A New Construction for Decompositions of λK_n into LE Graphs

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ABSTRACT. In this paper, we revisit LE graphs, find the minimum λ for decomposition of λK_n into these graphs, and show that for all viable values of λ, the necessary conditions are sufficient for LE–decompositions using cyclic decompositions from base graphs.

1. Introduction

Decompositions of graphs into subgraphs is a well–known classical problem; for an excellent survey on graph decompositions, see [**1**]. Recently, several people including Chan [**2**], El–Zanati, Lapchinda, Tangsupphathawat and Wannasit [**3**], Hein [**4**], Hurd [**8**], Sarvate [**5**, **6**, **7**], Winter [**9**, **10**] and Zhang [**11**] have worked on decomposing λK_n into multigraphs. In fact, Sarvate, Winter and Zhang [**9**] found decompositions of λK_n into so–called LE graphs. A new construction for such decompositions is given in the sequel.

2. Preliminaries

For simplicity of notation, we adopt the "alphabetic labeling" used in [**4**, **5**, **6**, **7**, **9**, **10**, **11**]:

DEFINITION 1. *An* LE graph $[a, b, c]$ *on* $V = \{a, b, c\}$ *is a graph with 4 edges where the frequencies of edges* $\{a, b\}$ *and* $\{b, c\}$ *are 1 and 3 (respectively).*

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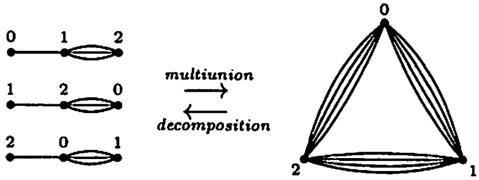
DEFINITION 2. *For positive integers $n \geq 3$ and $\lambda \geq 3$, an* LE–decomposition *of λK_n (denoted $LE(n, \lambda)$) is a collection of LE graphs such that the multiunion of their edge sets contains λ copies of all edges in a K_n.*

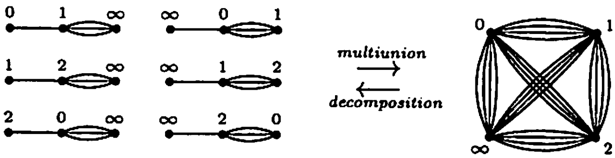
One of the powerful techniques to construct combinatorial designs is based on *difference sets* and *difference families*; see Stinson [**12**] for details. This technique is modified to achieve our decompositions of λK_n — in general, we exhibit the *base graphs*, which can be developed to obtain the decomposition.

EXAMPLE 1. *Considering the set of points to be $V = \mathbb{Z}_3$, the LE base graph* $[0, 1, 2]$ *(when developed modulo 3) constitutes an* $LE(3, 4)$.



We note that special attention is needed with the base graphs containing the "dummy element" ∞; the non–∞ elements are developed, while ∞ is simply rewritten each time.

EXAMPLE 2. *Considering the set of points to be $V = \mathbb{Z}_3 \cup \{\infty\}$, the LE base graphs* $[0, 1, \infty]$ *and* $[\infty, 0, 1]$ *(when developed modulo 3) constitute an* $LE(4, 4)$.



3. LE–Decompositions

We first address the minimum values of λ in an LE(n, λ). Recall that $\lambda \geq 3$.

THEOREM 3.1. *Let $n \geq 3$. The minimum values of λ for which an $LE(n, \lambda)$ exists are $\lambda = 3$ when $n \equiv 0, 1 \pmod 8$ and $\lambda = 4$ when $n \not\equiv 0, 1 \pmod 8$.*

PROOF. Since there are $\frac{\lambda n(n-1)}{2}$ edges in a λK_n, and 4 edges in an LE graph, we must have that $\lambda n(n-1) \equiv 0 \pmod 8$ (where $\lambda \geq 3$

and $n \geq 3$) for LE–decompositions. The result follows from cases on $n \pmod 8$. ■

We are now in a position to prove the main results of the paper. We first remark that an LE graph has 3 vertices; that is, we consider $n \geq 3$. Also, necessarily $\lambda \geq 3$. We note that we use difference sets to achieve our decompositions of λK_n. In general, we exhibit the base graphs, which can be developed (modulo either n or $n-1$) to obtain the decomposition. We also note that the frequency of the edges is fixed by position, as per the LE graph.

THEOREM 3.2. *The minimum number copies of K_n (as given in Theorem 3.1) can be decomposed into graphs of the LE–type.*

PROOF. Let $n \geq 3$. We proceed by cases on $n \pmod 8$.

If $n = 8t$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{8t-1} \cup \{\infty\}$. The number of graphs required for LE$(8t, 3)$ is $\frac{3(8t)(8t-1)}{8} = 3t(8t-1)$. Thus, we need $3t$ base graphs (modulo $8t-1$). Then, the differences we must achieve (modulo $8t-1$) are $1, 2, \ldots, 4t-1$. For the first three base graphs, use $[1, 0, \infty]$, $[0, 1, 4t]$ and $[0, 1, 4t-1]$. We also use the $3t-3$ base graphs $[0, 2, 4t-1]$, $[0, 2, 4t-2]$, $[0, 2, 4t-3]$, $[0, 3, 4t-3]$, $[0, 3, 4t-4]$, $[0, 3, 4t-5], \ldots, [0, t, 2t+3]$, $[0, t, 2t+2]$ and $[0, t, 2t+1]$ if necessary. Hence, LE$(8t, 3)$ exists.

If $n = 8t+1$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{8t+1}$. The number of graphs required for LE$(8t+1, 3)$ is $\frac{3(8t+1)(8t)}{8} = 3t(8t+1)$. Thus, we need $3t$ base graphs (modulo $8t+1$). Then, the differences we must achieve (modulo $8t+1$) are $1, 2, \ldots, 4t$. We use the base graphs $[0, 1, 4t+1]$, $[0, 1, 4t]$, $[0, 1, 4t-1]$, $[0, 2, 4t-1]$, $[0, 2, 4t-2]$, $[0, 2, 4t-3], \ldots, [0, t, 2t+3]$, $[0, t, 2t+2]$ and $[0, t, 2t+1]$. Hence, LE$(8t+1, 3)$ exists.

If $n = 8t+2$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{8t+1} \cup \{\infty\}$. The number of graphs required for LE$(8t+2, 4)$ is $\frac{4(8t+2)(8t+1)}{8} = (4t+1)(8t+1)$. Thus, we need $4t+1$ base graphs (modulo $8t+1$). Then, the differences we must achieve (modulo $8t+1$) are $1, 2, \ldots, 4t$. For the first five base graphs, use $[1, 0, \infty]$, $[\infty, 0, 1]$, $[0, 2, 4]$, $[0, 3, 6]$ and $[0, 4, 8]$. We also use the $4t-4$ base graphs $[0, 5, 10], \ldots, [0, 4t-3, 8t-6]$, $[0, 4t-2, 8t-4]$, $[0, 4t-1, 8t-2]$ and $[0, 4t, 8t]$ if necessary. Hence, LE$(8t+2, 4)$ exists.

If $n = 8t+3$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{8t+3}$. The number of graphs required for LE$(8t+3, 4)$ is $\frac{4(8t+3)(8t+2)}{8} = (4t+1)(8t+3)$. Thus, we need $4t+1$ base graphs (modulo $8t+3$). Then, the differences we must achieve (modulo $8t+3$) are $1, 2, \ldots, 4t+1$. We use the base graphs $[0,1,2]$, $[0,2,4], \ldots, [0,4t+1,8t+2]$. Hence, LE$(8t+3,4)$ exists.

If $n = 8t+4$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{8t+3} \cup \{\infty\}$. The number of graphs required for LE$(8t+4,4)$ is $\frac{4(8t+4)(8t+3)}{8} = (4t+2)(8t+3)$. Thus, we need $4t+2$ base graphs (modulo $8t+3$). Then, the differences we must achieve (modulo $8t+3$) are $1, 2, \ldots, 4t+1$. For the first two base graphs, we use $[1,0,\infty]$ and $[\infty,0,1]$. We also use the $4t$ base graphs $[0,2,4]$, $[0,3,6], \ldots, [0,4t+1,8t+2]$ if necessary. Hence, LE$(8t+4,4)$ exists.

If $n = 8t+5$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{8t+5}$. The number of graphs required for LE$(8t+5,4)$ is $\frac{4(8t+5)(8t+4)}{8} = (4t+2)(8t+5)$. Thus, we need $4t+2$ base graphs (modulo $8t+5$). Then, the differences we must achieve (modulo $8t+5$) are $1, 2, \ldots, 4t+2$. We use the base graphs $[0,1,2]$, $[0,2,4], \ldots, [0,4t+2,8t+4]$. Hence, LE$(8t+5,4)$ exists.

If $n = 8t+6$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{8t+5} \cup \{\infty\}$. The number of graphs required for LE$(8t+6,4)$ is $\frac{4(8t+6)(8t+5)}{8} = (4t+3)(8t+5)$. Thus, we need $4t+3$ base graphs (modulo $8t+5$). Then, the differences we must achieve (modulo $8t+5$) are $1, 2, \ldots, 4t+2$. For the first three base graphs, we use $[1,0,\infty]$, $[\infty,0,1]$ and $[0,2,4]$. We also use the $4t$ base graphs $[0,3,6]$, $[0,4,8], \ldots, [0,4t+2,8t+4]$ if necessary. Hence, LE$(8t+6,4)$ exists.

If $n = 8t+7$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{8t+7}$. The number of graphs required for LE$(8t+7,4)$ is $\frac{4(8t+7)(8t+6)}{8} = (4t+3)(8t+7)$. Thus, we need $4t+3$ base graphs (modulo $8t+7$). Then, the differences we must achieve (modulo $8t+7$) are $1, 2, \ldots, 4t+3$. We use the base graphs $[0,1,2]$, $[0,2,4], \ldots, [0,4t+3,8t+6]$ if necessary. Hence, LE$(8t+7,4)$ exists. ∎

We now address the sufficiency of existence of LE(n, λ).

THEOREM 3.3. *Let $n \geq 3$ and $\lambda \geq 3$. For LE(n,λ), the necessary conditions for n are that $n \equiv 0, 1 \pmod 8$ when $\lambda \equiv 1, 3 \pmod 4$ and $n \equiv 0, 1 \pmod 4$ when $\lambda \equiv 2 \pmod 4$. There is no condition for n when $\lambda \equiv 0 \pmod 4$.*

PROOF. Similar to the proof of Theorem 3.1, but by cases on $\lambda \pmod 8$. ∎

LEMMA 3.1. *There exists an LE$(n,3)$ for the necessary $n \geq 3$.*

PROOF. From Theorem 3.3, the necessary condition is $n \equiv 0, 1 \pmod 8$. In these cases, LE$(n, 3)$ exists from Theorem 3.2. ∎

LEMMA 3.2. *There exists an LE$(n, 4)$ for any $n \geq 3$.*

PROOF. From Theorem 3.3, there is no condition for n. We consider cases when $n \geq 3$ is odd or even.

If $n = 2t + 1$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{2t+1}$. The number of graphs required for LE$(2t + 1, 4)$ is $\frac{4(2t+1)(2t)}{8} = t(2t + 1)$. Thus, we need t base graphs (modulo $2t + 1$). The differences we must achieve (modulo $2t + 1$) are $1, 2, \ldots, t$. We use the base graphs $[0, 1, 2]$, $[0, 2, 4], \ldots, [0, t, 2t]$. Hence, LE$(2t + 1, 4)$ exists.

If $n = 2t$ (for $t \geq 2$), we consider the set V as $\mathbb{Z}_{2t-1} \cup \{\infty\}$. The number of graphs required for LE$(2t, 4)$ is $\frac{4(2t)(2t-1)}{8} = t(2t - 1)$. Thus, we need t base graphs (modulo $2t - 1$). The differences we must achieve (modulo $2t - 1$) are $1, 2, \ldots, t - 1$. For the first two base graphs, we use $[1, 0, \infty]$ and $[\infty, 0, 1]$. We also use the $t - 2$ base graphs $[0, 2, 4]$, $[0, 3, 6], \ldots, [0, t - 1, 2t - 2]$ if necessary. Hence, LE$(2t, 4)$ exists. ∎

LEMMA 3.3. *There does not exist an LE$(n, 5)$.*

PROOF. The only edge frequencies in an LE graph are 1 and 3. The only way to write $\lambda = 5$ as a sum of 1s and 3s (that includes a '3') is as $5 = 3 + 1 + 1$. In an LE$(n, 5)$, the number of times each edge needs to occur triply is half the number of times it should occur singly. However, as there are equal numbers of single edges and triple edges in an LE graph, such a decomposition is not possible. ∎

LEMMA 3.4. *There exists an LE$(n, 6)$ for necessary $n \geq 3$.*

PROOF. From Theorem 3.3, the necessary condition is $n \equiv 0, 1 \pmod 4$.

If $n = 4t$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{4t-1} \cup \{\infty\}$. The number of graphs required for LE$(4t, 6)$ is $\frac{6(4t)(4t-1)}{8} = 3t(4t - 1)$. Thus, we need $3t$ base graphs (modulo $4t - 1$). The differences we must achieve (modulo $4t - 1$) are $1, 2, \ldots, 2t - 1$. For the first three base graphs, use $[0, 1, 2]$ and $[1, 0, \infty]$ twice. For the last $3t - 3$ base graphs, use $[0, 2, 4]$, $[0, 2, 2t + 1]$ twice, $[0, 3, 6]$, $[0, 3, 2t + 1]$ twice, $\ldots, [0, t, 2t]$ and $[0, t, 2t + 1]$ twice, if necessary. Hence, LE$(4t, 6)$ exists.

If $n = 4t + 1$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{4t+1}$. The number of graphs required for LE$(4t + 1, 6)$ is $\frac{6(4t+1)(4t)}{8} = 3t(4t + 1)$.

Thus, we need $3t$ base graphs (modulo $4t + 1$). The differences we must achieve (modulo $4t+1$) are $1, 2, \ldots, 2t$. We use the base graphs $[0, 1, 2]$, $[0, 1, 2t+1]$ twice, $[0, 2, 4]$, $[0, 2, 2t+1]$ twice, $\ldots$, $[0, t, 2t]$ and $[0, t, 2t + 1]$ twice. Hence, LE$(4t + 1, 6)$ exists. ∎

LEMMA 3.5. *There exists an LE$(n, 9)$ for necessary $n \geq 3$.*

PROOF. From Theorem 3.3, the necessary condition is $n \equiv 0, 1 \pmod 8$.

If $n = 8t$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{8t-1} \cup \{\infty\}$. The number of graphs required for LE$(8t, 9)$ is $\frac{9(8t)(8t-1)}{8} = 9t(8t - 1)$. Thus, we need $9t$ base graphs (modulo $8t - 1$). The differences we must achieve (modulo $8t-1$) are $1, 2, \ldots, 4t-1$. For the first nine base graphs, use $[0, 1, 2]$ twice, $[1, 0, \infty]$, $[0, 2, 4]$ twice, $[2, 0, \infty]$, $[0, 3, 6]$ twice and $[3, 0, \infty]$. For the last $9t-9$ base graphs, use $[0, 4, 8]$ twice, $[0, 4, 4t + 3]$, $[0, 5, 10]$ twice, $[0, 5, 4t + 4]$, $[0, 6, 12]$ twice, $[0, 6, 4t + 5], \ldots, [0, 3t-2, 6t-4]$ twice, $[0, 3t-2, 6t-1]$, $[0, 3t-1, 6t-2]$ twice, $[0, 3t - 1, 6t]$, $[0, 3t, 6t]$ twice and $[0, 3t, 6t + 1]$ if necessary. Hence, LE$(8t, 9)$ exists.

If $n = 8t + 1$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{8t+1}$. The number of graphs required for LE$(8t+1, 9)$ is $\frac{9(8t+1)(8t)}{8} = 9t(8t+1)$. Thus, we need $9t$ base graphs (modulo $8t + 1$). The differences we must achieve (modulo $8t+1$) are $1, 2, \ldots, 4t$. We use the base graphs $[0, 1, 2]$ twice, $[0, 1, 4t + 1]$, $[0, 2, 4]$ twice, $[0, 2, 4t + 2]$, $[0, 3, 6]$ twice, $[0, 3, 4t + 3], \ldots, [0, 3t - 2, 6t - 4]$ twice, $[0, 3t - 2, 6t - 1]$, $[0, 3t - 1, 6t-2]$ twice, $[0, 3t-1, 6t]$, $[0, 3t, 6t]$ twice and $[0, 3t, 6t+1]$. Hence, LE$(8t + 1, 9)$ exists. ∎

THEOREM 3.4. *An LE(n, λ) exists for all $\lambda \geq 3$ (except $\lambda = 5$, according to Lemma 3.3) and necessary $n \geq 3$.*

PROOF. We proceed by cases on $\lambda \pmod 4$.

For $\lambda \equiv 0 \pmod 4$ (so that $\lambda = 4t$ for $t \geq 1$), by taking t copies of an LE$(n, 4)$ (given in Lemma 3.2), we have an LE$(n, 4t)$.

For $\lambda \equiv 1 \pmod 4$ (so that $\lambda = 4t + 1 = 4(t - 2) + 9$ for $t \geq 2$), we first take an LE$(n, 9)$ (given in Lemma 3.5). (This gives us $\lambda = 9$ thus far.) We then adjoin this to $t - 2$ copies of an LE$(n, 4)$ (given in Lemma 3.2) if necessary. Hence, we have an LE$(n, 4t + 1)$.

For $\lambda \equiv 2 \pmod 4$ (so that $\lambda = 4t + 2 = 4(t - 1) + 6$ for $t \geq 1$), we first take an LE$(n, 6)$ (given in Lemma 3.4). (This gives us $\lambda = 6$ thus far.) We then adjoin this to $t - 1$ copies of an LE$(n, 4)$ (given in Lemma 3.2) if necessary. Hence, we have an LE$(n, 4t + 2)$.

For $\lambda \equiv 3 \pmod 4$ (so that $\lambda = 4t + 3$ for $t \geq 0$), we first take an LE$(n, 3)$ (given in Lemma 3.1). (This gives us $\lambda = 3$ thus far.) We then adjoin this to t copies of an LE$(n, 4)$ (given in Lemma 3.2) if necessary. Hence, we have an LE$(n, 4t + 3)$. ■

4. Conclusion

We have revisited LE graphs, found the minimum λ for decomposition of λK_n into these graphs, and showed that for all viable values of λ, the necessary conditions are sufficient for LE–decompositions.

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