66	11	2202	16	49338
2	1	7	142	12	4188	17	89585
3	6	8	290	13	7871	18	161646
4	12	9	582	14	14639	19	290036
5	31	10	1141	15	26982	20	517768

Table 3

The results in Table 3 suggest the following.

Theorem 12: For $n \ge 1$, (a) s_{3n} is divisible by 6; and, (b) s_{4n} is even.
Proof: (a) From Theorem 11 it follows that

$$\begin{aligned} s_{3n} &= \frac{6}{25}F_{3n} + \frac{3}{5}(3n)F_{3n} - \frac{7}{50}(3n)L_{3n} + \frac{2}{5}(3n)^2F_{3n} - \frac{1}{10}(3n)^2L_{3n} \\ &= \left(\frac{6}{25} + \frac{9n}{5} + \frac{18n^2}{5}\right)F_{3n} - \left(\frac{21n}{50} + \frac{9n^2}{10}\right)L_{3n}. \end{aligned}$$

Since $L_{3n} = 2F_{3n+1} - F_{3n}$, we find that

$$\begin{aligned} s_{3n} &= \left(\frac{6}{25} + \frac{111n}{50} + \frac{45n^2}{10}\right)F_{3n} - 2\left(\frac{21n}{50} + \frac{9n^2}{10}\right)F_{3n+1}, \text{ so} \\ 50s_{3n} &= (225n^2 + 111n + 12)F_{3n} - 2(45n^2 + 21n)F_{3n+1}. \end{aligned}$$

We need to show that $(225n^2 + 111n + 12)F_{3n} - 2(45n^2 + 21n)F_{3n+1}$ is divisible by 12 for $n \ge 1$. For then [$50s_{3n}$ divisible by 12] $\Longrightarrow$ [$25s_{3n}$ divisible by 6] $\Longrightarrow$ [s_{3n} divisible by 6, since the $gcd(25, 6) = 1$].

Once again, we use the result of Theorem 16.1 on P. 196 of [3] and note here that F_3 $(= 2)$ divides F_{3n} for $n \ge 1$. Further, $(225n^2 + 111n + 12) = 3(75n^2 + 37n + 4)$. If n is even, then $(75n^2 + 37n + 4) = n(75n + 37) + 4$ is even. For n odd, $75n^2$ and $37n$ are both odd, so $75n^2 + 37n$ is even, as is $75n^2 + 37n + 4$. Consequently, $(225n^2 + 111n + 12)F_{3n}$ is divisible by 12, for $n \ge 1$. Also, $2(45n^2 + 21n) = (2)(3)(n)(15n + 7)$, which is divisible by 12 when n is even. If n is odd, then $15n + 7$ is even. Consequently, for $n \ge 1$, $2(45n^2 + 21n)F_{3n+1}$ is divisible by 12. Therefore, for $n \ge 1$, $50s_{3n} = (225n^2 + 111n + 12)F_{3n} - 2(45n^2 + 21n)F_{3n+1}$ is divisible by 12, and it follows that s_{3n} is divisible by 6.

Proof: (b) Again from Theorem 11 we start with

$$s_{4n} = \frac{6}{25}F_{4n} + \frac{3}{5}(4n)F_{4n} - \frac{7}{50}(4n)L_{4n} + \frac{2}{5}(4n)^2F_{4n} - \frac{1}{10}(4n)^2L_{4n}.$$

Since $L_{4n} = F_{4n+1} + F_{4n-1} = 2F_{4n+1} - F_{4n}$, it follows that

$$\begin{aligned} s_{4n} &= \left(\frac{6}{25} + \frac{12n}{5} + \frac{32n^2}{5}\right)F_{4n} - \left(\frac{14n}{25} + \frac{8n^2}{5}\right)(2F_{4n+1} - F_{4n}) \\ &= \left(\frac{6}{25} + \frac{74n}{25} + \frac{40n^2}{5}\right)F_{4n} - \left(\frac{28n}{25} + \frac{16n^2}{5}\right)F_{4n+1}, \text{ so} \\ 25s_{4n} &= (6 + 74n + 200n^2)F_{4n} - (28n + 80n^2)F_{4n+1}, \end{aligned}$$

which is even. Consequently, since the $gcd(2, 25) = 1$, we have s_{4n} even.

7. The Unimodality of the Numbers $a(n, k)$

Examining the binomial coefficients in a fixed row of Pascal's triangle, we find that the numbers increase to a maximum — that maximum occurring once for n even and twice for n odd. Following (the last occurrence of) the maximum, the numbers then decrease. Consequently we say that for a fixed integer $n \geq 0$, the binomial coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ form a unimodal sequence. In general, a sequence $x_0, x_1, \ldots, x_n$ is called *unimodal* if there is an integer r, where (i) $0 \leq r \leq n$; (ii) $x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_r$; and (iii) $x_r \geq x_{r+1} \geq x_{r+2} \geq \ldots \geq x_n$.

The results in Table 1 suggest that for a fixed positive integer n, the sequence $a(n, 1), a(n, 2), \ldots, a(n, n)$ is unimodal. In this section we shall establish that this is true for all n. Along the way various properties of the numbers $a(n, k)$ will come to the forefront. We start with the following.

Theorem 13: Let $n = 2i$, for $i \geq 1$. If $i + 1 \leq k \leq n$, then $a(n, k) = F_{n-2}$.

Proof: Here we find that

$$a(n, k) = 1 + \binom{n-4}{1} + \binom{n-5}{2} + \binom{n-6}{3} + \cdots + \binom{n-(i-1)}{i-2},$$

where (i) 1 accounts for the extraordinary subset $\{2, k\}$, (ii) $\binom{n-4}{1}$ accounts for the extraordinary subsets containing 3 and k and any element in $[n]$, other than $1, 2, 3, k$, and (iii) $\binom{n-(i-1)}{i-2}$ accounts for the extraordinary subsets that contain k and i, and any $i - 2$ of the $i - 1$ $(= n - (i - 1))$ elements in $[n]$, excluding $1, 2, 3, \ldots, i$, and k. Consequently, by Lucas's Formula

(Theorem 12.4 on Pp. 155 – 156 of [3]),

$$a(n,k) = \binom{n-3}{0} + \binom{n-4}{1} + \binom{n-5}{2} + \binom{n-6}{3} + \cdots + \binom{\frac{n}{2}-1}{\frac{n}{2}-2}$$
$$= \sum_{j=0}^{\frac{n}{2}-2} \binom{(n-3)-j}{j} = F_{(n-3)+1} = F_{n-2}.$$

This explains why the entries in the second half of each row of Table 1, for n even, all equal F_{n-2}.

Referring back to Theorem 13, we now want to know what happens when $k = i$.

Theorem 14: Let $n = 2k$, for $k \geq 1$. Then $a(n,k) = F_{n-2} - 1$.
Proof: Here

$$\begin{aligned} a(n,k) &= a(2k,k) \\ &= 1 + \binom{n-4}{1} + \binom{n-5}{2} + \cdots + \binom{n-k}{k-3} + \binom{n-k}{k-1}, \end{aligned}$$

where $\binom{n-k}{k-3}$ accounts for the extraordinary subsets of $[n]$ that contain $k-1$ and k, and any $k-3$ of the $n-k (= k)$ elements of $[n]$ excluding $1, 2, 3, \ldots, k-2$. The summand $\binom{n-k}{k-1}$ counts the extraordinary subsets of $[n]$ where k is minimal.

Since $\binom{n-k}{k-1} = \binom{k}{k-1} = \binom{k-1}{k-2} + \binom{k-1}{k-1} = \binom{n-(k+1)}{k-2} + \binom{n-(k+1)}{k-1}$, it follows that

$$a(n,k) = \binom{n-3}{0} + \binom{n-4}{1} + \binom{n-4}{1} + \binom{n-5}{2} + \cdots + \binom{n-k}{k-3}$$
$$+ \binom{n-(k+1)}{k-2} + \binom{n-(k+1)}{k-1}$$
$$= \sum_{j=0}^{\frac{n}{2}-2} \left(\binom{(n-3)-j}{j} + \binom{2k-(k+1)}{k-1} \right)$$
$$= \sum_{j=0}^{\frac{n}{2}-2} \binom{(n-3)-j}{j} + 1 = F_{(n-3)+1} + 1 = F_{n-2} + 1.$$

Turning now to the case where n is odd, we find comparable results in Theorems 15 and 16.

Theorem 15: Let $n = 2i - 1$, for $i \geq 1$. If $i \leq k \leq 2i - 1$, then $a(n,k) = F_{n-2}$.

Proof: (i) For $i+1 \le k \le 2i-1$, it follows that $i+1 \le k \le 2i$, so from the result in Section 2, we find that

$$a(n,k) = a(2i-1,k) = a(2i,k) - a(2(i-1),k-1).$$

Then from Theorem 13 we have

$$a(n,k) = F_{2i-2} - F_{2(i-1)-2} = F_{2i-2} - F_{2i-4} = F_{2i-3} = F_{(2i-1)-2} = F_{n-2}.$$

(ii) When $k = i$, the result in Theorem 14 shows us that $a(n,k) = a(2i-1,i) = a(2i,i) - a(2(i-1),i-1) = (F_{2i-2}+1) - (F_{2(i-1)-2}+1) = F_{2i-2} - F_{2i-4} = F_{2i-3} = F_{(2i-1)-2} = F_{n-2}$.

Theorem 16: For $n = 2k-1$, where $k \ge 2$, $a(n,k-1) = F_{n-2} + (k-2)$.

Proof: This result follows by induction on k. When $k = 2$ (and $n = 3$), $a(n,k-1) = a(3,1) = 1 = F_1 = F_1 + 0 = F_1 + (2-2) = F_{n-2} + (k-2)$. Assuming $a(n,k-1) = a(2k-1,k-1) = F_{n-2} + (k-2)$ for some fixed (but arbitrary) $k\,(\ge 2)$, then for $k+1$ and $n = 2(k+1)-1 = 2k+1$, we have

$$\begin{aligned} a(n,(k+1)-1) &= a(n,k) = a(n-1,k) + a(n-2,k-1) \\ &= a(2k,k) + a(2k-1,k-1) \\ &= (F_{(n-1)-2}+1) + (F_{(n-2)-2} + (k-2)), \end{aligned}$$

where the first summand follows from Theorem 14 and the second from the induction hypothesis. Consequently,

$$a(n,k) = (F_{n-3} + F_{n-4}) + (k-1) = F_{n-2} + (k-1) = F_{n-2} + ((k+1)-2),$$

and the result follows for $k \ge 2$ (and $n = 2k-1$).

Our next three results will help to verify that, for m even, the entries

$$a(3m,1), a(3m,2), a(3m,3), \ldots, a(3m,3m)$$

form a unimodal sequence where the maximum value is

$$a(3m,m) = a(3m,m+1).$$

Theorem 17: For $m \ge 1$, $a(3m,m) = a(3m,m+1)$.

Proof: We find that

$$\begin{aligned} a(3m,m) =& 1 + \binom{3m-4}{1} + \binom{3m-5}{2} + \binom{3m-6}{3} + \cdots + \\ & \binom{3m-m}{m-3} + \binom{3m-m}{m-1}, \end{aligned}$$

where (i) the summand 1 accounts for the extraordinary subset $\{2, m\}$, the only subset of $[3m]$ which contains m and where 2 is minimal; (ii) the summand $\binom{3m-4}{1}$ accounts for the extraordinary subsets of $[3m]$ which contain $3, m$, and one of the elements from $[3m]$, other than $1, 2, 3, m$; (iii) the summand $\binom{3m-5}{2}$ accounts for the extraordinary subsets of $[3m]$ of size 4, which contain m; (iv) the summand $\binom{3m-m}{m-3}$ accounts for the extraordinary subsets of $[3m]$ of size $m-1$, which contain m; and, (v) the summand $\binom{3m-m}{m-1}$ accounts for those extraordinary subsets of $[3m]$, where m is minimal.

Meanwhile, a similar argument provides

$$a(3m, m+1) = 1 + \binom{3m-4}{1} + \binom{3m-5}{2} + \binom{3m-6}{3} + \cdots + \binom{3m-m}{m-3} + \binom{3m-(m+1)}{m-2} + \binom{3m-(m+1)}{m}.$$

From the last two summands in $a(3m, m+1)$, we now find that

$$\binom{3m-(m+1)}{m-2} + \binom{3m-(m+1)}{m} = \binom{2m-1}{m-2} + \binom{2m-1}{m}$$
$$= \binom{2m-1}{m-2} + \binom{2m-1}{m-1} = \binom{2m}{m-1} = \binom{3m-m}{m-1},$$

the last summand in $a(3m, m)$. Consequently, it follows that

$$a(3m, m) = a(3m, m+1), \text{ for } m \geq 1.$$

Theorem 18: For $m \geq 2$ and $1 \leq k \leq m-1$, $a(3m, k) < a(3m, k+1)$.

Proof: For $k = 1$ (and $m \geq 2$), $a(3m, 1) = 1 < 3m - 2 = a(3m, 2)$, as the extraordinary subsets of $[3m]$ which contain 2 are those where 2 is minimal. These subsets consist of 2 and one other element selected from $[3m] - \{1, 2\}$.

For $k = 2$ (and $m \geq 3$), $a(3m, 2) = 3m - 2 = 1 + (3m - 3) < 1 + (3m-3)(\frac{5}{2})$. Now $((m-1) \geq k) \implies (m \geq 3) \implies ((\frac{3m-4}{2}) \geq \frac{5}{2})$. So $a(3m, 2) < 1 + (3m-3)(\frac{5}{2}) \leq 1 + (3m-3)(\frac{3m-4}{2}) = 1 + \binom{3m-3}{2} = a(3m, 3)$.

For $k \geq 3$ (and $m \geq 4$), $a(3m, k) = 1 + \binom{3m-4}{1} + \binom{3m-5}{2} + \cdots + \binom{3m-k}{k-3} + \binom{3m-k}{k-1}$, where (i) the summand 1 accounts for the extraordinary subset $\{2, k\}$, the only subset of $[3m]$ which contains k and for which 2 is minimal; (ii) the summand $\binom{3m-4}{1}$ accounts for the extraordinary subsets of $[3m]$ of size 3 which contain $3, k$, and one of the elements from $[3m] - \{1, 2, 3, k\}$; (iii) the summand $\binom{3m-5}{2}$ accounts for the extraordinary subsets of $[3m]$ of size 4 which contain $4, k$, and two of the elements from $[3m] - \{1, 2, 3, 4, k\}$; (iv) the summand $\binom{3m-k}{k-3}$ accounts for the extraordinary subsets of $[3m]$ which contain the minimal element $k-1$, along with k, and $k-3$ elements from $[3m]-[k]$; and, (v) the summand $\binom{3m-k}{k-1}$ accounts for the extraordinary subsets of $[3m]$ where k is minimal.

Meanwhile, for $k \geq 3$ (and $m \geq 4$),
$a(3m,k+1) = 1+\binom{3m-4}{1}+\binom{3m-5}{2}+\cdots+\binom{3m-k}{k-3}+\binom{3m-(k+1)}{k-2}+\binom{3m-(k+1)}{k}$, where (i) the summand 1 accounts for the extraordinary subset $\{2, k+1\}$, the only subset of $[3m]$ which contains $k+1$ and for which 2 is minimal; (ii) the summand $\binom{3m-4}{1}$ accounts for the extraordinary subsets of $[3m]$ of size 3 which contain $3, k+1$, and one of the elements from $[3m]-\{1,2,3,k+1\}$; (iii) the summand $\binom{3m-5}{2}$ accounts for the extraordinary subsets of $[3m]$ of size 4 which contain $4, k+1$, and two of the elements from $[3m]-\{1,2,3,4,k+1\}$; (iv) the summand $\binom{3m-(k+1)}{k-2}$ accounts for the extraordinary subsets of $[3m]$ which contain the minimal element k, along with $k+1$, and $k-2$ elements from $[3m]-[k+1]$; and, (v) the summand $\binom{3m-(k+1)}{k}$ accounts for the extraordinary subsets of $[3m]$ where $k+1$ is minimal.

Consider the last summand in $a(3m,k)$ —namely, $\binom{3m-k}{k-1}$. We find that

$$\binom{3m-k}{k-1} = \binom{3m-k-1}{k-1} + \binom{3m-k-1}{k-2}.$$

For the last two summands in $a(3m, k+1)$ we have

$$\binom{3m-(k+1)}{k-2} + \binom{3m-(k+1)}{k} = \binom{3m-k-1}{k-2} + \binom{3m-k-1}{k}.$$

From the $(3m-k-1)$th row of Pascal's triangle, where $1 \leq k \leq m-1$, the values of the binomial coefficients increase, so it follows that $\binom{3m-k-1}{k-1} < \binom{3m-k-1}{k}$. Consequently, for $m \geq 2$ and $1 \leq k \leq m-1$, $a(3m,k) < a(3m,k+1)$.

Focusing on the sequence $a(3m,1), a(3m,2), \ldots, a(3m,3m)$, for m even, we see that for $m=2$, the resulting sequence — namely, $1,4,4,3,3,3$ — is unimodal, with maximum value $a(6,3) = a(6,2) = 4$. Our next result helps to answer what happens when $m \geq 4$.

Theorem 19: Let m be even with $m \geq 4$. Then for $m+1 \leq k \leq \frac{3m}{2}-1$, $a(3m,k) > a(3m,k+1)$.

Proof: For $m+1 \leq k \leq \frac{3m}{2}-1$,

$$a(3m,k) = 1+\binom{3m-4}{1}+\binom{3m-5}{2}+\cdots+\binom{3m-k}{k-3}+\binom{3m-k}{k-1},$$

where (i) the summand 1 accounts for the extraordinary subset $\{2, k\}$; (ii) the summand $\binom{3m-4}{1}$ accounts for the extraordinary subsets containing $3, k$, and one element from $[3m]-\{1,2,3,k\}$; (iii) the summand $\binom{3m-5}{2}$ accounts for the extraordinary subsets containing $4, k$, and two elements from

$[3m] - \{1,2,3,4,k\}$; (iv) the summand $\binom{3m-k}{k-3}$ accounts for the extraordinary subsets containing $k-1, k$, and $k-3$ elements from $[3m] - [k]$; and, (v) the summand $\binom{3m-k}{k-1}$ accounts for the extraordinary subsets where k is minimal.

Meanwhile, for $m+1 \le k \le \frac{3m}{2} - 1$, a similar argument leads to

$$a(3m, k+1) = 1 + \binom{3m-4}{1} + \binom{3m-5}{2} + \cdots + \binom{3m-k}{k-3} + \binom{3m-(k+1)}{k-2} + \binom{3m-(k+1)}{k}.$$

So the result will follow if we can show that $\binom{3m-k}{k-1} > \binom{3m-(k+1)}{k-2} + \binom{3m-(k+1)}{k}$. Since $\binom{3m-k}{k-1} = \binom{3m-k-1}{k-1} + \binom{3m-k-1}{k-2}$, we need to determine when $\binom{3m-k-1}{k-1} > \binom{3m-(k+1)}{k}$. We find that

$$\binom{3m-k-1}{k-1} > \binom{3m-(k+1)}{k} \iff$$
$$\frac{(3m-k-1)!}{(k-1)!(3m-2k)!} > \frac{(3m-k-1)!}{k!(3m-2k-1)!} \iff$$
$$k!(3m-2k-1)! > (k-1)!(3m-2k)! \iff$$
$$k > (3m-2k) \iff 3k > 3m \iff k > m.$$

Since $m+1 \le k$, it follows that $k > m$. But to guarantee all of the logical equivalences we need to have $3m-2k-1 \ge 0$ — that is, we need to have $3m-1 \ge 2k$ or $k \le \frac{1}{2}(3m-1) = \frac{3m}{2} - \frac{1}{2}$. This follows because $k \le \frac{3m}{2} - 1$. Consequently, all of the above inequalities are valid in reverse order, from which it follows that $a(3m,k) > a(3m,k+1)$ for $m+1 \le k \le \frac{3m}{2} - 1$.

The results in Theorems 13, 14, 17, 18, and 19 now lead to the following.

Theorem 20: For m even, with $m \ge 2$, the $3m$ entries

$$a(3m,1), a(3m,2), \ldots, a(3m,3m)$$

form a unimodal sequence with maximum value $a(3m,m+1) = a(3m,m)$.

Before proceeding, let us demonstrate the result of Theorem 20 for the case where $n = 12$. From Theorem 13, with $i = 6$, we have $a(12,k) = F_{12-2} = F_{10} = 55$, for $7 \le k \le 12$. Theorem 14, with $i = k = 6$, tells us that $a(12,6) = F_{12-2} + 1 = F_{10} + 1 = 56 > 55 = a(12,7)$. When $m = 4$ in Theorem 19, we learn that $a(12,k) > a(12,k+1)$ for all $5 \le k \le 5$, so we have $a(12,5) > a(12,6)$. Theorem 17 tells us that when $m = 4$, $a(12,4) = a(12,5)$. Finally, Theorem 18, for $m = 4$, implies that $a(12,1) < a(12,2) < a(12,3) < a(12,4)$.

It then follows that the sequence $a(12,1), a(12,2), \ldots, a(12,12)$ is unimodal, and that $a(12,4) = a(12,5)$ is the maximum value that occurs in the sequence.

To deal with the case where m is odd, we need the following result, comparable to Theorem 19.

Theorem 21: Let m be odd with $m \geq 5$. Then for $m+1 \leq k \leq \frac{3m-1}{2}$, $a(3m,k) > a(3m,k+1)$.

Proof: The proof here is similar to the one given for Theorem 19.

The results in Theorems $15, 16, 17, 18$, and 21 now lead to the following.

Theorem 22: For m odd, $m \geq 1$, the $3m$ entries

$$a(3m,1), a(3m,2), \ldots, a(3m,3m)$$

form a unimodal sequence with maximum value $a(3m,m+1) = a(3m,m)$.

Now we'll consider the case where $n = 3m+1$. The results in Table 1 (of Section 2), for $m = 1, 2$, and 3, demonstrate that the sequence in the rows for $n = 4, 7$, and 10, are unimodal with a maximum value that only occurs once in each sequence. We shall prove this result for $m \geq 4$ by considering the following eight theorems.

Theorem 23: For $m \geq 1$, $a(3m+1,m) = a(3m+1,m+2)$.

Proof: From Table 1 we see that this result is true for $m = 1, 2$, and 3, so we shall consider $m \geq 4$. We find that

$$\begin{aligned} a(3m+1,m) =& 1 + \binom{(3m+1)-4}{1} + \binom{(3m+1)-5}{2} + \cdots \\ & + \binom{(3m+1)-m}{m-3} + \binom{(3m+1)-m}{m-1}, \end{aligned}$$

where (i) the summand 1 accounts for the extraordinary subset $\{2,m\}$, the only subset of $[3m+1]$ which contains m and where 2 is minimal; (ii) the summand $\binom{(3m+1)-4}{1}$ accounts for the extraordinary subsets of $[3m+1]$ which contain $3, m$, and one of the elements in $[3m+1]$, other than $1, 2, 3, m$; (iii) the summand $\binom{(3m+1)-m}{m-3}$ accounts for the extraordinary subsets of $[3m+1]$ of size $m-1$, which contain m; and, (iv) the summand $\binom{(3m+1)-m}{m-1}$ accounts for the extraordinary subsets of $[3m+1]$ where m is minimal.

A similar argument provides

$$a(3m+1,m+2) = 1 + \binom{(3m+1)-4}{1} + \binom{(3m+1)-5}{2} + \cdots$$
$$+\binom{(3m+1)-m}{m-3} + \binom{(3m+1)-m-1}{m-2} + \binom{(3m+1)-m-2}{m-1}$$
$$+\binom{(3m+1)-m-2}{m+1}.$$

When we examine the last summand for $a(3m+1,m)$, we see that

$$\begin{aligned}\binom{(3m+1)-m}{m-1} &= \binom{2m+1}{m-1} = \binom{2m}{m-1} + \binom{2m}{m-2}\\ &= \binom{2m}{m-2} + \binom{2m-1}{m-1} + \binom{2m-1}{m+1}\\ &= \binom{(3m+1)-m-1}{m-2} + \binom{(3m+1)-m-2}{m-1}\\ &+\binom{(3m+1)-m-2}{m+1},\end{aligned}$$

the last three summands in $a(3m+1,m+2)$. This establishes that $a(3m+1,m) = a(3m+1,m+2)$.

But how is $a(3m+1,m+1)$ related to $a(3m+1,m)$? The following will answer this question and show us one way that the Catalan numbers arise from the entries in Table 1. (The Catalan numbers appear as sequence *A*000108 in [4]. Numerous examples where these numbers occur can be found in [5, 6].)

Theorem 24: For $m \geq 1, a(3m+1,m+1) > a(3m+1,m)$. In fact, $a(3m+1,m+1) - a(3m+1,m) = \frac{1}{m+1}\binom{2m}{m}$, the mth Catalan number.
Proof: Here we find that

$$a(3m+1,m+1) - a(3m+1,m)$$
$$= \binom{(3m+1)-(m+1)}{m-2} + \binom{(3m+1)-(m+1)}{m} - \binom{(3m+1)-m}{m-1}$$
$$= \binom{2m}{m-2} + \binom{2m}{m} - \binom{2m+1}{m-1} = \binom{2m}{m} - \binom{2m}{m-1} = \frac{1}{m+1}\binom{2m}{m}.$$

From Theorems 23 and 24 we also learn the following.

Theorem 25: For $m \geq 1$, $a(3m+1,m+1) > a(3m+1,m+2)$.

The next result is comparable to Theorem 18.

Theorem 26: For $m \geq 2$ and $1 \leq k \leq m-1$, $a(3m+1,k) < a(3m+1,k+1)$.

Proof: The results in Table 1 confirm that this is true for the cases where $m = 2$ and $m = 3$. For $m \geq 4$, we need to compare

$$a(3m+1,k) = 1 + \binom{(3m+1)-4}{1} + \binom{(3m+1)-5}{2} + \cdots + \binom{(3m+1)-k}{k-3} + \binom{(3m+1)-k}{k-1} \text{ with}$$

$$a(3m+1,k+1) = 1 + \binom{(3m+1)-4}{1} + \binom{(3m+1)-5}{2} + \cdots + \binom{(3m+1)-k}{k-3} + \binom{(3m+1)-(k+1)}{k-2} + \binom{(3m+1)-(k+1)}{k}.$$

Comparing the last summand in $a(3m+1,k)$ with the last two summands in $a(3m+1,k+1)$, we find that

$$\begin{aligned}\binom{(3m+1)-k}{k-1} &= \binom{3m-k}{k-1} + \binom{3m-k}{k-2} \\ &= \binom{(3m+1)-(k+1)}{k-2} + \binom{(3m+1)-(k+1)}{k-1} \\ &< \binom{(3m+1)-(k+1)}{k-2} + \binom{(3m+1)-(k+1)}{k},\end{aligned}$$

since $\binom{(3m+1)-(k+1)}{k-1} < \binom{(3m+1)-(k+1)}{k}$, for $1 \leq k \leq m-1$. Consequently, for $m \geq 2$ and $1 \leq k \leq m-1$, $a(3m+1,k) < a(3m+1,k+1)$.

Focusing now on the sequence $a(3m+1,1), a(3m+1,2), \ldots, a(3m+1,3m+1)$, for m even, we see that for $m = 2$, the resulting sequence — namely, $1,5,7,5,5,5,5$ — is unimodal, with maximum value $a(7,3) = 7$. As with Theorem 19, our next result helps to answer what happens when $m \geq 4$.

Theorem 27: Let m be even with $m \geq 4$. Then for $m+1 \leq k \leq \frac{3m}{2} - 1$, $a(3m+1,k) > a(3m+1,k+1)$.

Proof: For $m+1 \leq k \leq \frac{3m}{2} - 1$,

$$a(3m+1,k) = 1 + \binom{(3m+1)-4}{1} + \binom{(3m+1)-5}{2} + \cdots + \binom{(3m+1)-k}{k-3} + \binom{(3m+1)-k}{k-1}, \text{ while}$$

$$a(3m+1,k+1) = 1 + \binom{(3m+1)-4}{1} + \binom{(3m+1)-5}{2} + \cdots + \binom{(3m+1)-k}{k-3} + \binom{(3m+1)-(k+1)}{k-2} + \binom{(3m+1)-(k+1)}{k}.$$

As in the proof of Theorem 19, here the result will follow if we can show that $\binom{(3m+1)-k}{k-1} > \binom{(3m+1)-(k+1)}{k-2} + \binom{(3m+1)-(k+1)}{k}$. Since $\binom{(3m+1)-k}{k-1} = \binom{(3m+1)-(k+1)}{k-1} + \binom{(3m+1)-(k+1)}{k-2}$, we need to determine when $\binom{(3m+1)-(k+1)}{k-1} > \binom{(3m+1)-(k+1)}{k}$. We find that

$$\binom{(3m+1)-(k+1)}{k-1} > \binom{(3m+1)-(k+1)}{k} \iff$$
$$\frac{(3m-k)!}{(k-1)!(3m-2k+1)!} > \frac{(3m-k)!}{k!(3m-2k)!} \iff$$
$$k!(3m-2k)! > (k-1)!(3m-2k+1)! \iff$$
$$k > (3m-2k+1) \iff$$
$$3k > 3m+1 \iff k > m + \frac{1}{3}.$$

Since $m+1 \leq k$, it follows that $k > m + \frac{1}{3}$. But in order to guarantee all of the logical equivalences we need to have $3m - 2k \geq 0$ — that is, we need to have $3m \geq 2k$ or $k \leq \frac{3m}{2}$. This follows because $k \leq \frac{3m}{2} - 1$. Consequently, all of the above inequalities are true in reverse order, from which it follows that $a(3m+1,k) > a(3m+1,k+1)$ for $m+1 \leq k \leq \frac{3m}{2} - 1$.

The next result now follows from Theorems $15, 16, 23, 26$, and 27.

Theorem 28: For m even with $m \geq 2$, the $3m+1$ entries $a(3m+1,1), a(3m+1,2), \ldots, a(3m+1,3m+1)$ form a unimodal sequence with maximum value $a(3m+1,3m+1)$.

To deal with the case where m is odd we need the following result, comparable to Theorem 21.

Theorem 29: Let m be odd with $m \geq 5$. Then for $m+1 \leq k \leq \frac{3m-1}{2}$, $a(3m+1,k) > a(3m+1,k+1)$.

Proof: The proof here is similar to the one given for Theorem 27.

The results in Theorems $13, 14, 23, 24, 25, 26$, and 29 now lead to the following.

Theorem 30: For m odd, $m \geq 1$, the $3m+1$ entries $a(3m+1,1), a(3m+1,2), \ldots, a(3m+1,3m+1)$ form a unimodal sequence with maximum value $a(3m+1,m+1)$.

Finally, let us consider the case where $n = 3m+2$. The results in Table 1, for $m = 1, 2$, and 3, demonstrate that the sequences in the rows for $n = 5, 8$, and 11, are unimodal with a maximum value that only occurs once in each sequence — at the entry $a(3m+2,m+1)$. That this is also true for $m \geq 4$ follows from Theorems $20, 22, 28$, and 30, and the result in Section 2. Consequently, the following two results complete this section.

Theorem 31: For $m \geq 1$, the $3m+2$ entries $a(3m+2,1), a(3m+2,2), \ldots, a(3m+2,3m+2)$ form a unimodal sequence with maximum value $a(3m+2,m+1)$.

Theorem 32: For $n \geq 1$, the n entries $a(n,1), a(n,2), \ldots, a(n,n)$ form a unimodal sequence with maximum value

$$a\left(n, \frac{n}{3}\right) \left(= a\left(n, \frac{n}{3}+1\right)\right), \; n \equiv 0 \pmod 3$$

$$a\left(n, \frac{n-1}{3}+1\right), \; n \equiv 1 \pmod 3$$

$$a\left(n, \frac{n-2}{3}+1\right), \; n \equiv 2 \pmod 3.$$

8. Further Instances where the Catalan Numbers Arise

In Theorem 24 we found that for $m \geq 1$, $a(3m+1,m+1) - a(3m+1,m) = \frac{1}{m+1}\binom{2m}{m}$, the mth Catalan number. From Theorem 23 it then follows that for $m \geq 1$, $a(3m+1,m+1) - a(3m+1,m+2)$ is also $\frac{1}{m+1}\binom{2m}{m}$. Our next result provides two more instances where the Catalan numbers arise.

Theorem 33: For $m \geq 1$,

$$a(3m+2, m+1) - a(3m+2, m+2) = \frac{1}{m+1}\binom{2m}{m}, \text{ while}$$
$$a(3m+2, m+1) - a(3m+2, m) = \frac{1}{((m+1)+1)}\binom{2(m+1)}{m+1}.$$

Proof: After eliminating common summands we find that

$$\begin{aligned}
&a(3m+2, m+1) - a(3m+2, m+2) \\
&= \binom{(3m+2)-(m+1)}{(m+1)-1} \\
&- \left(\binom{(3m+2)-(m+2)}{(m+1)-2} + \binom{(3m+2)-(m+2)}{m+1}\right) \\
&= \binom{2m+1}{m} - \left(\binom{2m}{m-1} + \binom{2m}{m+1}\right) \\
&= \left(\binom{2m}{m} + \binom{2m}{m-1}\right) - \left(\binom{2m}{m-1} + \binom{2m}{m+1}\right)
\end{aligned}$$

$$= \binom{2m}{m} - \binom{2m}{m+1} = \frac{1}{m+1}\binom{2m}{m}, \text{ the } m\text{th Catalan number, while}$$

$$\begin{aligned}
&a(3m+2, m+1) - a(3m+2, m) \\
&= \left(\binom{2m+1}{m-2} + \binom{2m+1}{m}\right) - \binom{2m+2}{m-1} \\
&= \left(\binom{2m+1}{m-2} + \binom{2m+1}{m}\right) - \left(\binom{2m+1}{m-1} + \binom{2m+1}{m-2}\right) \\
&= \binom{2m+1}{m} - \binom{2m+1}{m-1} \\
&= \frac{1}{(m+1)+1}\binom{2(m+1)}{m+1}, \text{ the } (m+1)\text{st Catalan number.}
\end{aligned}$$

Lastly, upon examining three consecutive entries in certain columns of Table 1, we have the following.

Theorem 34: For $m \geq 1$,

$$a(3m-1,m)+a(3m,m)-a(3m+1,m)$$
$$=\frac{1}{m+1}\binom{2m}{m}, \text{ while}$$
$$a(3m+1,m+1)-a(3m,m+1)-a(3m-1,m+1)$$
$$=\frac{1}{((m-1)+1)}\binom{2(m-1)}{m-1}.$$

Proof:

$$a(3m-1,m)+a(3m,m)-a(3m+1,m)$$
$$=a(3m-1,m)+a(3m,m)-(a(3m,m)-a(3m-1,m-1))$$
$$=a(3m-1,m)-a(3m-1,m-1)$$
$$=a(3(m-1)+2,(m-1)+1)-a(3(m-1)+2,m-1)$$
$$=\frac{1}{m+1}\binom{2m}{m}, \text{ the } m\text{th Catalan number,}$$

from the second result in Theorem 33,

while

$$a(3m+1,m+1)-a(3m,m+1)-a(3m-1,m+1)$$
$$=(a(3m,m+1)+a(3m-1,m))-a(3m,m+1)-a(3m-1,m+1)$$
$$=a(3m-1,m)-a(3m-1,m+1)$$
$$=a(3(m-1)+2,(m-1)+1)-a(3(m-1)+2,(m-1)+2)$$
$$=\frac{1}{(m-1)+1}\binom{2(m-1)}{m-1}, \text{ the } (m-1)\text{st Catalan number,}$$

from the first result in Theorem 33.

9. References

1. Brualdi, Richard A. *Introductory Combinatorics*, fifth edition. Upper Saddle River, New Jersey: Pearson Prentice Hall, 2010.

2. Grimaldi, Ralph P. *Discrete and Combinatorial Mathematics*, fifth edition. Boston, Massachusetts: Pearson Addison Wesley, 2004.

3. Koshy, Thomas. *Fibonacci and Lucas Numbers with Applications.* New York, New York: John Wiley & Sons, Inc., 2001.

4. Sloane, Neil James Alexander. The On-Line Encyclopedia of Integer Sequences.
http://www.research.att.com/~njas/sequences/, 2007.

5. Stanley, Richard Peter. *Enumerative Combinatorics*, Volume 2. Cambridge, United Kingdom: Cambridge University Press, 1999.

6. Stanley, Richard Peter. *Catalan Addendum.*
http://www-math.mit.edu/~rstan/ec/catadd.pdf

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