Downhill and Uphill Domination in Graphs

Jessie Deering[1], Teresa W. Haynes[1],
Stephen T. Hedetniemi[2], and William Jamieson[1]

[1]Department of Mathematics and Statistics
East Tennessee State University
Johnson City, TN 37614 USA
Email: haynes@etsu.edu

[2]Professor Emeritus
School of Computing
Clemson University
Clemson, SC 29634 USA
Email: hedet@clemson.edu

Abstract

Placing degree constraints on the vertices of a path yields the definitions of uphill and downhill paths. Specifically, we say that a path $\pi = v_1, v_2, \dots v_{k+1}$ is a *downhill path* if for every i, $1 \leq i \leq k$, $\deg(v_i) \geq \deg(v_{i+1})$. Conversely, a path $\pi = u_1, u_2, \dots u_{k+1}$ is an *uphill path* if for every i, $1 \leq i \leq k$, $\deg(u_i) \leq \deg(u_{i+1})$. The downhill domination number of a graph G is defined to be the minimum cardinality of a set S of vertices such that every vertex in V lies on a downhill path from some vertex in S. The uphill domination number is defined as expected. We explore the properties of these invariants and their relationships with other invariants. We also determine a Vizing-like result for the downhill (respectively, uphill) domination numbers of Cartesian products.

Keywords: downhill path, uphill path, downhill domination number, uphill domination number, Cartesian product.

1 Introduction

In a graph $G = (V, E)$, the *degree* of a vertex v is given by $\deg(v) = |\{u : uv \in E\}|$. The minimum and maximum degrees of vertices in a graph G are denoted $\delta(G)$ and $\Delta(G)$, respectively. A graph G is *r-regular* if $\deg(v) = r$ for every vertex $v \in V$. A *path of length k* in G is a sequence of distinct vertices $v_1, v_2, \ldots, v_{k+1}$, such that for every i, $1 \leq i \leq k$, $v_i v_{i+1} \in E$.

To these standard definitions above, we introduce the following concepts. We say that a path $v_1, v_2, \ldots v_{k+1}$ is a *downhill path* if for every i, $1 \leq i \leq k$, $\deg(v_i) \geq \deg(v_{i+1})$. Similarly, we define an *uphill path* to be a path $v_1, v_2, \ldots v_{k+1}$ having the property that for every i, $1 \leq i \leq k$, $\deg(v_i) \leq \deg(v_{i+1})$. For example, in Figure 1, several downhill paths are given for the same graph. Note that any downhill path can be reversed to create an uphill path. A downhill path $P : u = v_1, v_2, \ldots, v_{k+1} = v$ is called a u-v downhill path and is said to *orginate* at u and *terminate* at v. We say that a vertex v_j is downhill from a vertex v_i if v_i and v_j are on P and $j \geq i$. Similar terminology is used for uphill paths.

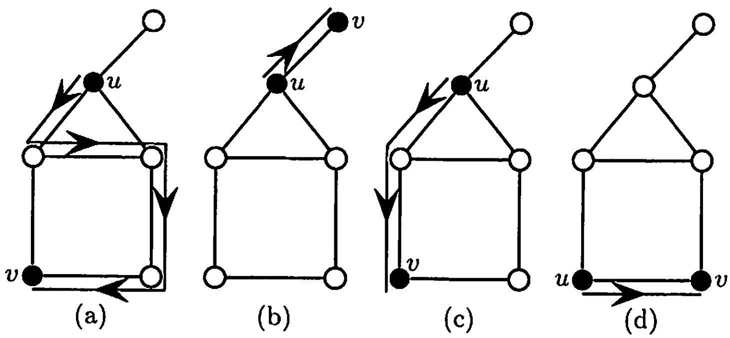

Figure 1: Examples of downhill paths.

We should observe that although the definition of a downhill (uphill) path is given in terms of the degrees of the vertices on the path, a similar definition can be given in terms of any function that assigns weights to the vertices of a graph, as is done in surveying when assigning elevations to the points of a topographic map, or in thermal imaging, in which the values assigned to the points in an image are a measure of their heat content.

A vertex u is said to *downhill dominate* a vertex v if there exists a u-v downhill path. Note that a vertex downhill dominates itself. A *downhill dominating set*, abbreviated DDS, is a set $S \subseteq V$ having the property that every vertex $v \in V$ is downhill dominated by some vertex of S, that is, every vertex $v \in V$ lies on a downhill path originating from some vertex in S. The *downhill domination number* $\gamma_{dn}(G)$ equals the minimum cardinality of a DDS of G. A DDS S having minimum cardinality is called a *γ_{dn}-set.*

An *uphill dominating set*, abbreviated UDS, is a set $S \subseteq V$ having the property that every vertex $v \in V$ lies on a uphill path originating from some vertex in S. The *uphill domination number* $\gamma_{up}(G)$ equals the minimum cardinality of a UDS of G. An UDS S having minimum cardinality is called a *γ_{up}-set.*

Although the definitions of downhill and uphill paths are similar, somewhat surpisingly, the parameters $\gamma_{dn}(G)$ and $\gamma_{up}(G)$ are incomparable. To see this, we note that $\gamma_{dn}(G) = 1 = \gamma_{up}(G)$ for connected, regular graphs G, the graph G in Figure 2 has $\gamma_{dn}(G) < \gamma_{up}(G)$, and the graph H in Figure 2 has $\gamma_{dn}(H) > \gamma_{up}(H)$. In these figures, the darkened vertices form a γ_{up}-set and the circled vertices form a γ_{dn}-set.

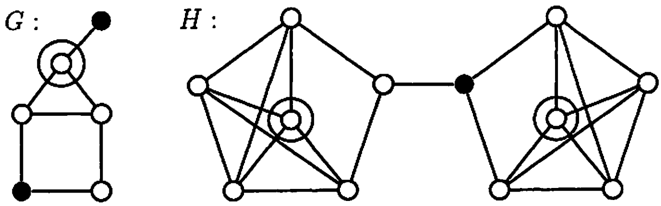

Figure 2: A graph G with $\gamma_{dn}(G) = 1 < 2 = \gamma_{up}(G)$ and a graph H with $\gamma_{up}(H) = 1 < 2 = \gamma_{dn}(H)$.

A set S is a *dominating set* of a graph G if every vertex in $V \setminus S$ has a neighbhor in S, and is an *independent dominating set* if it is both dominating and independent. The domination number $\gamma(G)$ (respectively, independent domination number $i(G)$) is the minimum cardinality of a dominating (respectively, independent dominating) set of G. We note here that downhill and uphill domination numbers are also incomparable with these standard domination parameters. For instance, let G be the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ for $n \geq 6$. If n is even, then $\gamma(G) = 2 > 1 = \gamma_{dn}(G) = \gamma_{up}(G)$. On the other hand, for odd $n \geq 7$, $\gamma(G) = 2 < \lfloor \frac{n}{2} \rfloor = \gamma_{dn}(G) < \lceil \frac{n}{2} \rceil = \gamma_{up}(G)$. Additionally, $\gamma_{dn}(G)$ and $\gamma_{up}(G)$ are also incomparable with $i(G)$. To see this, let H be the graph formed from the complete bipartite graph $K_{r,s}$, for $r \geq 3$ and $s \geq r + 2$, by deleting an arbitrary edge. In this case, $i(H) = 2 < r = \gamma_{dn}(H) < s = \gamma_{up}(H)$. On the other hand, the graph H given in Figure 2 has $1 = \gamma_{up}(H) < \gamma_{dn}(H) = 2 < 3 = i(H)$. For more details on domination, see [1].

Strong and weak domination were defined in [2]. A set S is a *strong dominating set* (respectively, *weak dominating set*) if every vertex $u \in V \setminus S$ has a neighbor $v \in S$ such that $\deg(v) \geq \deg(u)$ (respectively, $\deg(v) \leq \deg(u)$).

For a graph G, the *strong domination number* $\gamma_S(G)$ is the minimum cardinality of a strong dominating set of G, and the *weak domination number* $\gamma_W(G)$ is the minimum cardinality of a weak dominating set of G. We note that downhill and uphill domination are generalizations of these concepts, that is, a strong dominating set is also a downhill dominating set, and a weak dominating set is also an uphill dominating set. Hence, $\gamma_{dn}(G) \le \gamma_S(G)$ and $\gamma_{up}(G) \le \gamma_W(G)$.

In Section 2, we determine properties of downhill/uphill dominating sets. Among other results, we show that every minimal DDS (respectively, UDS) is a minimum DDS (respectively, UDS). As we have seen, the downhill and uphill domination are incomparable. In fact, we show in Section 3 that for any pair of positive integers d and u, there exists an infinite family of graphs G having $\gamma_{dn}(G) = d$ and $\gamma_{up}(G) = u$. In Section 4, we determine a Vizing-type result for the the downhill (respectively, uphill) domination number of Cartesian products. We then conclude with a list of open questions.

2 Properties of DDS and UDS

We begin this section with straightforward observations.

Observation 1. *For a graph G, a γ_{dn}-set contains at least one vertex with degree $\Delta(G)$, and a γ_{up}-set contains at least one vertex of degree $\delta(G)$.*

Observation 2. *If G is a connected graph of order at least 3 and S is a γ_{dn}-set of G, then* $\deg(v) \ge 2$ *for every $v \in S$.*

Observation 3. *If v is a vertex with $deg(v) \le 1$ in a graph G, then v is in every UDS of G.*

Observation 4. *For trees with l leaves, $\gamma_{up}(T) \ge l$.*

Our final observation gives examples.

Observation 5.

1. *For a connected r-regular graph G, $\gamma_{dn}(G) = \gamma_{up}(G) = 1$.*
2. *For a path P_n, $\gamma_{dn}(P_n) = 1$ and $\gamma_{up}(P_n) = 2$.*
3. *For a complete k-partite graph $G = K_{n_1,n_2,\ldots,n_k}$ where $n_i \le n_{i+1}$ for $1 \le i \le k-1$, $\gamma_{dn}(G) = n_1$ and $\gamma_{up}(G) = n_k$, if $n_1 \ne n_k$; otherwise, $\gamma_{dn}(G) = \gamma_{up}(G) = 1$.*

Unlike standard domination, where a graph can have different sizes of minimal dominating sets, we next show that any minimal DDS (respectively, UDS) of a graph G is a γ_{dn}-set (respectively, γ_{up}-set) of G.

Theorem 6. *Every minimal DDS of a graph G is a minimum DDS of G.*

Proof. Suppose to the contrary that there exists a minimal DDS, say D, of G, such that $|D| > \gamma_{dn}(G)$. Among all γ_{dn}-sets of G, select D' to be one that has the maximum number of vertices in common with D, that is, $|D' \cap D|$ is maximized.

Since $|D'| < |D|$, there exists a vertex $u \in (D \setminus D')$. Thus, u is downhill dominated by a vertex, say d', in D'. Then u and all the vertices downhill from u are downhill dominated by d'. If $d' \in D$, then $D \setminus \{u\}$ is a DDS with cardinality less than $|D|$, contradicting the minimality of D. Hence we may assume that $d' \notin D$.

Thus there exists a vertex $v \in D$ that downhill dominates d' and all of the vertices downhill from d'. Suppose $u \neq v$. Then v downhill dominates u and so, again, $D \setminus \{u\}$ is a DDS, contradicting the minimality of D. If $u = v$, then since v downhill dominates d' and d' downhill dominates u, it follows that $\deg(u) = \deg(d')$. Moreover, u downhill dominates d' and the vertices downhill dominated by d'. Thus, $D'' = (D' \setminus \{d'\}) \cup \{u\}$ is a γ_{dn}-set of G such that $|D'' \cap D| > |D' \cap D|$, contradicting our choice of D'. □

An analogous argument shows that any minimal UDS of a graph G is a γ_{up}-set of G.

Theorem 7. *Every minimal UDS of a graph G is a minimum UDS of G.*

Our next result shows that any minimal DDS (respectively, UDS) of a graph is an independent set.

Theorem 8. *Any minimal downhill (respectively, uphill) dominating set is an independent set.*

Proof. Assume S is a minimal DDS of G. If two vertices u and v of S are adjacent, then, without loss of generality, there exists a downhill path from u through v to all vertices which are downhill from v. Thus, $S \setminus \{v\}$ is a DDS of G, contradicting the minimality of S. Hence, S is an independent set. An analogous argument holds for a minimal UDS. □

The *independence number* of G, denoted $\beta_0(G)$, is the maximum number of vertices in an independent set of vertices of G.

Corollary 9. *For any graph G, $\gamma_{dn}(G) \leq \beta_0(G)$ and $\gamma_{up}(G) \leq \beta_0(G)$.*

To see the sharpness of the bounds of Corollary 9, consider the complete graph K_n, for which $\gamma_{dn}(K_n) = \beta_0(K_n) = 1$, and the complete bipartite graph $K_{r,s}$, $r \leq s$, for which $\gamma_{up}(K_{r,s}) = \beta_0(K_{r,s}) = s$. In fact, we next show that every possible pair of $\gamma_{dn}(G)$ (respectively, $\gamma_{up}(G)$) and $\beta_0(G)$ is realizable by an infinite number of graphs.

Theorem 10. *Given positive integers a and b such that $a \leq b$,*

1. *there exists an infinite family of graphs G with $\gamma_{dn}(G) = a$ and $\beta_0(G) = b$, and*
2. *there exists an infinite family of graphs G with $\gamma_{up}(G) = a$ and $\beta_0(G) = b$.*

Proof. Let G be the join $K_a + bK_c$ where $c \geq a$. (See graph G in Figure 3 for an example where $a = c = 2$ and $b = 4$). Then, $\gamma_{dn}(G) = a$ and $\beta_0(G) = b$.

Next we consider the uphill domination number. For $a = 1$, we let G be the lexicographic product of the cycle C_{2b} and the complete graph K_c, that is, $G = C_{2b}[K_c]$ is the graph obtained from a cycle C_{2b} by replacing each vertex with a complete graph K_c, where $c \geq 1$, and adding all edges between the vertices of two copies of K_c if and only if they correspond to adjacent vertices on the cycle. Then, $\gamma_{up}(G) = a = 1$ and $\beta_0(G) = b$. For $a > 1$, construct G from a cycle $C_{2b-a} = v_1, v_2, ..., v_{2b-a}, v_1$ as follows: For each v_i, $1 \leq i \leq a-1$, add a copy of K_c, where $c \geq 2$, by identifying one vertex of the K_c with v_i. (For example see the graph H in Figure 3, where $a = 3$, $b = 4$, and $c = 2$). Let $v_i' \neq v_i$ be a vertex in the copy of K_c containing v_i. Then $\{v_i', v_a \mid 1 \leq i \leq a-1\}$ is a γ_{up}-set of G. Also the set $\{v_i' \mid 1 \leq i \leq a-1\}$ unioned with $\lceil \frac{2b-2a+1}{2} \rceil$ independent vertices from the path $v_a, v_{a+1}, ..., v_{2b-a}$ is a maximum independent set. Thus, $\gamma_{up}(G) = a$ and $\beta_0(G) = b$. □

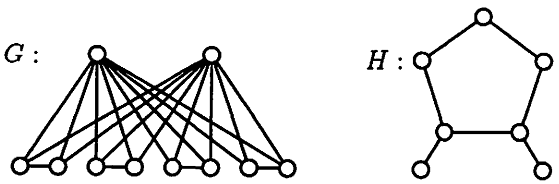

Figure 3: A graph $G = \overline{K}_2 + 4K_2$ with $\gamma_{dn}(G) = 2$ and $\beta_0(G) = 4$, and a graph H with $\gamma_{up}(H) = 3$ and $\beta_0(H) = 4$.

3 Realizability

As we noted in the introduction, although their definitions and the proofs of results involving them are analogous, the parameters $\gamma_{dn}(G)$ and $\gamma_{up}(G)$ are incomparable. In fact, next we show that for any pair of positive integers d and u, there exists an infinite family of graphs having $\gamma_{dn}(G) = d$ and $\gamma_{up}(G) = u$.

Theorem 11. *Given positive integers d and u, there exists infinitely many graphs G for which $\gamma_{dn}(G) = d$ and $\gamma_{up}(G) = u$.*

Proof. Let $x.K_r$ denote a complete graph K_r, for any $r \geq 5$, one edge of which has been subdivided and x denotes the subdivision vertex.

Let d and u be any two positive integers. If $d = u = 1$, then the graph $G = x.K_r$ has $\gamma_{dn}(G) = \gamma_{up}(G) = 1$. Hence, we may assume that $d+u \geq 3$. We construct G from a cycle C_{d+u} of order $d+u$ as follows. To any u of the vertices on this cycle attach a leaf. To the remaining d vertices on this cycle attach a copy of $x.K_r$ by adding an edge between the vertex on the cycle and the subdivision vertex x. See Figure 4 for an example, where $d = 4$ and $u = 3$.

Note that every vertex on the cycle C_{d+u} has degree 3 in G. Also, G has precisely u vertices of degree 1, and this independent set of vertices U forms a unique γ_{up}-set of G.

Let $D = \{v_1, v_2, \ldots, v_d\}$ denote any set of d vertices of degree $r-1$, each of which is selected from one the d copies of $x.K_r$ that are attached to d distinct vertices of the cycle. Then D is a γ_{dn}-set of G. □

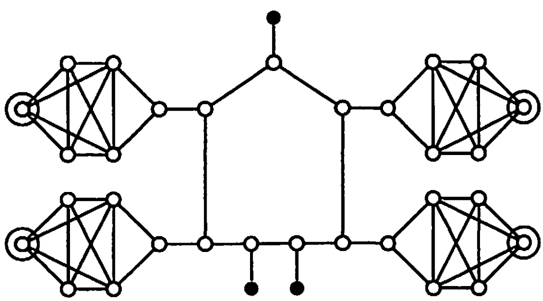

Figure 4: A graph G having $\gamma_{dn}(G) = 4$ and $\gamma_{up}(G) = 3$. The circled vertices form a γ_{up}-set and the darkened vertices form a γ_{dn}-set.

4 Cartesian Products

The well-known conjecture of Vizing on the domination number of Cartesian product graphs claims that for any two graphs G and H, $\gamma(G \square H) \geq \gamma(G)\gamma(H)$. We give a Vizing-type result for the downhill and uphill domination numbers of Cartesian products.

Theorem 12. *For any two graphs G and H, $\gamma_{dn}(G \square H) = \gamma_{dn}(G)\gamma_{dn}(H)$.*

Proof. We first show that $\gamma_{dn}(G\Box H) \leq \gamma_{dn}(G)\gamma_{dn}(H)$. Let S_1 and S_2 be a γ_{dn}-set of G and a γ_{dn}-set of H, respectively. Let $S = \{(u, v) \mid u \in S_1 \text{ and } v \in S_2\}$ be a set of vertices in $G\Box H$. In order to show that S is a DDS of $G\Box H$, it suffices to show that every element in $V(G\Box H)\backslash S$ is downhill from a vertex in S. Let (x, y) be an arbitrary vertex of $V(G\Box H)\backslash S$. By symmetry, we consider two cases:

Case 1: *$x \in S_1$ and $y \notin S_2$.* Then x is in a γ_{dn}-set S_1 of G and y is on a downhill path $P = (v = v_1, v_2, ..., v_k = y)$ originating from a vertex $v \in S_2$ in H. Thus, $(x, v_1), (x, v_2), \ldots, (x, v_k)$ is a downhill path from (x, v) to (x, y) in $G\Box H$.

Case 2: *$x \notin S_1$ and $y \notin S_2$.* Thus x is downhill from some vertex, say $u \in S_1$, and y is downhill from some vertex, say $v \in S_2$. As before, we deduce there is a (u, v)-(x, v) downhill path and a (x, v)-(x, y) downhill path in $G\Box H$. Thus, (x, y) is downhill from (u, v) in $G\Box H$, implying that S is a DDS of $G\Box H$. Hence, $\gamma_{dn}(G\Box H) \leq |S| = |S_1||S_2|$.

Since S is a DDS of G, Theorem 7 implies that to complete the proof, it suffices to show that S is a minimal DDS of G. Assume to the contrary that S is not minimal. Then $S \setminus \{(x, y)\}$ is a DDS for $G\Box H$, that is, (x, y) is downhill from some vertex, say (u, v), in $S \setminus \{(x, y)\}$. Let $P : (u, v) = (u_1, v_1), (u_2, v_2), \ldots, (u_k, v_k) = (x, y)$ be a (u, v)-(x, y) downhill path. Since for any (u_i, v_i) on P, either $u_{i+1} = u_i$ and $\deg_H(v_{i+1}) \leq \deg_H(v_i)$, or $v_{i+1} = v_i$ and $\deg_G(u_{i+1}) \leq \deg_G(u_i)$, it follows that x is downhill from u in G and y is downhill from v in H. Recall that $u, x \in S_1$. But then $S_1 \setminus \{x\}$ is a DDS of G having cardinality less than $\gamma_{dn}(G)$, a contradiction. Thus, we conclude that S is a minimal DDS of $G\Box H$, and so by Theorem 7, S is γ_{dn}-set of $G\Box H$. Hence, $\gamma_{dn}(G\Box H) = |S| = |S_1||S_2| = \gamma_{dn}(G)\gamma_{dn}(H)$. ❑

Once again, an analogous argument gives the result for uphill domination.

Theorem 13. *For any two graphs G and H, $\gamma_{up}(G\Box H) = \gamma_{up}(G)\gamma_{up}(H)$.*

5 Open Problems

The concept of downhill and uphill paths may suggest many different avenues for future research. We conclude this paper by listing a few open problems.

1. Characterize the graphs G for which $\gamma_{dn}(G) = 1$ and those for which $\gamma_{up}(G) = 1$.

2. Investigate the downhill/uphill domination numbers of self-complementary graphs.

3. Investigate downhill/uphill domination for other types of graph products and operations.

4. Characterize the non-regular connected graphs G for which $\gamma_{dn}(G) = \gamma_{up}(G)$.

5. Characterize the graphs for which $\gamma(G) = \gamma_{dn}(G)$ and those for which $\gamma(G) = \gamma_{up}(G)$.

6. Determine bounds on the downhill/uphill domination numbers.

References

[1] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater. *Fundamentals of Domination in Graphs.* Marcel Dekker, New York, 1998.

[2] E. Sampathkumar and L. Pushpa Latha. Strong weak domination and domination balance in a graph. *Discrete Math.* 161:235-242 (1996).