Gossiping and Set-to-Set Broadcasting in Weighted Graphs

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Abstract

Set-to-Set Broadcasting is an information distribution problem in a connected graph, $G = (V, E)$, in which a set of vertices A, called *originators* distributes messages to a set of vertices B called *receivers*, such that by the end of the broadcasting process each *receiver* has received the messages of all the *originators*. This is done by placing a series of calls among the communication lines of the graph. Each call takes place between two adjacent vertices, which share all the messages they have. Gossiping is a private case of set-to-set broadcasting, where $A = B = V$. We use $F(A, B, G)$ to denote the length of the shortest sequence of calls that completes the set to set broadcast from a set A of *originators* to a set B of *receivers*, within a connected graph G. $F(A, B, G)$ is also called the *cost* of an algorithm. We present bounds on $F(A, B, G)$ for weighted and for non-weighted graphs.

Keywords: Broadcasting, set-to-set broadcasting, gossiping.

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1 Introduction

Set to Set Broadcasting is an information distribution problem in a connected graph, in which a set of vertices, say A, called the *originators*, has to distribute messages to all vertices of a given set, say B, called the *receivers*, such that each vertex in B ultimately receives all the information held by the vertices in A.

This is done by placing a series of *calls* among the edges of the graph. Each call consists of two vertices that share information with each other. Vertices that receive information may distribute it further, thereby aiding the originators in transmitting their messages to the receivers. This is assumed to take place in discrete time units. There are some constraints on the set to set broadcast:

1. Each call involves two vertices that are adjacent to each other in the graph.
2. The two vertices share all the information they have, during a call between them.
3. Each call requires one time unit.
4. A vertex can participate in at most one call at each time unit.
5. At each time unit many calls can be performed in parallel.

Gossiping is an information distribution problem in a connected graph, in which each vertex in the graph has a piece of information to distribute to all other vertices in the graph. At the end of the gossip process, each vertex has all the information of all other vertices. The gossiping problem is in fact a special case of set-to-set broadcasting, in which $A = B = V$.

Broadcasting and gossiping have been extensively studied in the past years. However, in this paper we shall concentrate only on the works in which the gossip and the set-to-set models are involved.

Richards and Liestman [5] defined $F(A, B, G)$ to be the length of the shortest sequence of calls to complete set to set broadcasting. They calculated $F(A, B, K_n)$, where K_n is the complete graph on n vertices, namely,

$$F(A, B, K_n) = \begin{cases} |A| + |B| - |A \cap B| - 1, & \text{if } 0 \leq |A \cap B| \leq 2 \\ |A| + |B| - 3, & \text{if } |A \cap B| = 3 \end{cases}$$

They also showed that $F(A, B, K_n) \leq |A| + |B| - 4$ for $|A \cap B| \geq 4$ and conjectured that $F(A, B, K_n) = |A| + |B| - 4$. Li, Zhang and Xu [4] later confirmed this conjecture. Lee and Chang [3] studied $F(A, B, T)$, where T is a tree.

In this paper we compute $F(A, B, G)$ for set-to-set broadcasting and for gossiping on any weighted or non-weighted graph that is simple and connected.

In the following theorem we provide an upper bound for $F(A, B, G)$, where G is a connected weighted graph. The definition of W, W', d and D are in the next section.

Theorem 1.1. *Let G be a connected weighted graph. Let $AB = |A \cup B|$ and $E_{AB} = E_A \cup E_B$. Then,*

$$F(A, B, G) \leq \begin{cases} W'(T_A) - D + W'(T_B), & \textit{if } AB = 0 \textit{ and } AB \neq \phi \\ W'(T_A) + d + W'(T_B), & \textit{if } AB = 0 \textit{ and } E_{AB} = \phi \\ W'(T_A) - D + W'(T_B), & \textit{if } AB = 1 \textit{ and } E_{AB} \neq \phi \\ W'(T_A) + W'(T_B), & \textit{if } AB = 1 \textit{ and } E_{AB} = \phi \\ W'(T_A) + W'(T_B) - w(uv), & \textit{if } AB \geq 2, E_{AB} \neq \phi \\ & \{u, v\} \subseteq A \cap B \\ & \textit{and } uv \in E(T_{AB}) \\ W'(T_A) + W'(T_B), & \textit{if } AB \geq 2 \textit{ and } E_{AB} = \phi \end{cases}$$

The theorem holds for non-weighted graphs, as well, which can be represented as weighted graphs, in which the weight of each edge is 1, $W'(A) = |E'(A)|$, $W'(B) = |E'(B)|$ and $w(uv) = 1$.

2 Definitions and Notation

In this section we present definitions and notation that are used in what follows. As usual, $\mathbb{R}^+$ shall denote the set of all positive real numbers.

Let $G = (V, E)$ be a connected undirected graph.

1. For a vertex $a \in V$ define $N(a) = \{v \in V | av \in E\}$.

2. For each $u, v \in V$ define $d_G(u, v)$ or $d(u, v)$ (if there is no question about G) as the length of the shortest path from u to v in G.

3. The *diameter* of G, denoted $diam(G)$, is $max\{d(u, v) | u, v \in V\}$.

4. For each $u, v \in V$ define $P_{uv} = (V_{uv}, E_{uv})$ as the path from u to v.

5. Define the union of two graphs $H = (V_1, E_1)$, $G = (V_2, E_2)$ to be $H \cup G = (V_1 \cup V_2, E_1 \cup E_2)$.

6. The graph $G = (V, E)$ is called an *edge-weighted graph* if there is a function $w : E \to \mathbb{R}^+$, called a *weight function* such that for each $xy \in E$, $w(xy) = d_w(x, y)$, where $d_w(x, y)$ is the minimum weighted path between x and y.

   (a) For each $x \in V$, define $W(x) = \Sigma_{y \in N(x)} w(xy)$.

   Notice: Since G is connected and not the trivial graph on one vertex, the functions w and W are well defined.

   (b) The weight of G, denoted $W'(G) = \Sigma_{e \in E(G)} w(e)$.

   (c) For $u, v \in V$, define $P(u, v)$ a path between u and v and define the weight of a path to be $W'(P) = \sum_{e \in E(P)} w(e)$.

   Observation 2.1. *If G is not a weighted graph then $W'(P)$ is the length of the path P.*

7. Let S be a set, $S \subseteq V$. Denote by $GP(G, S)$ the number of calls needed to complete a gossip process among the members of S in the graph G.

8. Let S be a set, $S \subseteq V$ and let $w : E \to \mathbb{R}^+$ be a weight function on E. Denote by $GP_w(G, S)$ the total cost associated with a complete gossip process among the members of S in the graph G, where the total cost is the sum of the weights of the edges that participated in the gossip.

   Note: If an edge is used k times, where k is a positive integer, in the gossip scheme, the edge weight is included k times in the total cost.

9. Let $A, B \subseteq V$, where A is the set of originators and B is the set of receivers. Define $F(A, B, G)$ to be the minimum number of calls needed to accomplish set-to-set broadcasting from A to B.

   For a connected weighted graph $G = (V, E)$ with a weight function $w : E \to \mathbb{R}^+$, define $F_w(A, B, G)$, accordingly.

10. Let $G = (V, E)$ be a connected edge weighted graph. Let $A, B \subseteq V$, where A is the set of originators and B is the set of receivers. Define a tree $T_A = (V_A, E_A)$, where $A \subseteq V_A \subseteq V$, $E_A \subseteq E$. Define $T_B = (V_B, E_B)$, accordingly.

    Given two trees $T_A = (V_A, E_A)$ and $T_B = (V_B, E_B)$,

    (a) Let $x \in V_A$, $y \in V_B$ such that $d_w(x, y) = \min\{d_w(u, v) | u \in V_A, v \in V_B\}$. Denote $d_w(x, y)$ as d. Notice that the path that connects x to y with weight $d_w(x, y)$ is in $G \setminus (T_A \cup T_B)$.

    (b) Let $x \in V_A$, $y \in V_B$ such that $D_w(x, y) = max\{d_w(u, v) | u \in V_A, v \in V_B \wedge \forall wz \in E(P_{uv}), wz \in E_A \cap E_B\}$. Denote $D_w(x, y)$ as D.

    (c) For $x \in V_A$, T_x is the tree T_A rooted at x and for $y \in V_B$, T_y is the tree T_B rooted at y.

For other graph theoretical definitions one may look at [6].

3 Gossiping in a Weighted Graph

Let $G = (V, E)$, $|V| = n$, be a connected weighted graph. In this section we provide lower and upper bounds for gossiping in weighted graphs. We describe a process GW for gossiping between the vertices of V, which provides the upper bound for $GP_w(G, V)$.

Following is a lemma that presents a lower bound for gossiping in a weighted graph G.

Lemma 3.1. *Let $G = (V, E)$, $|V| = n$, be a connected weighted graph with a weight function $w : E \to \mathbb{R}^+$. Let T be a minimum weighted spanning tree of G. Then,*

$$GP_w(G, V) \geq 2W'(T) - max\{w(e)|e \in E(T)\}.$$

Proof. Each optimal algorithm for gossiping in a weighted (or non-weighted) graph executes at least $n-1$ calls. After the first $n-1$ calls, exactly two vertices x and y have all the information. The set of the first $n-1$ calls uses a set of edges, say S_1, where for each vertex $v \in V$ there is a vertex $u \in V$ such that $uv \in S_1$. Thus, $T_1 = (V, S_1)$ is a connected graph, specifically, a spanning tree. It is obvious that if T_1 is a minimum spanning tree, the sum of weights of the edges of S_1 is minimal. As noted, after the first $n-1$ calls, exactly two vertices have all the information, meaning that the remaining $n-2$ vertices do not have all the information. Thus, at least $n-2$ additional calls are needed to accomplish gossiping in G using a set of $n-2$ edges, say S_2. For each vertex $v \in V \setminus \{x, y\}$ there is a vertex $u \in V$ such that $uv \in S_2$ and $T_2 = (V, S_2 \cup \{xy\})$ is a spanning tree. Therefore, the minimal weight of such a set S_2 is the sum of weights of a set of edges forming a minimum spanning tree, say T, excluding the edge with the highest maximal weight. Thus, $GP_w(G, V) \geq 2W'(T) - max\{w(e)|e \in E(T)\}$.

□

Following is an algorithm GW for gossiping in a weighted graph G.

1. Find a minimum weighted spanning tree of G, $T = (V, E')$ (use either the Kruskal algorithm or the Prim algorithm [1]).

2. Find an edge, $e' = uv \in E'$ such that $w(e') = max\{w(e)|e \in E'\}$.

3. Let T_u and T_v be the two trees rooted at u and v respectively, obtained by deleting e' from T. Then,

   3.1 Each vertex in T_u and T_v (except the roots) calls its parent. The transmission in each of the trees T_u and T_v is from the lowest level towards the roots u and v (bottom up). Thus each edge in T_u and T_v is used only once.

   3.2 u calls v.

   3.3 Each vertex in T_u and T_v (except the leaves) calls its children. The transmission in each of the trees T_u and T_v is from the roots u and v towards the leaves (top down). Thus each edge in T_u and T_v is used only once.

The cost of the gossiping process, GW, obtained by the algorithm in G is:

$$GP_w(G, V) \leq 2W'(T) - w(uv), \tag{1}$$

where phase 3.1 contributes $W'(T) - w(uv)$, phase 3.2 contributes $w(uv)$ and phase 3.3 contributes $W'(T) - w(uv)$.

The algorithm complexity is:

phase 1 - $O(|E| \log |V|)$ (Kruskal algorithm) or $O(|E| + |V| \log |V|)$ (Prim algorithm).

phase 2 - $O(|E|)$.

phase 3 - $O(|V| + |E|)$.

Thus, the complexity of the algorithm is $O(|E| \log |V|)$ or $O(|E| + |V| \log |V|)$.

Lemma 3.2. *Let $G = (V, E)$ be a connected weighted graph. Then,*

$$GP_w(G, V) = 2W'(T) - w(uv).$$

Proof. The proof of the lemma is obtained combining both the upper bound obtained by the algorithm in (1) and the lower bound obtained in Lemma 3.1. □

Corollary 3.1. *If G is a non-weighted graph, then $GP(G, V) = 2n - 3$.*

Proof. Since we assume that the weight of each edge is 1, the result follows from Lemma 3.2, where $W'(T) = n - 1$. □

4 Set-to-Set Broadcast Algorithm in Weighted Graphs

In this section we give lower bounds for the set-to-set broadcasting problem and describe set-to-set broadcasting algorithm on a weighted graph yielding the results of Theorem 1.1.

We start by establishing a lower bound for the cost of the set-to-set broadcasting process in a weighted connected graph.

Denote by $\min(v) = \min\{w(vu) | vu \in E(G)\}$.

Lemma 4.1. *Let $G = (V, E)$ be a connected weighted graph. Let $A, B \subseteq V$. Then, $F(A, B, G) \geq \sum_{v \in (A \cup B) \setminus (A \cap B)} \min(v)$.*

Proof. Each vertex in A must participate in at least one call in order to send its message. Each vertex in B must participate in at least one call in order to receive the messages. The cost of a call involving vertex $v \in (A \cup B) \setminus (A \cap B)$ is at least $\min(v)$. □

Remark 4.1. *Notice that applying Lemma 4.1 with Theorem 1.1 on the complete non-weighted graph K_n, we obtain the results of the complete graph.*

We now derive a lower bound for the cost of the set-to-set broadcasting in a non-weighted connected graph.

Lemma 4.2. *Let $G = (V, E)$ be a non-weighted connected graph. Let $A, B \subseteq V$ and A is the set of originators while B is the set of receivers.*

$$\text{Then, } F(A,B,G) \geq \begin{cases} |A|+|B|-1, & \text{if } |A \cap B| = 0 \\ |A|+|B|-2, & \text{if } |A \cap B| = 1 \\ |A|+|B|-3, & \text{if } |A \cap B| = 2,3 \\ |A|+|B|-4, & \text{if } |A \cap B| \geq 4 \end{cases}$$

Proof. The lower bound is a lower bound for the case where all calls are possible, i.e. G is the complete graph. In case where G is not the complete graph, the number of calls needed to complete the set-to-set broadcast might increase and thus the lower bound increases.

□

We now present an algorithm for finding minimal spanning trees of a subset of V possibly with additional vertices.

Let $G = (V, E)$ be a weighted graph and let $A \subseteq V$. Define a tree $T_A = (V_A, E_A)$, where $A \subseteq V_A \subseteq V$, $E_A \subseteq E$. We describe an algorithm that finds a tree T_A for a given graph $G = (V, E)$ and a set $A \subseteq V$.

1. Find a minimal spanning tree, $T = (V_T, E_T)$, of G (use either the Kruskal algorithm or the Prim algorithm[1]).

2. Define a set of trees $T_i = (V_i, E_i)$, for $0 \leq i \leq k$, where $T_0 = T$ and $T_k = (V_k, E_k)$ is a tree such that for each $v \in V_k$ where $d(v) = 1$, it follows that $v \in A$. In other words, all the leaves in the tree T_k are members in A.

   For each j, $1 \leq j \leq k$, define $A_{j-1} = \{x \in V_{j-1} \setminus A | d(x) = 1\}$, $B_{j-1} = \{xu \in E_{j-1} | x \in A_{j-1}\}$ and $T_j = (V_j, E_j)$, where $V_j = V_{j-1} \setminus A_{j-1}$ and $E_j = E_{j-1} \setminus B_{j-1}$. In other words, T_j is the tree that is obtained from T_{j-1} by removing all leaves in T_{j-1} that are not in A.

3. $j = 1$. If $T_j \neq T_k$,

    (a) obtain T_j from T_{j-1}

    (b) $j = j + 1$

4. Return T_k.

The algorithm complexity is as follows:

phase 1 - $O(|E| \log |V|)$ (Kruskal algorithm) or $O(|E| + |V| \log |V|)$ (Prim algorithm).

phases 2,3 - $O(|E|)$.

Thus, the complexity of the algorithm is $O(|E| \log |V|)$ or $O(|E| + |V| \log |V|)$.

Next, we present an algorithm for set-to-set broadcasting in a weighted connected graph, which applies the proof to theorem 1.1.

1. If there are two trees $T_A = (V_A, E_A)$ and $T_B = (V_B, E_B)$ such that $E_A \cap E_B \neq \phi$, then find two trees T_A and T_B with minimal $W'(T_A \cup T_B)$ and D. Let $x \in V_A$, $y \in V_B$ such that $D = D_w(x, y)$.

    Otherwise, find two trees T_A and T_B with minimal $W'(T_A \cup T_B)$ and d. Let $x \in V_A$, $y \in V_B$ such that $d = d_w(x, y)$. (Notice that if $V_A \cap V_B \neq \phi$ and $x = y$, then $d = 0$).

2. Each vertex in T_x, except the root x, calls its parent. The transmission is from the lowest level towards the root x (bottom up).

3. If $x \neq y$, x calls y.

4. Each vertex in T_y calls its children. The transmission is from the root y towards the lowest level (top down).

4.1 Proof of Theorem 1.1

Proof. case 1: $|A \cap B| = 0$ and $E_A \cap E_B \neq \phi$. Let $x \in A$, $y \in B$ such that $xy \in E_A \cap E_B$ and let $d(x, y) = D$. First, all the vertices in A transmits their messages via the edges of T_A to x, where y is the last vertex that calls x. This phase costs $W'(T_A)$. Then y broadcasts all the messages to all the vertices in B using the edges of $T_B \setminus \{xy\}$, which costs $W'(T_B) - D$. Thus, $F(A, B, G) \leq W'(T_A) - D + W'(T_B)$.

case 2: $|A \cap B| = 0$ and $E_A \cap E_B = \phi$. Let $x \in T_A$, $y \in T_B$ such that $d(x, y) = d$. $x \in A$ receives all the messages of the vertices in A. This phase costs $W'(T_A)$. Then x calls $y \in B$. Finally, all the messages are transmitted to all the vertices in B, which costs $W'(T_B)$. Thus, $F(A, B, G) \leq W'(T_A) + d + W'(T_B)$.

case 3: $|A \cap B| = 1$ and $E_A \cap E_B \neq \phi$. Let $xy \in E_A \cap E_B$, where $x \in A$, $y \in A \cap B$ and $d(x, y) = D$. First, all the vertices in $A \setminus \{y\}$ transmit their messages via T_A to x. This phase costs $W'(T_A) - D$. Then, x calls y, which costs D, and y broadcasts all the messages to all the vertices in B via T_B, which costs $W'(T_B)$. Thus, $F(A, B, G) \leq W'(T_A) - D + W'(T_B)$.

case 4: $|A \cap B| = 1$ and $E_A \cap E_B = \phi$. Notice that if $|A \cap B| = 1$ and $E_A \cap E_B = \phi$, $A \cap B = \{x = y\}$ and therefore $d = 0$. A vertex $x \in A \cap B$ receives all the messages of the vertices in A. This phase costs $W'(T_A)$. Then, all the messages are transmitted from x to all the vertices in B, which costs $W'(T_B)$. Thus, $F(A, B, G) \leq W'(T_A) + W'(T_B)$.

case 5: $|A \cap B| \geq 2$ and $\{x, y\} \subseteq A \cap B$ and $xy \in E(T_A) \cap E(T_B)$. First, all the vertices in $A \setminus \{y\}$ transmit their messages to x, using the edges of T_A. This phase costs $W'(T_A) - w(xy)$. Then, x calls y, which costs $w(xy)$. Finally, y broadcasts all the messages to all the vertices in B using the edges of T_B, which costs $W'(T_B)$. Thus, $F(A, B, G) \leq W'(T_A) + W'(T_B) - w(xy)$.

case 6: $|A \cap B| \geq 2$ and $E_A \cap E_B = \phi$. A vertex $x \in A \cap B$ receives all the messages of the vertices in A. This phase costs $W'(T_A)$. Then, all the messages are transmitted from x to all the vertices in B, which costs $W'(T_B)$. Thus, $F(A, B, G) \leq W'(T_A) + W'(T_B)$. □

The following figures illustrate examples of case 2, described above.

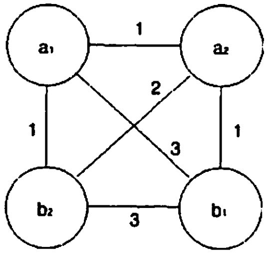

Figure 1: This example demonstrates case 2, where choosing x, y such that $x = y$ and $d = 0$, minimizes the total cost. $W'(T_A) = 1$ and $W'(T_B) = 3$. $E_A = \{a_1a_2\}$, but there are two options for E_B namely, $E_B = \{b_1b_2\}$ or $E_B = \{a_1b_1, a_1b_2\}$. On one hand, choosing $E_B = \{b_1b_2\}$, $x = a_1$, $y = b_1$ and $d = 2$. The total cost of the set-to-set broadcast is $w(a_1a_2)$ + $w(b_1b_2)$ + $w(a_1b_2)$ = $1 + 3 + 1 = 5$. On the other hand, choosing $E_B = \{b_1a_1, b_2a_1\}$, minimizes the total cost of the set-to-set broadcast to $w(a_1a_2)+w(a_1b_1)+w(a_1b_2) = 1+2+1 = 4$. Here, $|A \cap B| = 0$, $E_A \cap E_B = \phi$ and $x = y = a_1$ and therefore $d = 0$.

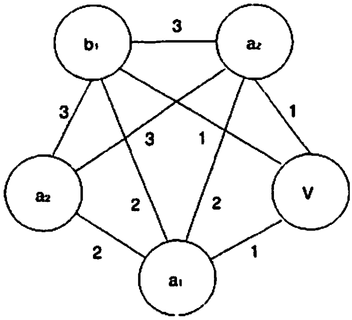

Figure 2: This example demonstrates case 2, where $d \neq 0$ and $V_A \cup V_B \neq A \cup B$, but $A \cup B \subset V_A \cup V_B$. $E_A = \{a_1a_2\}$, $E_B = \{b_1v, b_2v\}$, $W'(T_A) = 2$ and $W'(T_B) = 2$. The total cost of the set-to-set broadcast is $w(a_1a_2) + w(b_1v) + w(b_2v) + w(a_1v) = 2+1+1+1 = 5$. Here, $|A \cap B| = 0$, $E_A \cap E_B = \phi$, $x = a_1$, $y = v$ and $d = 1$.

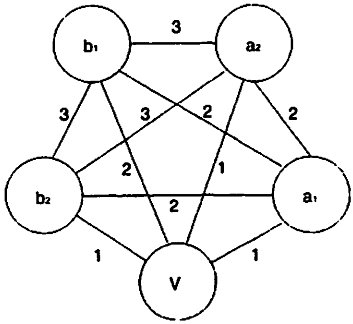

Figure 3: This example demonstrates case 2, where $d \neq 0$ and $V_A \cup V_B = A \cup B$. $E_A = \{a_1a_2\}$, $E_B = \{b_1v, b_2v\}$. $w(T_A) = 2$, $w(T_B) = 4$. If we use E_B the total cost will be $w(a_1a_2) + w(b_1v) + w(b_2v) + w(a_1v) = 2 + 2 + 2 + 3 = 9$, but if we use $E_B = \{b1b2\}$, the total cost will be $w(a_1a_2) + w(b_1b_2) + w(a_1b_1) = 2 + 5 + 1 = 8$.

In the next proposition we show that applying our set-to-set broadcasting algorithm to a complete non-weighted graph yields the result of [5] and [4]. In this case T_A contains only vertices from A and T_B contains only vertices from B and thus $|E(T_A)| = |A| - 1$ and $|E(T_B)| = |B| - 1$.

4.2 Set-to-set broadcasting in complete bipartite graph

Theorem 4.1. *Let $G = (X \cup Y, E)$, $X \cap Y = \phi$ be a complete bipartite graph and let $A, B \subseteq X \cup Y$. Then, $F(A, B, G) = |A| + |B| - i - j$, where*

$$i = \begin{cases} 0, & \text{if } A, B \subseteq X \text{ or } A, B \subseteq Y \\ 1, & \text{otherwise} \end{cases}$$

$$j = \begin{cases} 0, & \text{if } A \cap B = \phi \\ 2, & \text{if } \exists v \in (A \cap B) \cap X \text{ and } \exists u \in (A \cap B) \cap Y \\ 1, & \text{otherwise} \end{cases}$$

Note that the case where $i = 0$ and $j = 2$ is not possible.

Proof. 1. $i = j = 0$.

Lower bound. $A, B \subseteq X$ or $A, B \subseteq Y$ and $A \cap B = \phi$. Assume W.L.O.G. that $A, B \subseteq X$. Then, at least $|A|$ calls are needed in order to transfer all the messages from X to Y and at least $|B|$ calls are needed in order to transfer all the messages from Y to the vertices of $B \subseteq X$. Therefore, at least $|A| + |B|$ calls are needed to complete a set-to-set broadcast from A to B.

Upper bound. Assume W.L.O.G. that $A, B \subseteq X$. Then both T_A and T_B are trees that contain exactly one common vertex v, $v \in Y$ and all the other vertices are in $A, B \subseteq X$. Then, $W'(T_A) = |A|$ and $W'(T_B) = |B|$, $|A \cap B| = 0$ and $E_A \cap E_B = \phi$. By Theorem 1.1 it follows that $F(A, B, G) \leq W'(T_A) + d + W'(T_B)$. Since the graph is a complete bipartite graph, $v = x = y$ and thus, $d = 0$. Therefore our algorithm performs $|A| + |B| = |A| + |B| - i - j$ calls in order to complete the set-to-set broadcast.

2. $i = 0$ and $j = 1$.

Lower bound. $\exists v \in (A \cap B) \cap X$ or $\exists u \in (A \cap B) \cap Y$, but not both. Assume W.L.O.G. that $\exists v \in (A \cap B) \cap X$. Then, $|A|$ calls are needed to transfer the messages of the vertices of A to Y. Let v be the last vertex that calls a vertex in Y; then v may know all the messages of A, and then at least $|B| - 1$ more calls are needed to complete the set-to-set broadcast from A to B.

Upper bound. Assume W.L.O.G. that $A, B \subseteq X$. Then both T_A and T_B are trees that contain exactly one common vertex v, $v \in Y$, and all other vertices are in $A, B \subseteq X$. Since $j = 1$ there exists one common vertex u, $u \in X$. Then, $uv \in E_A \cap E_B$ and therefore $E_A \cap E_B \neq \phi$. $W'(T_A) = |A|$ and $W'(T_B) = |B|$, and according to Theorem 1.1, if $|A \cap B| = 1$, $F(A, B, G) \leq W'(T_A) - D + W'(T_B)$, and if $|A \cap B| \geq 2$, $F(A, B, G) \leq W'(T_A) - w(uv) + W'(T_B)$. Since the graph is a complete non-weighted bipartite graph, $u = x$, $v = y$, $D = 1$ and $w(uv) = 1$. Therefore our algorithm performs $|A| + |B| - 1 = |A| + |B| - i - j$ calls in order to complete the set-to-set broadcast.

3. $i = 1$ and $j = 0$.

   Lower bound. $A \cap B = \phi$, but $\exists a \in A \subseteq X$ and $\exists b \in B \subseteq Y$ or $\exists a \in A \subseteq Y$ and $\exists b \in B \subseteq X$. Assume W.L.O.G. that $\exists a \in A \subseteq X$ and $\exists b \in B \subseteq Y$. Then, after $|A|$ calls, $b \in B \subseteq Y$ knows all the messages of A, and therefore at least $|B| - 1$ more calls are needed to complete the set-to-set broadcast.

   Upper bound. $W'(T_A) = |A| - 1$, $W'(T_B) = |B| - 1$, $|A \cap B| = 0$, and $E_A \cap E_B = \phi$. By Theorem 1.1, $F(A, B, G) \leq W'(T_A) - D + W'(T_B)$. Since the graph is a complete bipartite graph and $i = 1$, there exist $x \in A \subseteq X$ and $y \in B \subseteq Y$ or there exist $x \in A \subseteq Y$ and $y \in B \subseteq X$. Therefore, $D = 1$, and our algorithm performs $|A| + |B| - 1 = |A| + |B| - i - j$ calls in order to complete the set-to-set broadcast.

4. $i = 1$ and $j = 1$.

   Lower bound. $\exists v \in (A \cap B) \cap X$ or $\exists u \in (A \cap B) \cap Y$, but not both. Assume W.L.O.G. that $\exists v \in (A \cap B) \cap X$. After $|A|$ calls, all the messages of A have been transmitted to the vertices of Y. Let $y \in (A \cap B) \cap Y$ be a vertex such that the last call is from v to y. It is possible that v knows all the messages of the vertices in A, and therefore $|B| - 2$ more calls are needed to complete the set-to-set broadcast from A to B.

Upper bound. $W'(T_A) = |A| - 1$, $W'(T_B) = |B| - 1$, $|A \cap B| \neq 0$ and $E_A \cap E_B = \phi$. According to theorem 1.1 $F(A, B, G) \leq W'(T_A) + W'(T_B)$. Since the graph is a complete bipartite graph and $i = 1$, there exist $x \in A \subseteq X$ and $y \in B \subseteq Y$ or there exist $x \in A \subseteq Y$ and $y \in B \subseteq X$. Therefore, our algorithm performs $|A| + |B| - 2 = |A| + |B| - i - j$ calls in order to complete the set to set broadcast.

5. $i = 1$ and $j = 2$.

   Lower bound. $\exists v \in (A \cap B) \cap X$ and $\exists u \in (A \cap B) \cap Y$. Then, $|A| - 1$ calls are needed in order to obtain two vertices v and u that have all the messages of A (the last call should be between v and u). Then, $|B| - 2$ calls are needed to complete the set-to-set broadcast.

   Upper bound. $W'(T_A) = |A| - 1$, $W'(T_B) = |B| - 1$, $|A \cap B| \geq 2$ and $E_A \cap E_B = \phi$. According to Theorem 1.1, $F(A, B, G) \leq W'(T_A) - w(uv) + W'(T_B)$. Since the graph is a complete bipartite graph, $i = 1$ and $j = 2$, there exist $u \in (A \cap B) \subseteq X$ and $v \in (A \cap B) \subseteq Y$, and $w(uv) = 1$. Therefore, our algorithm performs $|A| - 1 + |B| - 1 - 1 = |A| + |B| - 3 = |A| + |B| - i - j$ calls in order to complete the set-to-set broadcast.

□

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