# Some formulas related to residue method *

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**Abstract**

In this paper, with the help of residue method we find some interesting formulas relate residue and ordinary Bell polynomials $\hat{B}_{n,k}(x_1, x_2, \ldots)$. Further, we prove identities involving some combinatorial numbers to demonstrate the application of the formulas.

**Keywords:** residue; formal power series; formal Laurent series; generating function; ordinary Bell polynomials
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## 1 Introduction and preliminaries

G. P. Egorychev [1] introduced us a new method which transforms combinatorial sums into integrals. Upon using substitution or residue-calculus we can simplify these integrals. Today, residue is also one of the most effective theoretical tools to handle common problems in mathematical physics, even in engineering. In [2], Christoph Fürst also demonstrates the application of residue method that how one can obtain closed forms for combinatorial sums. That was shown that residue method is quite useful for studying combinatorics.

In this paper, the formula

$$\sum_{k=0}^{\infty} h_k \operatorname*{res}_t g^r(t)(f(t))^k t^{-n-1} = \operatorname*{res}_t g^r(t)h(f(t))t^{-n-1}$$

comes up naturally in an application of residue method.

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Let $\mathbb{N}, \mathbb{Q}, \mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathbb{K}, \mathbb{K}[[x]]$ and $\mathbb{K}((x))$ be, respectively, the set of natural numbers, the set of rational numbers, the set of integers, the set of real numbers, the set of complex numbers, a field containing the field $\mathbb{Q}$ as a subfield, the ring of formal power series over $\mathbb{K}$ and the field of formal Laurent series over $\mathbb{K}$. A brief introduction about res-operator:

Let $f(x) = \sum_{k=-\infty}^{\infty} f_k x^k \in \mathbb{K}((x))$, then the formal residue res of $f(x)$ is

$$\underset{x}{res}\, f(x) = [x^{-1}]f(x) = f_{-1}\,,$$

where $[x^n]$ is the coefficient of $x^n$, $n \in \mathbb{Z}$. If the generating function $f(x) \in \mathbb{K}[[x]]$ for the sequence $(f_k)_{k\geq 0}$ is $f(x) = \sum_{k=0}^{\infty} f_k x^k$, then we have

$$f_k = [x^k]f(x) = \underset{x}{res}\, f(x)x^{-k-1},\ k \geq 0\,.$$

In [2], several rules for the res-functional have been listed. For $f(x),\ g(x) \in \mathbb{K}[[x]]$:

We get $f(x) = g(x)$ if and only if

$$\underset{x}{res}\, f(x)x^{-k-1} = \underset{x}{res}\, g(x)x^{-k-1},\ k \geq 0\ (\textit{Removal of res})\,. \qquad (1)$$

For $\alpha, \beta \in \mathbb{K}$, have

$$\alpha \underset{x}{res}\, f(x)x^{-k-1} + \beta \underset{x}{res}\, g(x)x^{-k-1} = \underset{x}{res}\, (\alpha f(x) + \beta g(x))x^{-k-1}, \qquad (2)$$

$k \geq 0$ (*Linearity*). Let $h(t) = \sum_{k=1}^{\infty} h_k t^k \in \mathbb{K}((t))$ be a delta series, then

$$\sum_{k=0}^{\infty} h^k(t) \underset{x}{res}\, f(x)x^{-k-1} = f(h(t))\ (\textit{Substitution})\,. \qquad (3)$$

Well-known generating functions of the unsigned Stirling numbers of the first kind $\bar{s}(n,k)$, the Stirling numbers of second kind $S(n,k)$, the n-ordered Bell numbers $b_n$, the Bernoulli polynomials of the first kind $B_n(x)$ and the ordinary Bell polynomials $\hat{B}_{n,k}$ are given as:

$$\sum_{n=k}^{\infty} \bar{s}(n,k)\frac{t^n}{n!} = \frac{1}{k!}(log\frac{1}{1-t})^k\,,$$

$$\sum_{n=k}^{\infty} S(n,k)\frac{t^n}{n!} = \frac{1}{k!}(e^t - 1)^k\,,$$

$$\sum_{n=0}^{\infty} b_n\frac{t^n}{n!} = \frac{1}{2-e^t}\,,$$

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{te^{xt}}{e^t - 1}\,,$$

$$\sum_{n=k}^{\infty} \hat{B}_{n,k}(x_1, x_2, \ldots)t^n = (\sum_{m=1}^{\infty} x_m t^m)^k .$$

From [3, 4], we have

$$\hat{B}_{n,k}(1,1,1,\ldots) = \binom{n-1}{k-1},$$

$$\hat{B}_{n,k}(1, \frac{1}{2}, \frac{1}{3}, \ldots) = \frac{k!}{n!}\bar{s}(n,k),$$

$$\hat{B}_{n,k}(\frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \ldots) = \frac{k!}{n!}S(n,k).$$

## 2 The main identity

In this section, we obtain some identities by means of the method residue. We also give a new proof of the conclusion that formula $\underset{t}{res}\ g(t) = \underset{t}{res}\ g(f(t))f'(t)$ is equivalent to the Lagrange inversion formula. Our main identity is as follows:

**Theorem 2.1** Let $(\omega_n)_{n \in \mathbb{N}}$. Suppose we are given a delta series $f(t) = \sum_{k=1}^{\infty} f_k \frac{t^k}{\omega_k} \in \mathbb{K}((t))$, the generating function $h(t) = \sum_{k=0}^{\infty} h_k \frac{t^k}{\omega_k} \in \mathbb{K}[[x]]$ for the sequence $(h_k)_{k \geq 0}$ and $g^r(t) = (\sum_{k=0}^{\infty} g_k \frac{t^k}{\omega_k})^r \in \mathbb{K}((t))$ with $g_0 = 1,\ r \geq 0$, then the identity

$$\sum_{k=0}^{\infty} \frac{h_k}{\omega_k} \underset{t}{res}\ g^r(t)(f(t))^k t^{-n-1} = \underset{t}{res}\ g^r(t)h(f(t))t^{-n-1}, \tag{4}$$

holds for every $n \geq 0$.

**Proof:** By (3), we have

$$\sum_{k=0}^{\infty} f^k(t) \underset{t}{res}\ h(t)t^{-k-1} = h(f(t)),$$

then

$$\sum_{k=0}^{\infty} \frac{h_k}{\omega_k}(g(t))^r(f(t))^k = (g(t))^r h(f(t)).$$

Since the function $(g(t))^r(f(t))^k t^{-n-1}$ is a formal power series when $k \geq n+1$, then $\underset{t}{res}\ (g(t))^r(f(t))^k t^{-n-1} = 0,\ k \geq n+1$ and

$$\underset{t}{res}\left(\sum_{k=n+1}^{\infty} \frac{h_k}{\omega_k}(g(t))^r(f(t))^k t^{-n-1}\right) = 0.$$

Hence by (1) and (2), we get

$$
\begin{aligned}
&\operatorname*{res}_t\left((g(t))^r(f(t))^k t^{-n-1}\right)\\
= &\operatorname*{res}_t\left(\sum_{k=0}^{\infty}\frac{h_k}{\omega_k}(g(t))^r(f(t))^k t^{-n-1}\right)\\
= &\operatorname*{res}_t\left(\sum_{k=0}^{n}\frac{h_k}{\omega_k}(g(t))^r(f(t))^k t^{-n-1}\right)+\operatorname*{res}_t\left(\sum_{k=n+1}^{\infty}\frac{h_k}{\omega_k}(g(t))^r(f(t))^k t^{-n-1}\right)\\
= &\operatorname*{res}_t\left(\sum_{k=0}^{n}\frac{h_k}{\omega_k}(g(t))^r(f(t))^k t^{-n-1}\right)\\
= &\sum_{k=0}^{n}\frac{h_k}{\omega_k}\operatorname*{res}_t\left((g(t))^r(f(t))^k t^{-n-1}\right)\\
= &\sum_{k=0}^{n}\frac{h_k}{\omega_k}\operatorname*{res}_t\left((g(t))^r(f(t))^k t^{-n-1}\right)+\sum_{k=n+1}^{\infty}\frac{h_k}{\omega_k}\operatorname*{res}_t\left((g(t))^r(f(t))^k t^{-n-1}\right)\\
= &\sum_{k=0}^{\infty}\frac{h_k}{\omega_k}\operatorname*{res}_t\left((g(t))^r(f(t))^k t^{-n-1}\right).
\end{aligned}
$$

This gives (4).

In the special case, $\omega_n = 1$ in Theorem 2.1, we obtain

**Corollary 2.2** A delta series $f(t) = \sum_{k=1}^{\infty} f_k t^k \in \mathbb{K}((t))$, the generating function $h(t) = \sum_{k=0}^{\infty} h_k t^k \in \mathbb{K}[[x]]$ for the sequence $(h_k)_{k\geq 0}$ and $g^r(t) = (\sum_{k=0}^{\infty} g_k t^k)^r \in \mathbb{K}((t))$ with $g_0 = 1,\ r \geq 0$, then

$$\sum_{k=0}^{\infty} h_k \operatorname*{res}_t g^r(t)(f(t))^k t^{-n-1} = \operatorname*{res}_t g^r(t)h(f(t))t^{-n-1}\,,\ n \geq 0, \tag{5}$$

where

$$g^r(t) = (\sum_{n=0}^{\infty} g_n t^n)^r = 1 + \sum_{n=1}^{\infty} \hat{P}_n^{(r)}(g_1, g_2, g_3, \ldots)t^n,\ g_0 = 1,$$

$$\hat{P}_n^{(r)}(g_1, g_2, g_3, \ldots) = \sum_{k=1}^{n}(r)_k \hat{B}_{n,k}(g_1, g_2, g_3, \ldots),\ \hat{P}_0^{(r)} = 1 \text{ [3, p. 141]},$$

when r=1, (5) is equivalent to the identity in Theorem 1.1([5]).

We also get:

$$
\begin{aligned}
&\sum_{k=0}^{n}\sum_{i=0}^{n}\hat{P}_i^{(r)}(g_1,g_2,g_3,\ldots)\hat{B}_{n-i,k}(f_1,f_2,f_3,\ldots)\\
=\ & \underset{t}{res}\ \frac{g^r(t)}{1-f(t)}t^{-n-1}, \qquad (6)\\
&\sum_{k=0}^{n}(-1)^k\sum_{i=0}^{n}\hat{P}_i^{(r)}(g_1,g_2,g_3,\ldots)\hat{B}_{n-i,k}(f_1,f_2,f_3,\ldots)\\
=\ & \underset{t}{res}\ \frac{g^r(t)}{1+f(t)}t^{-n-1}, \qquad (7)\\
&\sum_{k=0}^{n}\frac{(-1)^k}{k+1}\ \sum_{i=0}^{n}\hat{P}_i^{(r)}(g_1,g_2,g_3,\ldots)\hat{B}_{n-i,k}(f_1,f_2,f_3,\ldots)\\
=\ & \underset{t}{res}\ \frac{-g^r(t)}{f(t)}log\frac{1}{1+f(t)}t^{-n-1}, \qquad (8)\\
&\sum_{k=0}^{n}\ k\sum_{i=0}^{n}\hat{P}_i^{(r)}(g_1,g_2,g_3,\ldots)\hat{B}_{n-i,k}(f_1,f_2,f_3,\ldots)\\
=\ & \underset{t}{res}\ \frac{g^r(t)f(t)}{(1-f(t))^2}t^{-n-1}, \qquad (9)\\
&\sum_{k=0}^{n}\frac{1}{k!}\sum_{i=0}^{n}\hat{P}_i^{(r)}(g_1,g_2,g_3,\ldots)\hat{B}_{n-i,k}(f_1,f_2,f_3,\ldots)\\
=\ & \underset{t}{res}\ g^r(t)e^{f(t)}t^{-n-1}. \qquad (10)
\end{aligned}
$$

For $r=0$ in equation (5), we get

$$\sum_{k=0}^{\infty}h_k\underset{t}{res}\ (f(t))^k t^{-n-1}=\underset{t}{res}\ h(f(t))t^{-n-1},$$

from Lagrange inversion formula (see [3]) we obtain

$$\underset{t}{res}\ (f(t))^k t^{-n-1}=\frac{k}{n}\underset{t}{res}\ t^{k-1}(\overline{f}(t))^{-n},$$

where $f(\bar{f}(t))=\bar{f}(f(t))=t$, then

$$\sum_{k=0}^{\infty}h_k\cdot k\ \underset{t}{res}\ t^{k-1}(\overline{f}(t))^{-n}=\sum_{k=0}^{\infty}h_k\cdot n\ \underset{t}{res}\ (f(t))^k t^{-n-1},$$

since $n \operatorname*{res}_t t^{-n-1} f^k = \operatorname*{res}_t t^{-n} D(f)^k$, we have

$$\operatorname*{res}_t g(t) = \operatorname*{res}_t g(f(t))f'(t),$$

where $\sum_{k=0}^{\infty} h_k \cdot k\ t^{k-1}(\overline{f}(t))^{-n} = g(t)$.

## 3 Combinatorial sums

In this section, we establish some identities involving the binomial coefficient, the unsigned Stirling numbers of the first kind, the Stirling numbers of the second kind, the n-ordered Bell numbers, the Bernoulli polynomials of the first kind, the ordinary Bell polynomials, and the Riemann Zeta Function.

**Proposition 3.1** For $m \geq 0,\ r \geq 1$, we have

$$\sum_{k=0}^{n-m} \binom{n-k+r-1}{mr+r-1} = \binom{n+r}{mr+r}, \tag{11}$$

when $r = 1$, (11) comes to $\sum_{k=0}^{n-m} \binom{n-k}{m} = \binom{n+1}{m+1}$.

**Proof:** Consider

$$g^r(t) = \frac{t^{mr}}{(1-t)^{mr+r}},\ f(t) = t,$$

from Theorem 2.2, we obtain

$$\sum_{k=0}^{\infty} \operatorname*{res}_t \frac{t^{mr+k}}{(1-t)^{mr+r}} t^{-n-1} = \operatorname*{res}_t \frac{t^{mr}}{(1-t)^{mr+r+1}} t^{-n-1},\ h_k = 1,$$

then

$$\sum_{k=0}^{n-m} \binom{n-k+r-1}{mr+r-1} = \binom{n+r}{mr+r}.$$

**Proposition 3.2** Let $r \geq 0$. Then

$$\sum_{k=0}^{n} \sum_{i=0}^{n} \sum_{j=0}^{i} (r)_j j!k! \binom{n}{i} S(i,j)S(n-i,k) = \sum_{k=0}^{n} \binom{n}{k} r^{n-k} b_k, \tag{12}$$

$\sum_{k=0}^{n} k!S(n,k) = b_n$, $r = 0$, where $b_n$ is the n-ordered Bell numbers.

**Proof:** Setting $g^r(t) = e^{rt}$, $f(t) = e^t - 1$ in (6), we obtain

$$\sum_{k=0}^{n}\sum_{i=0}^{n}\sum_{j=0}^{i}(r)_j \hat{B}_{i,j}(\frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \ldots)\hat{B}_{n-i,k}(\frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \ldots)$$

$$= \underset{t}{res}\ t^{-n-1}\frac{e^{rt}}{2-e^t},$$

then

$$\sum_{k=0}^{n}\sum_{i=0}^{n}\sum_{j=0}^{i}\frac{(r)_j j!k!}{i!(n-i)!}S(i,j)S(n-i,k) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}r^{n-k}b_k,$$

which yields (12). Setting $r = 0$ in (12) gives $\sum_{k=0}^{n} k!S(n,k) = b_n$.

Similarly, by (7) and (8), we obtain Proposition 3.3 and Proposition 3.4.

**Proposition 3.3** Let $r \geq 0$. The following relation holds:

$$\sum_{k=0}^{n}\sum_{i=0}^{n}\sum_{j=0}^{i}(-1)^k(r)_j j!k!\binom{n}{i}S(i,j)S(n-i,k) = (r-1)^n, \tag{13}$$

$$\sum_{k=0}^{n}(-1)^k k!S(n,k) = (-1)^n,\ r = 0. \tag{14}$$

**Proposition 3.4** For $r \geq 0$, we have the following identity

$$\sum_{k=0}^{n}\sum_{i=0}^{n}\sum_{j=0}^{i}\frac{(-1)^k}{k+1}(r)_j j!k!\binom{n}{i}S(i,j)S(n-i,k) = B_n(r), \tag{15}$$

where $B_n(x)$ is the Bernoulli polynomials of the first kind.

**Proposition 3.5** As $r \geq 0$, we have

$$\sum_{k=0}^{n}\sum_{i=0}^{n}\sum_{j=0}^{i}(-1)^k k(r)_j\binom{i-1}{j-1}\binom{n-i-1}{k-1} = -\binom{n+r-3}{n-1}. \tag{16}$$

**Proof:** Let $g^r(t) = \frac{1}{(1-t)^r}$, $f(t) = \frac{-t}{1-t}$ , and by (9)

$$\sum_{k=0}^{n}\sum_{i=0}^{n}\sum_{j=0}^{i}(-1)^k k(r)_j\hat{B}_{i,j}(1,1,1,\ldots)\hat{B}_{n-i,k}(1,1,1,\ldots) = \underset{t}{res}\ t^{-n-1}\frac{\frac{-t}{(1-t)^{r+1}}}{(1+\frac{t}{1-t})^2},$$

$$\sum_{k=0}^{n}\sum_{i=0}^{n}\sum_{j=0}^{i}(-1)^k k(r)_j\binom{i-1}{j-1}\binom{n-i-1}{k-1} = -\binom{n+r-3}{n-1}.$$

This completes the proof.

**Proposition 3.6** Let $r \geq 0$, we have

$$\sum_{k=0}^{n}\sum_{i=0}^{n}\sum_{j=0}^{i}\frac{(r)_j}{(n-i)!}\binom{i-1}{j-1}\bar{s}(n-i,k)=\binom{n+r}{r}. \quad (17)$$

**Proof:** By $g^r(t)=\frac{1}{(1-t)^r}$, $f(t)=-log(1-t)$ and (10), we obtain

$$\sum_{k=0}^{n}\sum_{i=0}^{n}\sum_{j=0}^{i}\frac{(r)_j}{k!}\hat{B}_{i,j}(1,1,1,\ldots)\hat{B}_{n-i,k}(1,\frac{1}{2},\frac{1}{3},\ldots)$$

$$= \underset{t}{res}\; t^{-n-1}\frac{1}{(1-t)^r}\cdot\frac{1}{1-t},$$

then

$$\sum_{k=0}^{n}\sum_{i=0}^{n}\sum_{j=0}^{i}\frac{(r)_j}{k!}\frac{k!}{(n-i)!}\binom{i-1}{j-1}\bar{s}(n-i,k)=\binom{n+r}{r}.$$

Hence (17) holds.

**Proposition 3.7** For $\zeta_n(s)=\sum_{j=1}^{n}j^{-s}$, we obtain the following identity relates the Stirling numbers of both kinds:

$$\sum_{k=0}^{n}\sum_{i=0}^{n}\sum_{j=0}^{i}\frac{n!j!}{k!i!}(r)_jS(i,j)\hat{B}_{n-i,k}(\zeta_n(1),-\frac{\zeta_n(2)}{2},\frac{\zeta_n(3)}{3},\ldots)$$

$$=\sum_{i=0}^{n}\frac{r^i}{i!}\bar{s}(n+1,n-i+1).$$

**Proof:** From [3], we know that the the unsigned Stirling numbers of the first kind $\bar{s}(n,k)$ satisfy

$$n!(1+t)(1+\frac{t}{2})\ldots(1+\frac{t}{n})=\sum_{k=0}^{n}\bar{s}(n+1,k+1)t^k.$$

By $g^r(t)=e^{rt}$, $f(t)=log(1+t)(1+\frac{t}{2})\ldots(1+\frac{t}{n})=\sum_{s\geq 1}\frac{(-1)^{s-1}\zeta_n(s)}{s}t^s$ and (10), we have

$$\sum_{k=0}^{n}\sum_{i=0}^{n}\sum_{j=0}^{i}\frac{(r)_j}{k!}\hat{B}_{i,j}(\frac{1}{1!},\frac{1}{2!},\frac{1}{3!},\ldots)\hat{B}_{n-i,k}(\zeta_n(1),-\frac{\zeta_n(2)}{2},\frac{\zeta_n(3)}{3},\ldots)$$

$$=\underset{t}{res}\; t^{-n-1}e^{rt}(1+t)(1+\frac{t}{2})\ldots(1+\frac{t}{n}),$$

$$\sum_{k=0}^{n}\sum_{i=0}^{n}\sum_{j=0}^{i}\frac{n!j!}{k!i!}(r)_j S(i,j)\hat{B}_{n-i,k}(\zeta_n(1),-\frac{\zeta_n(2)}{2},\frac{\zeta_n(3)}{3},\dots)$$

$$=\sum_{i=0}^{n}\frac{r^i}{i!}\bar{s}(n+1,n-i+1).$$

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