Neighborhoods in Maximum Packings of $2K_n$ with Triples

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Abstract

In this paper, we use a recent result of Bryant, Horsley, and Pettersson in [1] to give an alternate and simpler proof of results on neighborhood graphs in maximum packings of $2K_n$ with triples, some of which were only recently obtained.

In any partial triple system (V, B) of $2K_n$, the neighborhood of a vertex v is the subgraph induced by $\{\{x, y\} \mid \{v, x, y\} \in B\}$. In results by Colbourn and Rosa ($n \equiv 0, 1 \pmod 3$) and Chaffee and Rodger ($n \equiv 2 \pmod 3$), a complete characterization of the possible neighborhoods in a maximum packing of $2K_n$ is provided. In both papers, the authors make use of difference methods, as well as a pull-up technique which is used to modify the neighborhood of a vertex; yet neither approach seems to readily obtain the other result. Here, a simpler unified proof is presented proving both results primarily by using the aforementioned result in [1].

1 Introduction

A (partial) λ-fold triple system is an ordered pair (V, B) where V is a n-element set and B is a multiset of 3-element subsets of V called blocks such that each 2-element subset of V appears in (at most) λ blocks of B. The blocks of B are also called triples.

A K_3-decomposition of a graph G is a partition of $E(G)$ into sets, each of which induces a graph isomorphic to K_3. For the purposes of this paper, a packing of a graph G is a K_3-decomposition of a subgraph H of G. If

(V, B) is a (partial) triple system, define the multiset of edges $E(B) = \{\{x, y\}, \{x, z\}, \{y, z\} \mid \{x, y, z\} \in B\}$. If (V, B) is a packing of λK_n, then the leave of the packing is defined to be the multiset of edges $L = E(\lambda K_n) \setminus E(B)$. It will cause no confusion to also refer to the leave as being the subgraph induced by $E(\lambda K_n) \setminus E(B)$. In particular, in the special case where $L = \{\{a, b\}, \{a, b\}\}$, L is expressed as the 2-cycle (a, b). A vertex v is said to be in the leave if there is some edge in the leave that is incident with v. A maximum packing is a packing such that among all packings the number of edges in its leave is as small as possible. It is straightforward to see that a (partial) λ-fold triple system on n vertices is equivalent to a K_3-decomposition (packing) of λK_n.

In any partial triple system (V, B) of $2K_n$, the neighborhood of a vertex $v \in V$ is the graph induced by $\{\{x, y\} \mid \{v, x, y\} \in B\}$. If (V, B) is a maximum packing of $2K_n$ then the neighborhood of each vertex is a 2-regular (or quadratic) graph on either $n - 1$ or $n - 2$ vertices. It will cause no confusion to also refer to the neighborhood as a set of cycles, each being a component of the neighborhood.

When the vertex v is not in the leave of the maximum packing, the neighborhood graph is 2-regular on $n - 1$ vertices. If the vertex v is in the leave, the neighborhood graph is 2-regular on $n - 2$ vertices. A natural question is to ask for which 2-regular graphs Q on $n - 1$ or $n - 2$ vertices is there a maximum packing of $2K_n$ such that the neighborhood of some vertex is Q. This question was answered by Colbourn and Rosa, and Chaffee and Rodger, respectively, in two papers culminating in the following two theorems.

Theorem 1. [3] *Suppose $n \equiv 0$ or $1 \pmod 3$. A 2-regular graph Q on $n - 1$ vertices is the neighborhood of a vertex in a 2-fold triple system on n vertices if and only if $(n, Q) \notin \{(6, C_2 \cup C_3), (7, C_3 \cup C_3)\}$.*

Theorem 2. [2] *Let $n \equiv 2 \pmod 3$ with $n > 2$, and let Q be a 2-regular multigraph on either $n - 2$ or $n - 1$ vertices. Then there exists a maximum packing of $2K_n$ with leave a 2-cycle such that the neighborhood graph of some vertex is Q if and only if $(n, Q) \neq (5, C_2 \cup C_2)$.*

Surprisingly, it does not seem that the techniques used in either paper can be used to readily obtain the other result, even if one "allows" extreme cases (such as the case when each cycle in the neighborhood has length two) to be handled using alternate methods. In this paper, a new, simpler, and unified proof that obtains both results is provided (see Theorem 10). However, this new proof relies heavily on the following major result, namely a recent and quite powerful result due to Bryant, Horsley, and Pettersson (see Theorem 5). In Section 2, we will state some well-known lemmas that are useful in handling extreme cases of Theorem 10. In Section 3, we state the theorem of Bryant, Horsley, and Pettersson, and use it to establish three key lemmas that will provide the basis for the proof of our main theorem. Finally, in Section 4, we will provide the new proof of our main theorem.

2 Basic Lemmas

In this section, we state two well-known theorems, one being on idempotent quasigroups and the other on maximum partial triple systems. We will use these lemmas to handle extreme cases of our main theorem (specifically the cases where $n \equiv 1$ or $5 \pmod 6$ and Q contains only 2-cycles).

Lemma 3. [6] *There exists an idempotent quasigroup of order n for all $n \neq 2$.*

The second lemma is more extensive than what appears below; however, what appears below is sufficient for our purposes.

Lemma 4. [4] *The leave of a maximum partial triple system of λK_n is*

1. *$\varnothing$ if $\lambda = 2$ and $n \equiv 0, 1 \pmod 3$,*
2. *a 2-cycle if $\lambda = 2$ and $n \equiv 2 \pmod 3$,*
3. *a $K_{1,3}$ and $\frac{n-4}{2}$ independent edges if $\lambda = 1$ and $n \equiv 4 \pmod 6$, and*
4. *a 1-factor if $\lambda = 1$ and $n \equiv 2 \pmod 6$.*

3 Useful Decomposition

In this section, we first state a powerful cycle-decomposition theorem before using it to establish three key lemmas used in the proof of our main theorem. In addition to being useful in this paper, it should be noted that the following result solved the Alspach conjecture.

Theorem 5. *[1]*

1. *Let n be odd. There exists a decomposition of K_n into cycles of lengths $m_1, \ldots, m_t$ if and only if*
   (a) $3 \leq m_i \leq n$ for $1 \leq i \leq t$ and
   (b) $\sum_{i=1}^{t} m_i = \binom{n}{2}$.
2. *Let n be even. There exists a decomposition of K_n into cycles of lengths $m_1, \ldots, m_t$ and a 1-factor if and only if*
   (a) $3 \leq m_i \leq n$ for $1 \leq i \leq t$ and

(b) $\sum_{i=1}^{t} m_i = \binom{n}{2} - \frac{n}{2}$.

This result provides the backbone of the proof technique used in this paper, establishing that K_n minus the edges of a certain set of cycles and possibly a 1-factor can be decomposed into triples.

We don't need the full power of this result for our theorem, since we will never choose more than three cycle lengths for any particular case. However, while older and more basic results can be used in many of the cases, we don't know of any other result that provides us the necessary tools to complete all of the cases in the proof of the theorem. (For instance, we know of no other result that establishes that K_n can be decomposed into a 1-factor, a Hamilton cycle, a near Hamilton cycle, and triples.)

In addition to needing Theorem 5 to prove Lemmas 7, 8, and 9, the following result due to Petersen on the decomposition of even regular graphs into 2-factors is also needed.

Lemma 6. [5] *Let H be any $2k$-regular multigraph. There exists a 2-factorization of H (into k 2-factors).*

We now prove three lemmas that will be used extensively in the proof of Theorem 10.

Lemma 7. *Let Q be a 2-regular graph on n vertices, and suppose that there exist K_3-decompositions of $K_n - G_1$ and $K_n - G_2$ where $G_1 \cup G_2$ is $2k$-regular (on n vertices), G_1 contains a Hamilton cycle, and G_2 contains either a Hamilton cycle or a 1-factor. Then there exists a decomposition of $2K_n$ into triples and k 2-factors, one of which is Q.*

Proof. Since Q is a 2-regular graph on n vertices, it can be expressed as a set of cycles (some possibly having length 2); hence, let $Q = \{c_0, \ldots, c_{q-1}\}$, where for each $i \in \mathbb{Z}_q$, $c_i = (c_{i,1}, \ldots, c_{i,l_i})$ has length l_i. Let $(\mathbb{Z}_n, B_1)$ be a K_3-decomposition of $K_n - G_1$, where G_1 contains the Hamilton cycle H_1 which is named so that $H_1 = (c_{0,1}, \ldots, c_{0,l_0}, c_{1,1}, \ldots, c_{1,l_1}, \ldots, c_{q-1,1}, \ldots, c_{q-1,l_{q-1}})$. Let $(\mathbb{Z}_n, B_2)$ be a K_3-decomposition of $K_n - G_2$. If G_2 contains a Hamilton cycle, name it $H_2 = (c_{0,1}, c_{0,l_0}, c_{1,1}, c_{1,l_1}, \ldots, c_{q-1,1}, c_{q-1,l_{q-1}}, v_1, \ldots, v_{n-2-2q})$ where $v_1, \ldots, v_{n-2-2q}$ are arbitrarily named. Otherwise G_2 contains a 1-factor F_1; name the vertices so that q of the edges in F_1 are in $\{\{c_{i,1}, c_{i,l_i}\} \mid i \in \mathbb{Z}_q\}$.

Let H_1' be the 2-factor induced by $(E(H_1) \cup \{\{c_{i,1}, c_{i,l_i}\} \mid i \in \mathbb{Z}_q\}) \setminus \{\{c_{i,l_i}, c_{i+1,1}\} \mid i \in \mathbb{Z}_q\}$ reducing the sum in the subscript modulo q. Then $H_1' \cong Q$. The graph induced by $E(G_1) \cup E(G_2) - E(H_1')$ is $2k-2$ regular, so by Lemma 6, it can be decomposed into $k-1$ 2-factors, say $D_0, \ldots D_{k-2}$. Then $(\mathbb{Z}_n, B_1 \cup B_2 \cup H_1' \cup_{i \in \mathbb{Z}_{k-1}} D_i)$ is the required decomposition. □

Lemma 8. *Let Q be a 2-regular graph on $n+1$ vertices with a cycle of length at least 4 or 5 when n is odd or even respectively. Suppose that there exist K_3-decompositions of $K_n - G_1$ and $K_n - G_2$ where $G_1 \cup G_2$ is 4-regular (on n vertices), G_1 contains a Hamilton cycle, and G_2 contains a Hamilton cycle if n is odd and a 1-factor if n is even. Then, there exists a maximum packing of $2K_{n+2}$ with leave a 2-cycle on the vertex set $\mathbb{Z}_n \cup \{\infty_1, \infty_2\}$ such that the neighborhood of the vertex ∞_1 is Q.*

Proof. Let $Q = \{c_0, \dots, c_{q-1}\}$ where for each $i \in \mathbb{Z}_q$, $c_i = (c_{i,1}, \dots, c_{i,l_i})$ has length l_i, $c_{j,k} \in \mathbb{Z}_n$ unless $(j,k) = (0,3)$, $c_{0,3} = \infty_2$, and $l_0 \geq 4$ or 5 when n is odd or even respectively. Using an argument similar to Lemma 7, let $(\mathbb{Z}_n, B_1)$ be a K_3-decomposition of $K_n - G_1$, where G_1 contains the Hamilton cycle H_1 named so that $H_1 = (c_{0,1}, c_{0,2}, c_{0,4}, c_{0,5}, \dots, c_{0,l_0}, c_{1,1}, \dots, c_{1,l_1}, \dots, c_{q-1,1}, \dots, c_{q-1,l_{q-1}})$ (so $c_{0,3} = \infty_2$ is omitted). Let $(\mathbb{Z}_n, B_2)$ be a K_3-decomposition of $K_n - G_2$. If G_2 contains a Hamilton cycle, name it $H_2 = (c_{0,2}, c_{0,4}, c_{0,1}, c_{0,l_0}, c_{1,1}, c_{1,l_1}, c_{2,1}, c_{2,l_2} \dots, c_{q-1,1}, c_{q-1,l_{q-1}}, v_1, \dots, v_{n-2(q+1)})$ where $v_1, \dots, v_{n-2(q+1)}$ exclude $c_{0,3} = \infty_2$ and are otherwise arbitrarily named. Otherwise G_2 contains the 1-factor F_1; name the vertices so that $\{c_{0,2}, c_{0,4}\}$ is an edge in F_1 and q of the remaining edges in F_1 are in $\{\{c_{i,1}, c_{i,l_i}\} \mid i \in \mathbb{Z}_q\}$ (note that $c_{0,4} \neq c_{0,l_0}$ because G_2 contains a 1-factor so n is even so $l_0 \geq 5$ by assumption). Finally, observe that the edge $\{c_{0,2}, c_{0,4}\}$ appears in both G_1 and G_2.

Let H_1' be the graph induced by $(E(H_1) \cup \{\{c_{i,1}, c_{i,l_i}\} \mid i \in \mathbb{Z}_q\}) \setminus \{\{c_{i,l_i}, c_{i+1,1}\} \mid i \in \mathbb{Z}_q\}$ reducing the sum in the subscript modulo q. Let H_2' be the graph induced by $(G_1 \cup G_2) \setminus E(H_1')$. Then H_1' and H_2' are each 2-regular spanning subgraphs of K_n, and the set of cycles formed by the components in H_1' contains the cycles $c_1, \dots, c_{q-1}$ and the cycle c_0' where c_0' is formed from c_0 by deleting the edges $\{c_{0,2}, c_{0,3}\}$ and $\{c_{0,3}, c_{0,4}\}$ and adding the edge $\{c_{0,2}, c_{0,4}\}$. Note that the edge $\{c_{0,2}, c_{0,4}\}$ appears in both $E(H_1')$ and $E(H_2')$.

Then $(\mathbb{Z}_n \cup \{\infty_1, \infty_2\}, (B_1 \cup B_2 \cup \{\{\infty_i, a_i, b_i\} \mid \{a_i, b_i\} \in E(H_i'), 1 \leq i \leq 2\} \setminus \{\{\infty_i, c_{0,2}, c_{0,4}\} \mid i \in \{1,2\}\}) \cup \{\{\infty_1, \infty_2, c_{0,i}\} \mid i \in \{2,4\}\})$ is the required maximum packing (with leave $(c_{0,2}, c_{0,4})$). □

Lemma 9. *Let Q be a 2-regular graph on $n+1$ vertices with a cycle of length at least 4. Suppose that there exist K_3-decompositions of $K_n - G_1$ and $K_n - G_2$ where G_1 and G_2 are near-Hamilton cycles. Then there exists a K_3-decomposition of $2K_{n+2}$ on the vertex set $\mathbb{Z}_n \cup \{\infty_1, \infty_2\}$ such that the neighborhood of the vertex ∞_1 is Q.*

Proof. Let $Q = \{c_0, \dots, c_{q-1}\}$ where for each $i \in \mathbb{Z}_q$, $c_i = (c_{i,1}, \dots, c_{i,l_i})$, $c_{j,k} \in \mathbb{Z}_n$ unless $(j,k) = (0,2)$, $c_{0,2} = \infty_2$, and $l_0 \geq 4$. Let $(\mathbb{Z}_n, B_1)$ be a K_3-decomposition of $K_n - G_1$ (so $c_{0,2} = \infty_2$ is not in the vertex set), where G_1 is the near-Hamilton cycle H_1, named so that $H_1 =$

$(c_{0,3}, c_{0,4}, c_{0,5}, \dots, c_{0,l_0}, c_{1,1}, \dots, c_{1,l_1}, \dots, c_{q-1,1}, \dots, c_{q-1,l_{q-1}})$ (so $c_{0,1}$ is omitted). Let $(\mathbb{Z}_n, B_2)$ be a K_3-decomposition of $K_n - G_2$, where G_2 is the near-Hamilton cycle $H_2 = (c_{0,1}, c_{0,l_0}, c_{1,1}, c_{1,l_1}, \dots, c_{q-1,1}, c_{q-1,l_{q-1}}, v_1, \dots, v_{n-1-2q})$ where $v_1, \dots, v_{n-1-2q}$ omit $c_{0,3}$ and are otherwise arbitrarily named. Note that $c_{0,3} \neq c_{0,l_0}$ since $l_0 \geq 4$. Note that $c_{0,1}$ has degree 2 in H_2 and degree 0 in H_1, $c_{0,3}$ has degree 2 in H_1 and degree 0 in H_1, and every other vertex in $\mathbb{Z}_n$ has degree 2 in both H_1 and H_2.

Let H_1' be the graph induced by $(E(H_1) \cup \{\{c_{i,1}, c_{i,l_i}\} \mid i \in \mathbb{Z}_q\}) \setminus (\{\{c_{i,l_i}, c_{i+1,1}\} \mid i \in \mathbb{Z}_q\} \cup \{\{c_{0,3}, c_{q-1,l_{q-1}}\}\})$ reducing the sum in the subscript modulo q (and noting that $\{c_{q-1,l_{q-1}}, c_{0,1}\}$ is not an edge and hence not removed). Note that in the graph induced by H_1', every vertex in $\mathbb{Z}_n$ has degree 2 except $c_{0,1}$ and $c_{0,3}$, both of which have degree 1. Let H_2' be the graph induced by $(E(H_1) \cup E(H_2)) \setminus H_1'$. Note that in the graph induced by H_2', every vertex in $\mathbb{Z}_n$ has degree 2 except $c_{0,1}$ and $c_{0,3}$, both of which have degree 1. The set of cycles formed by the components in H_1' contains the cycles $c_1, \dots, c_{q-1}$ and c_0' where c_0' is formed from c_0 by deleting the edges $\{c_{0,1}, c_{0,2}\}$ and $\{c_{0,2}, c_{0,3}\}$.

Then $(\mathbb{Z}_n \cup \{\infty_1, \infty_2\}, B_1 \cup B_2 \cup \{\{\infty_i, a_i, b_i\} \mid \{a_i, b_i\} \in E(H_i'), 1 \leq i \leq 2\} \cup \{\{\infty_1, \infty_2, c_{0,i}\} \mid i \in \{1, 3\}\})$ is the required K_3-decomposition. □

4 Main Result

We now state and prove our main theorem, the results of which appear in [2] and [3].

Theorem 10. *1. Let $n \neq 2$, and let Q be a 2-regular multigraph on $n-1$ vertices. Then there exists a maximum packing of $2K_n$ (possibly the leave is empty) such that the neighborhood graph of some vertex is Q if and only if $(n, Q) \notin \{(5, C_2 \cup C_2), (6, C_2 \cup C_3), (7, C_3 \cup C_3)\}$.*

2. Let $n \equiv 2 \pmod 3$ with $n > 2$, and let Q be a 2-regular multigraph on $n-2$ vertices. Then there exists a maximum packing of $2K_n$ such that the neighborhood graph of some vertex is Q.

Proof. If $(n, Q) \in \{(5, C_2 \cup C_2), (6, C_2 \cup C_3), (7, C_3 \cup C_3)\}$ and there exists a maximum packing of $2K_n$ such that the leave of some vertex v is Q, then deleting the triples containing v leaves the graph $\overline{K_2} \bigvee_2 \overline{K_2}$, $\overline{K_2} \bigvee_2 K_3$, or $K_3 \bigvee_2 K_3$, when $n = 5, 6$ or 7 respectively. In each case, any triple that contains mixed edges contains 2 mixed edges and 1 pure edge. But in each case, there are more than twice as many mixed edges as pure edges remaining. Hence these cases are not possible.

To prove the sufficiency, we will first handle two extreme cases (Case 1) followed by 14 main cases. While there are a large number of cases, most follow from the lemmas in Section 3 or from similar ideas. Let Q be a 2-regular multigraph on either $n-2$ or $n-1$ vertices (with the size of Q specified in each case) such that $(n,Q) \notin \{(5, C_2 \cup C_2), (6, C_2 \cup C_3), (7, C_3 \cup C_3)\}$.

Case 1: Let $n \equiv 1$ or $5 \pmod 6$, let Q consist entirely of 2-cycles (so if $n \equiv 5 \pmod 6$, $|Q| = n-1$), and let $(n,Q) \neq (5, C_2 \cup C_2)$ (since this case is not possible). $n = 1$ is trivial. So for $n > 5$, let $n = 6l + 1 + \epsilon$ with $l \geq 1$ and $\epsilon \in \{0, 4\}$. Let $V = \{\infty_1\} \cup (\mathbb{Z}_{3l+\frac{\epsilon}{2}} \times \mathbb{Z}_2)$. By Lemma 3, let $(\mathbb{Z}_{3l+\frac{\epsilon}{2}}, \circ)$ be an idempotent quasigroup ($l \geq 1$, so $3l + \frac{\epsilon}{2} \geq 3$, and hence this quasigroup exists). By Lemma 4, there exists a maximum packing $(\mathbb{Z}_{3l+\frac{\epsilon}{2}} \times \{1\}, B_1)$ of $2K_{3l+\frac{\epsilon}{2}}$ with leave c where c is a 2-cycle if $\epsilon = 4$ and $c = \varnothing$ otherwise. Then $(\{\infty_1\} \cup \mathbb{Z}_{3l+\frac{\epsilon}{2}} \times \mathbb{Z}_2, B_1 \cup \{\{(a,0),(b,0),(a \circ b, 1)\}, \{(a,0),(b,0),(b \circ a, 1)\} \mid 0 \leq a < b \leq 3l - 1 + \frac{\epsilon}{2}\} \cup \{\{\infty_1, (a,0),(a,1)\}, \{\infty_1,(a,0),(a,1)\} \mid a \in \mathbb{Z}_{3l+\frac{\epsilon}{2}}\})$ is the required decomposition with leave c.

Case 2: Suppose $|Q| = n-2$ and $n \equiv 5 \pmod 6$. We construct the maximum packing on the vertex set $\mathbb{Z}_{n-2} \cup \{\infty_1, \infty_2\}$ where the neighborhood of ∞_1 is $Q = \{c_0, \ldots, c_{q-1}\}$, the q-cycles being defined on the vertex set $\mathbb{Z}_{n-2}$; so the leave of the maximum packing will be (∞_1, ∞_2).

In this case, $n-2 \equiv 0 \pmod 3$ and ≥ 3 so $\binom{(n-2)}{2} - (n-2) \equiv 0 \pmod 3$ and ≥ 0 respectively. Thus, for $k \in \{1,2\}$, by Theorem 5 let $(\mathbb{Z}_{n-2}, B_k)$ be a K_3-decomposition of $K_{n-2} - H_k$, where H_1 and H_2 are Hamilton cycles. Then by Lemma 7, there exists a K_3-decomposition $(\mathbb{Z}_{n-2}, B)$ of $2K_{n-2} - (H_1' \cup H_2')$ where H_1' and H_2' are 2-regular graphs and $H_1' \cong Q$.

Then $(\mathbb{Z}_{n-2} \cup \{\infty_1, \infty_2\}, B \cup \{\{\infty_i, a_i, b_i\} \mid \{a_i, b_i\} \in E(H_i'), 1 \leq i \leq 2\})$ is the required maximum packing.

Case 3: Suppose $|Q| = n-2$ and $n \equiv 2 \pmod 6$. We construct the maximum packing on the vertex set $\mathbb{Z}_{n-2} \cup \{\infty_1, \infty_2\}$ where the neighborhood of ∞_1 is $Q = \{c_0, \ldots, c_{q-1}\}$, the q-cycles being defined on the vertex set $\mathbb{Z}_{n-2}$; so the leave of the maximum packing will be (∞_1, ∞_2).

In this case, $n-2 \equiv 0 \pmod 6$ and ≥ 6 so $\binom{(n-2)}{2} - (n-2) - \frac{n-2}{2}$ and $\binom{(n-2)}{2} - \frac{n-2}{2}$ are both divisible by 3 and ≥ 0. So by Theorem 5, let $(\mathbb{Z}_{n-2}, B_1)$ be a K_3-decomposition of $K_{n-2} - (H_1 \cup F_1)$ and let $(\mathbb{Z}_{n-2}, B_2)$ be a K_3-decomposition of $K_{n-2} - F_2$ where H_1 is a Hamilton cycle and F_1 and F_2 are 1-factors. By Lemma 7, there exists a K_3-decomposition $(\mathbb{Z}_{n-2}, B)$ of $2K_{n-2} - (H_1' \cup H_2')$ where H_1' and H_2' are 2-regular graphs and $H_1' \cong Q$.

Then $(\mathbb{Z}_{n-2} \cup \{\infty_1, \infty_2\}, B \cup \{\{\infty_i, a_i, b_i\} \mid \{a_i, b_i\} \in E(H_i'), 1 \leq i \leq 2\})$ is the required maximum packing.

Case 4: Suppose $|Q| = n-1$, $n \equiv 5 \pmod 6$, and that Q has a cycle of length at least 4.

In this case, $n-2 \equiv 0 \pmod 3$ and ≥ 3 so $\binom{(n-2)}{2} - (n-2) \equiv 0 \pmod 3$ and ≥ 0 respectively. Thus, for $k \in \{1,2\}$, by Theorem 5, let $(\mathbb{Z}_{n-2}, B_k)$ be a K_3-decomposition of $K_{n-2} - H_k$, where H_1 and H_2 are Hamilton cycles. Then by Lemma 8, the required maximum packing exists.

Case 5: Suppose that $|Q| = n - 1$, $n \equiv 5 \pmod 6$ and that Q contains a 3-cycle.

We construct the maximum packing on the vertex set $\mathbb{Z}_{n-4} \cup \{\infty_j \mid 1 \le j \le 4\}$ where the neighborhood of ∞_1 is $Q = \{c_0, \ldots, c_{q-1}\}$, where $c_0 = (\infty_2, \infty_3, \infty_4)$, and the $q - 1$ other cycles are defined on the vertex set $\mathbb{Z}_{n-4}$ with the length of c_1 being odd (since $n - 1$ is even here, and Q contains a 3-cycle, Q must contain some other cycle of odd length). We will construct the maximum packing so that the leave will be $(\infty_2, c_{1,2})$, where $c_{1,2}$ is defined below.

For each $i \in \mathbb{Z}_q \setminus \{0\}$, let $c_i = (c_{i,1}, \ldots, c_{i,l_i})$ where l_i is the length of c_i. In this case, $n - 4 \equiv 1 \pmod 6$ and thus $\binom{(n-4)}{2} - 3(n-4)$ and $\binom{(n-4)}{2} - (n - 4 - 1)$ are both divisible by 3. Further, Q cannot contain a 3-cycle in this case if $n = 5$, so we have that $n - 4 \ge 7$, so both quantities are also nonnegative. So by Theorem 5, let $(\mathbb{Z}_{n-4}, B_1)$ be a K_3-decomposition of $K_{n-4} - (H_1 \cup H_3 \cup H_4)$ and let $(\mathbb{Z}_{n-4}, B_2)$ be a K_3-decomposition of $K_{n-4} - H_2$ where H_1, H_3, and H_4 are Hamilton cycles and H_2 is a near-Hamilton cycle with H_1 and H_2 named as follows: Let $H_1 = (c_{1,1}, \ldots, c_{1,l_1}, c_{2,1}, \ldots, c_{2,l_2}, \ldots, c_{q-1,1}, \ldots, c_{q-1,l_{q-1}})$. Let H_2 be defined by $H_2 = (c_{1,1}, c_{1,l_1}, c_{2,1}, c_{2,l_2}, \ldots, c_{q-1,1}, c_{q-1,l_{q-1}}, v_1, \ldots, v_{n-3-2q})$ where $v_1, \ldots, v_{n-3-2q}$ exclude $c_{1,2}$ and are otherwise arbitrarily named (note that $c_{1,2}$ is omitted from H_2 altogether since c_1 was assumed to have had odd length and hence $c_{1,2} \ne c_{1,l_1}$).

Let $H_1' = (E(H_1) \cup \{\{c_{i,1}, c_{i,l_i}\} \mid i \in \mathbb{Z}_q \setminus \{0\}\}) \setminus (\{\{c_{i,l_i}, c_{i+1,1}\} \mid i \in \mathbb{Z}_q \setminus \{0, q-1\}\} \cup \{\{c_{q-1,l_{q-1}}, c_{1,1}\}\})$. Let $H_2' = (E(H_1) \cup E(H_2)) \setminus H_1'$. Then H_1' and H_2' induce 2-regular spanning subgraphs of K_{n-4} and K_{n-5} respectively, the set of cycles formed by the components in H_1' being $Q \setminus c_0$. Let $H_3' = E(H_3)$ and $H_4' = E(H_4)$.

Let $(\{\infty_j \mid 1 \le j \le 4\}, B_3)$ be a K_3-decomposition of $2K_4$ (where the neighborhood of ∞_1 is the 3-cycle $(\infty_2, \infty_3, \infty_4)$).

Then $(\mathbb{Z}_{n-4} \cup \{\infty_j \mid 1 \le j \le 4\}, B_1 \cup B_2 \cup B_3 \cup \{\{\infty_i, a_i, b_i\} \mid \{a_i, b_i\} \in H_i'\}$ is the required packing.

Case 6: Suppose $|Q| = n - 1$, $n \equiv 2 \pmod 6$, and that Q has a cycle of length at least 5. $n \ne 2$ so $n \ge 8$ and hence, $n - 2 \equiv 0 \pmod 3$ and ≥ 6 so $\binom{(n-2)}{2} - (n-2) \equiv 0 \pmod 3$ and ≥ 0 respectively. By Theorem 5 let $(\mathbb{Z}_{n-2}, B_1)$ be a K_3-decomposition of $K_{n-2} - (H_1 \cup F_1)$ and $(\mathbb{Z}_{n-2}, B_2)$ be a K_3-decomposition of $K_{n-2} - F_2$ where H_1 is a Hamilton cycle and F_1 and F_2 are 1-factors. Then by Lemma 8, the required decomposition exists.

Case 7: Suppose that $|Q| = n - 1$, $n \equiv 2 \pmod 6$ (with $n \ne 2$) and that Q contains a 3-cycle.

First suppose $n = 8$. Let $B = \{\{0, \infty_1, 1\}, \{0, \infty_1, 1\}, \{\infty_1, 2, 3\},$

$\{\infty_1, 2, 3\}, \{\infty_1, 4, 5\}, \{\infty_1, 4, 6\}, \{\infty_1, 5, 6\}, \{0, 2, 4\}, \{0, 2, 5\}, \{0, 3, 4\}, \{0, 3, 5\}, \{1, 2, 4\}, \{1, 2, 6\}, \{1, 3, 5\}, \{1, 3, 6\}, \{1, 4, 5\}, \{2, 5, 6\}, \{3, 4, 6\}\}$. In B, the neighborhood of ∞_1 is $C_2 \cup C_2 \cup C_3$. Finally, the neighborhood of ∞_1 in $(B \setminus \{\{\infty_1, 1, 0\}, \{\infty_1, 2, 3\}\{0, 2, 5\}, \{1, 3, 5\}\}) \cup \{\{0, \infty_1, 2\}, \{\infty_1, 1, 3\}, \{0, 1, 5\}, \{2, 3, 5\}\}$ is $C_4 \cup C_3$.

For $n > 8$, we construct the maximum packing on the vertex set $\mathbb{Z}_{n-4} \cup \{\infty_j \mid 1 \leq j \leq 4\}$ where the neighborhood of ∞_1 is $Q = \{c_0, \ldots, c_{q-1}\}$, where $c_0 = (\infty_2, \infty_3, \infty_4)$, and the $q-1$ other-cycles are defined on the vertex set $\mathbb{Z}_{n-4}$. We will construct the maximum packing so that the leave will be (∞_3, a), where a is defined below.

For each $i \in (\mathbb{Z}_q \setminus \{0\})$, let $c_i = (c_{i,1}, \ldots, c_{i,l_i})$ where l_i is the length of c_i. In this case, $n - 4 \equiv 4 \pmod 6$ and ≥ 10 and thus $\binom{(n-4)}{2} - (n-4) - \frac{n-4}{2} - (n-4-1)$ and $\binom{(n-4)}{2} - (n-4) - \frac{n-4}{2}$ are both divisible by 3 and nonnegative. So by Theorem 5, let $(\mathbb{Z}_{n-4}, B_1)$ be a K_3-decomposition of $K_{n-4} - (H_1 \cup H_3 \cup F_1)$ and let $(\mathbb{Z}_{n-4}, B_2)$ be a K_3-decomposition of $K_{n-4} - (H_2 \cup F_2)$ where H_1 and H_2 are Hamilton cycles, H_3 is a near-Hamilton cycle with arbitrarily named vertex $a \in \mathbb{Z}_{n-4}$ omitted, F_1 and F_2 are 1-factors, and H_1 and H_2 are named as follows: Let $H_1 = (c_{1,1}, \ldots, c_{1,l_1}, c_{2,1}, \ldots, c_{2,l_2}, \ldots, c_{q-1,1}, \ldots, c_{q-1,l_{q-1}})$. Let $H_2 = (c_{1,1}, c_{1,l_1}, c_{2,1}, c_{2,l_2}, \ldots, c_{q-1,1}, c_{q-1,l_{q-1}}, v_1, \ldots, v_{n-2-2q})$ where $v_1, \ldots, v_{n-2-2q}$ are arbitrarily named.

Let $H_1' = (E(H_1) \cup \{\{c_{i,1}, c_{i,l_i}\} \mid i \in \mathbb{Z}_q\}) \setminus \{\{c_{i,l_i}, c_{i+1,1}\} \mid i \in \mathbb{Z}_q\}$ reducing the subscript modulo q. Let $H_2' = (E(H_1) \cup E(H_2)) \setminus H_1'$. Then H_1' and H_2' each induce 2-regular spanning subgraphs of K_{n-4}, the set of cycles formed by the components in H_1' being $Q \setminus c_0$. Let $H_3' = E(H_3)$ and $H_4' = E(F_1) \cup E(F_2)$.

Let $(\{\infty_j \mid 1 \leq j \leq 4\}, B_3)$ be a K_3-decomposition of $2K_4$ (where the neighborhood of ∞_1 is the 3-cycle $(\infty_2, \infty_3, \infty_4)$).

Then $(\mathbb{Z}_{n-4} \cup \{\infty_j \mid 1 \leq j \leq 4\}, B_1 \cup B_2 \cup B_3 \cup \{\{\infty_i, a_i, b_i\} \mid \{a_i, b_i\} \in H_i'\})$ is the required decomposition.

Case 8: Suppose $n \equiv 0 \pmod 6$ and that Q has a 2-cycle.

We construct the maximum packing on the vertex set $\mathbb{Z}_{n-3} \cup \{\{\infty_j\} \mid 1 \leq j \leq 3\}$ where the neighborhood of ∞_1 is $Q = \{c_0, \ldots, c_{q-1}\}$, where $c_0 = (\infty_2, \infty_3)$, and the $q-1$ other-cycles are defined on the vertex set $\mathbb{Z}_{n-3}$.

For each $i \in \mathbb{Z}_q \setminus \{0\}$, let $c_i = (c_{i,1}, \ldots, c_{i,l_i})$ where l_i is the length of c_i. Note that $n \neq 6$, since then $Q = C_2 \cup C_3$, and we assumed that $(n, Q) \neq (6, C_2 \cup C_3)$. In this case, $n - 3 \equiv 3 \pmod 6$ and thus $\binom{(n-3)}{2} - 2(n-3)$ and $\binom{(n-3)}{2} - (n-3)$ are both divisible by 3, and since $n - 3 \geq 9$, both quantities are nonnegative. So by Theorem 5, let $(\mathbb{Z}_{n-3}, B_1)$ be a K_3-decomposition of $K_{n-3} - (H_1 \cup H_3)$ and let $(\mathbb{Z}_{n-3}, B_2)$ be a K_3-decomposition of $K_{n-3} - H_2$ where H_1, H_2, and H_3 are Hamilton cycles.

Then by Lemma 7, there exists a K_3-decomposition $(\mathbb{Z}_{n-3}, B)$ of $2K_{n-3} - (H_1' \cup H_2' \cup H_3')$ where H_1', H_2', and H_3' are 2-regular graphs and $H_1' \cong Q$.

Let $(\{\{\infty_j\} \mid 1 \leq j \leq 3\}, B_3)$ be a K_3-decomposition of $2K_3$ (where the neighborhood of ∞_1 is the 2-cycle (∞_2, ∞_3)).

Then $(\mathbb{Z}_{n-3} \cup \{\{\infty_j\} \mid 1 \leq j \leq 3\}, B_1 \cup B_2 \cup B_3 \cup \{\{\infty_i, a_i, b_i\} \mid \{a_i, b_i\} \in H_i'\})$ is the required decomposition.

Case 9: Suppose $n \equiv 0 \pmod 6$ and that Q has no cycles of length 2 (and hence one of length at least 4).

First, suppose $n = 6$. Define $B = \{\{\infty_1, 5, 0\}, \{\infty_1, 5, 1\}, \{\infty_1, 0, 2\}, \{\infty_1, 1, 3\}, \{\infty_1, 2, 3\}, \{5, 0, 3\}, \{5, 1, 2\}, \{5, 2, 3\}, \{0, 1, 2\}, \{0, 1, 3\}\}$. Then the neighborhood of ∞_1 is C_5.

Otherwise for $n \geq 12$, we construct the maximum packing on the vertex set $\mathbb{Z}_{n-2} \cup \{\{\infty_j\} \mid 1 \leq j \leq 2\}$ where the neighborhood of ∞_1 is $Q = \{c_0, \ldots, c_{q-1}\}$.

For each $i \in \mathbb{Z}_q$, let $c_i = (c_{i,1}, \ldots, c_{i,l_i})$ where l_i is the length of c_i, $l_0 \geq 4$, $c_{j,k} \in \mathbb{Z}_{n-2}$ for all $(j,k) \neq (0,2)$ and $c_{0,2} = \infty_2$. In this case, $n - 2 \equiv 4 \pmod 6$ and thus $\binom{n-2}{2} - \frac{n-2}{2} - (n-2-3)$ and $\binom{n-2}{2} - \frac{n-2}{2} - 1$ are both divisible by 3, and since $n - 2 \geq 10$, both quantities are nonnegative. So by Theorem 5, let $(\mathbb{Z}_{n-2}, B_1)$ be a K_3-decomposition of $K_{n-2} - (H_1 \cup F_1)$ and by Lemma 4, let $(\mathbb{Z}_{n-2}, B_2)$ be a K_3-decomposition of $K_{n-2} - H_2$ where H_1 is a $n-5$ cycle, H_2 consists of a $K_{1,3}$ and $\frac{(n-2)-4}{2}$ independent edges, F_1 is a 1-factor, and H_1 and H_2 named as follows: If $l_0 = 4$, let $H_1 = (c_{1,1}, \ldots, c_{1,l_1}, c_{2,1}, \ldots, c_{2,l_2}, \ldots, c_{q-1,1}, \ldots, c_{q-1,l_{q-1}})$ (so that $c_{0,1}, c_{0,3}$, and $c_{0,4}$ are omitted) and if $l_0 \geq 5$, let $H_1 = (c_{0,4}, c_{0,5}, \ldots, c_{0,l_0}, c_{1,1}, \ldots, c_{1,l_1}, c_{2,1}, \ldots, c_{2,l_2}, \ldots, c_{q-1,1}, \ldots, c_{q-1,l_{q-1}-1})$ (so that $c_{0,1}, c_{0,3}$, and $c_{q-1,l_{q-1}}$ are omitted). If $l_0 = 4$, let H_2 be defined to contain the edges $\{c_{i,1}, c_{i,l_i}\}$ for each $i \in \mathbb{Z}_q \setminus \{0\}$ as well as the edges $\{c_{0,4}, c_{0,1}\}, \{c_{0,4}, c_{0,3}\}$, and $\{c_{0,4}, c_{1,2}\}$ (note that $c_{1,2} \neq c_{1,l_1}$ since all cycles have length greater than 2 and note that $c_{0,4}$ is the vertex of degree 3 in H_2) and finally $\frac{n-2q-2}{2}$ arbitrarily named edges. If $l_0 \geq 5$, let H_2 be defined to contain the edges $\{c_{i,1}, c_{i,l_i}\}$ for each $i \in \mathbb{Z}_q$ as well as the edges $\{c_{q-1,l_{q-1}}, c_{q-1,l_{q-1}-1}\}$ and $\{c_{q-1,l_{q-1}}, c_{1,2}\}$ (note that $c_{1,2} \neq c_{1,l_1}$ since all cycles have length greater than 2 and note that $c_{q-1,l_{q-1}}$ is the vertex of degree 3 in H_2) and finally $\frac{n-2q-2}{2}$ arbitrarily named edges.

Note that in each case $\cup_{i=1}^{q-1} E(c_i) \cup E(c_0') \subset E(H_1) \cup E(H_2)$ where c_0' is formed from c_0 by removing the edges $\{c_{0,1}, c_{0,2}\}$ and $\{c_{0,3}, c_{0,2}\}$. Let $H_1' = \cup_{i=1}^{q-1} E(c_i) \cup E(c_0')$ and let $H_2' = (E(H_1) \cup E(H_2)) \setminus H_1'$. Then every vertex has degree 2 in H_1' and H_2' except $c_{0,1}$ and $c_{0,3}$ both of which have degree 1 in both H_1' and H_2'.

Then $(\mathbb{Z}_{n-2} \cup \{\{\infty_j\} \mid 1 \leq j \leq 2\}, B_1 \cup B_2 \cup \{\{\infty_i, a_i, b_i\} \mid \{a_i, b_i\} \in H_i'\} \cup \{\{\infty_1, \infty_2, c_{0,l}\} \mid l \in \{1, 3\}\})$ is the required decomposition.

Case 10: Suppose $n \equiv 3 \pmod 6$ and that Q has a 2-cycle.

We construct the maximum packing on the vertex set $\mathbb{Z}_{n-3} \cup \{\{\infty_j\} \mid 1 \le j \le 3\}$ where the neighborhood of ∞_1 is $Q = \{c_0, \ldots, c_{q-1}\}$, where $c_0 = (\infty_2, \infty_3)$, and the $q-1$ other-cycles are defined on the vertex set $\mathbb{Z}_{n-3}$.

For each $i \in \mathbb{Z}_q \setminus \{0\}$ (if $n = 3$, no such i exists), let $c_i = (c_{i,1}, \ldots, c_{i,l_i})$ where l_i is the length of c_i. In this case, $n - 3 \equiv 0 \pmod 6$ and thus $\binom{(n-3)}{2} - (n-3) - \frac{n-3}{2}$ is divisible by 3, and since $n - 3 = 0$ or ≥ 6, the quantity is nonnegative. So by Theorem 5, for $k \in \{1, 2\}$, let $(\mathbb{Z}_{n-3}, B_k)$ be a K_3-decomposition of $K_{n-3} - (H_k \cup F_k)$ where H_1 and H_2 are Hamilton cycles and F_1 and F_2 are 1-factors.

Then by Lemma 7, there exists a K_3-decomposition $(\mathbb{Z}_{n-3}, B)$ of $2K_{n-3} - (H_1' \cup H_2' \cup H_3')$ where H_1', H_2', and H_3' are 2-regular graphs and $H_1' \cong Q$.

Let $(\{\{\infty_j\} \mid 1 \le j \le 3\}, B_3)$ be a K_3-decomposition of $2K_3$ (where the neighborhood of ∞_1 is the 2-cycle (∞_2, ∞_3)).

Then $(\mathbb{Z}_{n-3} \cup \{\{\infty_j\} \mid 1 \le j \le 3\}, B \cup B_3 \cup \{\{\infty_i, a_i, b_i\} \mid \{a_i, b_i\} \in H_i'\}$ is the required decomposition.

Case 11: Suppose $n \equiv 3 \pmod 6$ and that Q has a cycle of length at least 4.

In this case, $n - 2 \equiv 1 \pmod 6$ and ≥ 7 (if $n = 3$, Q can't have a cycle of length at least 4) so $\binom{(n-2)}{2} - (n-2-1) \equiv 0 \pmod 3$ and ≥ 0 respectively. Thus, for $k \in \{1, 2\}$, by Theorem 5, let $(\mathbb{Z}_{n-2}, B_k)$ be a K_3-decomposition of $K_{n-2} - H_k$, where H_1 and H_2 are near-Hamilton cycles. Then by Lemma 9, the required maximum packing exists.

Case 12: Suppose $n \equiv 4 \pmod 6$ and that Q has a 3-cycle.

We construct the maximum packing on the vertex set $\mathbb{Z}_{n-4} \cup \{\{\infty_j\} \mid 1 \le j \le 4\}$ where the neighborhood of ∞_1 is $Q = \{c_0, \ldots, c_{q-1}\}$, where $c_0 = (\infty_2, \infty_3, \infty_4)$, and the $q-1$ other-cycles are defined on the vertex set $\mathbb{Z}_{n-4}$.

For each $i \in \mathbb{Z}_q \setminus \{0\}$ (if $n = 4$, no such i exists), let $c_i = (c_{i,1}, \ldots, c_{i,l_i})$ where l_i is the length of c_i. In this case, $n - 4 \equiv 0 \pmod 6$ and thus both $\binom{(n-4)}{2} - (n-4) - \frac{n-4}{2}$ and $\binom{(n-4)}{2} - 2(n-4) - \frac{n-4}{2}$ are divisible by 3, and since $n - 4 = 0$ or ≥ 6, both quantities are nonnegative. So by Theorem 5, let $(\mathbb{Z}_{n-4}, B_1)$ be a K_3-decomposition of $K_{n-4} - (H_1 \cup F_1)$ and let $(\mathbb{Z}_{n-4}, B_2)$ be a K_3-decomposition of $K_{n-4} - (H_2 \cup H_3 \cup F_2)$ where H_1, H_2, and H_3 are Hamilton cycles and F_1 and F_2 are 1-factors.

Then by Lemma 7, there exists a K_3-decomposition $(\mathbb{Z}_{n-4}, B)$ of $2K_{n-4} - (H_1' \cup H_2' \cup H_3' \cup H_4')$ where H_1', H_2', H_3' and H_4' are 2-regular graphs and $H_1' \cong Q$.

Let $(\{\{\infty_j\} \mid 1 \le j \le 4\}, B_3)$ be a K_3-decomposition of $2K_4$ (where the neighborhood of ∞_1 is the 3-cycle $(\infty_2, \infty_3, \infty_4)$).

Then $(\mathbb{Z}_{n-4} \cup \{\{\infty_j\} \mid 1 \le j \le 4\}, B \cup B_3 \cup \{\{\infty_i, a_i, b_i\} \mid \{a_i, b_i\} \in H_i', 1 \le j \le 4\}$ is the required decomposition.

Case 13: Suppose $n \equiv 4 \pmod 6$ and that Q has a cycle of length at least 5.

We construct the maximum packing on the vertex set $\mathbb{Z}_{n-2} \cup \{\{\infty_j\} \mid 1 \le j \le 2\}$ where the neighborhood of ∞_1 is $Q = \{c_0, \dots, c_{q-1}\}$.

For each $i \in \mathbb{Z}_q$, let $c_i = (c_{i,1}, \dots, c_{i,l_i})$ where l_i is the length of c_i, $l_0 \ge 5$, $c_{j,k} \in \mathbb{Z}_{n-2}$ for all $(j,k) \ne (0,2)$, and $c_{0,2} = \infty_2$. In this case, $n-2 \equiv 2 \pmod 6$ and thus $\binom{(n-2)}{2} - \frac{n-2}{2} - (n-2-2)$ and $\binom{(n-2)}{2} - \frac{n-2}{2}$ are both divisible by 3, and since $n-2 \ge 8$, both quantities are nonnegative. So by Theorem 5, let $(\mathbb{Z}_{n-2}, B_1)$ be a K_3-decomposition of $K_{n-3} - (H_1 \cup F_1)$ and by Lemma 4, let $(\mathbb{Z}_{n-2}, B_2)$ be a K_3-decomposition of $K_{n-2} - (F_2)$ where H_1 is a $n-4$ cycle, F_1 and F_2 are 1-factors, and H_1 and F_2 are named as follows: Let $H_1 = (c_{0,4}, c_{0,5}, \dots, c_{0,l_0}, c_{1,1}, \dots, c_{1,l_1}, c_{2,1}, \dots, c_{2,l_2}, \dots, c_{q-1,1}, \dots, c_{q-1,l_{q-1}})$ (so that $c_{0,1}$ and $c_{0,3}$ are omitted). Let F_2 be defined to contain the edges $\{c_{i,1}, c_{i,l_i}\}$ for each $i \in \mathbb{Z}_q$ as well as the edge $\{c_{0,3}, c_{0,4}\}$ and finally $\frac{n-2q-2}{2}$ arbitrarily named edges.

Note that $\cup_{i=1}^{q-1} E(c_i) \cup E(c_0') \subset E(H_1) \cup E(H_2)$ where c_0' is formed from c_0 by removing the edges $\{c_{0,1}, c_{0,2}\}$ and $\{c_{0,3}, c_{0,2}\}$. Let $H_1' = \cup_{i=1}^{q-1} E(c_i) \cup E(c_0')$ and let $H_2' = (E(H_1) \cup E(H_2)) \setminus H_1'$. Then every vertex has degree 2 in H_1' and H_2' except $c_{0,1}$ and $c_{0,3}$ both of which have degree 1 in both H_1' and H_2'.

Then $(\mathbb{Z}_{n-2} \cup \{\{\infty_j\} \mid 1 \le j \le 2\}, B_1 \cup B_2 \cup \{\{\infty_i, a_i, b_i\} \mid \{a_i, b_i\} \in H_i'\} \cup \{\{\infty_1, \infty_2, c_{0,l}\} \mid l \in \{1,3\}\})$ is the required decomposition.

Case 14: Suppose $n \equiv 1 \pmod 6$ and that Q has a 3-cycle.

We construct the maximum packing on the vertex set $\mathbb{Z}_{n-4} \cup \{\{\infty_j\} \mid 1 \le j \le 4\}$ where the neighborhood of ∞_1 is $Q = \{c_0, \dots, c_{q-1}\}$, where $c_0 = (\infty_2, \infty_3, \infty_4)$, and the $q-1$ other cycles are defined on the vertex set $\mathbb{Z}_{n-4}$.

For each $i \in \mathbb{Z}_q \setminus \{0\}$, let $c_i = (c_{i,1}, \dots, c_{i,l_i})$ where l_i is the length of c_i. $n \ne 1$ clearly, and if $n = 7$, $Q = C_3 \cup C_3$ in this case, which is impossible, so we have $n \ge 13$. In this case, $n-4 \equiv 3 \pmod 6$ and thus $\binom{(n-4)}{2} - 2(n-4)$ is divisible by 3, and since $n-4 \ge 9$, the quantity is nonnegative. So by Theorem 5, let $(\mathbb{Z}_{n-4}, B_1)$ be a K_3-decomposition of $K_{n-4} - (H_1 \cup H_3)$ and let $(\mathbb{Z}_{n-4}, B_2)$ be a K_3-decomposition of $K_{n-4} - (H_2 \cup H_4)$ where H_1, H_2, H_3, and H_4 are Hamilton cycles.

Then by Lemma 7, there exists a K_3-decomposition $(\mathbb{Z}_{n-4}, B)$ of $2K_{n-4} - (H_1' \cup H_2' \cup H_3' \cup H_4')$ where H_1', H_2', H_3', and H_4' are 2-regular graphs and $H_1' \cong Q$.

Let $(\{\{\infty_j\} \mid 1 \le j \le 4\}, B_3)$ be a K_3-decomposition of $2K_4$ (where the neighborhood of ∞_1 is the 3-cycle $(\infty_2, \infty_3, \infty_4)$).

Then $(\mathbb{Z}_{n-4} \cup \{\{\infty_j\} \mid 1 \le j \le 4\}, B \cup B_3 \cup \{\{\infty_i, a_i, b_i\} \mid \{a_i, b_i\} \in H_i', 1 \le j \le 4\}$ is the required decomposition.

Case 15: Suppose $n \equiv 1 \pmod 6$ and that Q has a cycle of length at least 4.

In this case, $n-2 \equiv 5 \pmod 6$ and ≥ 5 (if $n=1$, Q can't have a cycle of length at least 4) so $\binom{(n-2)}{2} - (n-2-1) \equiv 0 \pmod 3$ and ≥ 0 respectively. Thus, for $k \in \{1,2\}$, by Theorem 5, let $(\mathbb{Z}_{n-2}, B_k)$ be a K_3-decomposition of $K_{n-2} - H_k$, where H_1 and H_2 are near-Hamilton cycles. Then by Lemma 9, the required decomposition exists.

Note that this covers all the cases since Q must contain an odd cycle when $n \equiv 2 \pmod 6$ and $|Q| = n-1$, Q cannot contain all 3-cycles when $n \equiv 0 \pmod 3$, and Q cannot consist entirely of even cycles when $n \equiv 4 \pmod 6$. □

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