On strict semibound graphs of posets

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Abstract

For a poset $P = (V(P), \leq_P)$, the *strict semibound graph* of P is the graph $\text{ssb}(P)$ on $V(\text{ssb}(P)) = V(P)$ for which vertices u and v of $\text{ssb}(P)$ are adjacent if and only if $u \neq v$ and there exists an element $x \in V(P)$ distinct from u and v such that $x \leq_P u, v$ or $u, v \leq_P x$. We prove that a poset P is connected if and only if the induced subgraph $\langle \max(P) \rangle_{\text{ssb}(P)}$ is connected. We also characterize posets whose strict semibound graphs are triangle-free.

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1 Introduction

In this paper we consider finite undirected simple graphs and finite posets. Throughout this paper, we use the graphs and posets terminology and notation introduced in [23] and [17].

In particular, given a graph G and a subset $S \subseteq V(G)$ of the set $V(G)$ of vertices of G, we denote by $\langle S \rangle_G$ the induced subgraph of G on S. Given a graph G and a vertex $v \in V(G)$, we set $N_G(v) = \{u : uv \in E(G)\}$.

Given a finite poset $P = (V(P), \leq_P)$, $\leq_P$ is a partial order relation on $V(P)$, and we denote by P^d the dual poset of P. For any $x \in V(P)$, we set $U_P(x) = \{y \in V(P) : x \leq_P y\}$, $U_P^-(x) = U_P(x) - \{x\}$, $L_P(x) = \{y \in V(P) : y \leq_P x\}$ and $L_P^-(x) = L_P(x) - \{x\}$. We denote by $\max(P)$ (resp. by $\min(P)$) the set of all maximal (resp. minimal) elements of P, and we set $\text{mid}(P) = V(P) - (\max(P) \cup \min(P))$.

In the paper we use the following definition.

Definition 1.1 *Assume that $P = (V(P), \leq_P)$ is a finite poset.*

(a) *The comparability graph of P is the graph* $\mathrm{com}(P)$ *on* $V(\mathrm{com}(P)) = V(P)$ *for which distinct vertices u and v of* $\mathrm{com}(P)$ *are adjacent if and only if $u \leq_P v$ or $v \leq_P u$.*

(b) *The strict semibound graph of P (ssb-graph, for short) is the graph* $\mathrm{ssb}(P)$, *with the set of vertices* $V(\mathrm{ssb}(P)) = V(P)$ *for which vertices u and v of* $\mathrm{ssb}(P)$ *are adjacent if and only if $u \neq v$ and there exists an element $x \in V(P)$ distinct from u and v such that $x \leq_P u, v$ or $u, v \leq_P x$. We say that a graph G is a strict semibound graph if there exists a poset whose strict semibound graph is isomorphic to G.*

In general we consider a food web as an acyclic digraph F whose vertices are species with an arc $u \to v$ whenever species u feeds on species v. Cohen [3] introduced a competition graph G on a food web F such that the vertices of G are the vertices of F and there is an edge from the species u to the species v if and only if for some species w, there are arcs $u \to w$ and $v \to w$. Cohen studied food webs in terms of competition graphs. Dutton, Brigham [6], Lundgren, Maybee [12] and Roberts [14] researched competition graphs as pure mathematical studies. Scott [16], Lundgren [11] and Cable, Jones, Lundgren, Seager [2] studied competition-common enemy graphs and niche graphs in the point of view of mathematics, whose graphs were also introduced for research on food webs.

Posets are reflexive, antisymmetric and transitive digraphs. McMorris, Zaslavsky [13] and Bergstrand, Jones [1] dealt with upper bound graphs and double bound graphs on posets, which correspond to competition graphs and competition-common enemy graphs, respectively. McMorris, Zaslavsky [13] and Diny [4] introduced the concept of strict-upper-bound graphs and strict-double-bound graphs, which correspond to competition graphs and competition-common enemy graphs on acyclic digraphs, respectively. Strict semibound graphs correspond to niche graphs on acyclic digraphs. Era, Ogawa, Tsuchiya [7], [8] dealt with semibound graphs, which correspond to niche graphs.

In this paper we consider properties of strict semibound graphs. In the present article, we examine the structure of strict semibound graphs as a way to study partially ordered sets. In Section 2, we characterize a poset P such that $\mathrm{ssb}(P)$ is a connected graph.

Except of the tools and techniques mentioned above, finite posets P are usually studied in the literature in terms of the *Hasse graph* $H(P)$ and the *Hasse quiver* $\mathcal{H}(P)$ associated to P. This type of technique is successfully used in the study of matrix representations and K-linear representations of finite posets P, see [5], [9], [18], [19] for details. Recently, the geometry and the structure of finite posets P are successfully studied in terms of edge-bipartite graphs Δ_P and the Coxeter (complex) spectrum $\mathbf{specc}_P \subset S^1$ associated with P using the Coxeter spectral analysis technique, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle, see [10], [15], [20], [21], and [22] for more details and applications.

2 Connected posets

In this section we use the following definition.

Definition 2.1

(a) Given a finite poset $P = (V(P), \leq_P)$ and elements $a, b \in V(P)$, a sequence of elements $a = c_1, c_2, ..., c_l = b$ is called an a-b comparable path if c_i is comparable with c_{i+1} for $i = 1, 2, ..., l-1$.

(b) Given a finite poset $P = (V(P), \leq_P)$ and $a, b \in V(P)$, a and b are connected if there exists an a-b comparable path.

(c) A finite poset $P = (V(P), \leq_P)$is connected if each pair of elements $a, b \in V(P)$ is connected.

Note that a finite poset P is connected if and only if the comparability graph $\mathrm{com}(P)$ of P is connected. If a finite poset P is disconnected, then $\mathrm{ssb}(P)$ is disconnected.

For strict semibound graphs of connected posets, we obtain the following results.

Theorem 2.2 *Let $P = (V(P), \leq_P)$ be a finite poset and $\mathrm{ssb}(P)$ is the strict semibound graph of P. Then the following are equivalent.*

1. *A poset P is connected, that is, the comparability graph $\mathrm{com}(P)$ is connected.*

2. *The induced subgraph $\langle \max(P) \rangle_{\mathrm{ssb}(P)}$ is a connected graph.*

3. *The induced subgraph $\langle \min(P) \rangle_{\mathrm{ssb}(P)}$ is a connected graph.*

Proof. First we show that the condition 1 implies the condition 2. For each pair $\alpha, \alpha' \in \max(P)$, there exists an α-α' comparable path $\alpha = c_1, c_2, ..., c_l = \alpha'$, because P is connected. For an α-α' comparable path $\alpha = c_1, c_2, ..., c_l = \alpha'$, we obtain a minimal α-α' comparable path $\alpha = c_1 = w_1, w_2, ..., w_m = c_l = \alpha'$ such that $w_1 = c_1 = \alpha$ and $w_{i+1} = c_{k_i}$, where $k_i = \max\{j : c_j$ is comparable with $w_i\}$. Then $w_m = c_l = \alpha'$ and w_i $(i = 1, 2, ..., m)$ is not comparable with w_k for $i + 1 < k$. Since $w_1 = \alpha \in \max(P)$, $w_{2r} \leq_P w_{2r-1}, w_{2r+1}$ for each $2r$. Since $w_m = \alpha' \in \max(P)$, upper bounds of w_m do not exist other than w_m. Thus m is not even.

For each w_i, there exists a maximal element α_i such that $w_i \leq_P \alpha_i$. If $w_i \in \max(P)$, $\alpha_i = w_i$. Then α_{2r-1} and α_{2r+1} have a common lower bound w_{2r}, because $w_{2r} \leq_P w_{2r-1} \leq_P \alpha_{2r-1}$ and $w_{2r} \leq_P w_{2r+1} \leq_P \alpha_{2r+1}$. Thus α_{2r-1} and α_{2r+1} are adjacent in ssb(P). Since $\{\alpha = w_1 = \alpha_1, \alpha_3, \alpha_5, ..., \alpha_m = w_m = \alpha'\} \subseteq \max(P)$ and α_{2r-1} and α_{2r+1} are adjacent in ssb(P), a sequence $\alpha = \alpha_1, \alpha_3, \alpha_5, ..., \alpha_m = \alpha'$ is an α-α' walk in $\langle \max(P) \rangle_{\mathrm{ssb}(P)}$. So α and α' are connected in $\langle \max(P) \rangle_{\mathrm{ssb}(P)}$ and $\langle \max(P) \rangle_{\mathrm{ssb}(P)}$ is a connected graph. Similarly if a poset P is connected, then $\langle \min(P) \rangle_{\mathrm{ssb}(P)}$ is connected. Therefore the condition 1 implies the conditions 2 and 3.

Next we show that the condition 2 implies the condition 1. We assume that the induced subgraph $\langle \max(P) \rangle_{\mathrm{ssb}(P)}$ is connected. For $u, v \in V(P)$, we show u and v are connected in P. We consider the following cases.

Case (a): $u, v \in \max(P)$.

Since $\langle \max(P) \rangle_{\mathrm{ssb}(P)}$ is connected, there exists a path $u = \alpha_1, \alpha_2, ..., \alpha_l = v$ of $\langle \max(P) \rangle_{\mathrm{ssb}(P)}$. Thus there exist $y_i \in V(P) - \max(P)$ such that $y_i \leq_P \alpha_i, \alpha_{i+1}$ for $i = 1, 2, ..., l-1$. So a sequence $u = \alpha_1, y_1, \alpha_2, y_2, ..., \alpha_{l-1}, y_{l-1}, \alpha_l = v$ is a u-v comparable path of P.

Case (b): $u \in \max(P)$ and $v \in V(P) - \max(P)$.

Then there exists an element $\alpha_l \in \max(P)$ such that $v \leq_P \alpha_l$. As is the case with Case (a), there exists a u-α_l comparable path $u = \alpha_1$, y_1, α_2, y_2 , ..., α_{l-1}, y_{l-1}, α_l. Thus there exists a u-v comparable path $u = \alpha_1$, y_1, α_2, y_2 , ..., α_{l-1}, y_{l-1}, α_l, v.

Case (c): $u \in V(P) - \max(P)$ and $v \in \max(P)$.

Similar to Case (b).

Case (d): $u, v \in V(P) - \max(P)$.

Then there exist elements $\alpha_1, \alpha_l \in \max(P)$ such that $u \leq_P \alpha_1$ and $v \leq_P \alpha_l$ and there exists an α_1-α_l comparable path $\alpha_1, y_1, \alpha_2, y_2, ..., \alpha_{l-1}, y_{l-1}, \alpha_l$. Thus there exists a u-v comparable path $u, \alpha_1, y_1, \alpha_2, y_2, ..., \alpha_{l-1}, y_{l-1}, \alpha_l, v$.

Therefore for each pair $u, v \in V(P)$, u and v are connected and a poset P is connected.

Similarly, the condition 3 implies the condition 1. □

Corollary 2.3 *Let $P = (V(P), \leq_P)$ be a finite poset. Then* $\text{ssb}(P)$ *is a connected graph if and only if P is connected and* $\text{mid}(P) \neq \emptyset$.

Proof. If $\text{mid}(P) \neq \emptyset$, then for all $x \in \text{mid}(P)$, there exist $\alpha \in \max(P)$ and $\beta \in \min(P)$ such that $\beta \leq_P x \leq_P \alpha$. So x is adjacent to a maximal element α and a minimal element β in $\text{ssb}(P)$, because $\beta \leq_P x, \alpha$ and $x, \beta \leq_P \alpha$.

By Theorem 2.2, for a connected poset P, any maximal elements are connected and any minimal elements are connected in $\text{ssb}(P)$. Therefore any elements of $\text{ssb}(P)$ are connected and $\text{ssb}(P)$ is a connected graph.

If a poset P is disconnected, then $\text{ssb}(P)$ is disconnected. If $\text{mid}(P) = \emptyset$, then $V(P) = \max(P) \cup \min(P)$. For $\alpha \in \max(P)$ and $\beta \in \min(P)$, α and β are not adjacent in $\text{ssb}(P)$, because $U_P^-(\alpha) = \emptyset$ and $L_P^-(\beta) = \emptyset$. So $\text{ssb}(P)$ is disconnected. □

A *clique* of a graph G is the vertex set of a maximal complete subgraph of G. Let $\mathcal{Q} = \{Q_1, Q_2, ..., Q_l\}$ be the family of all cliques of G noting that for each $uv \in E(G)$, there exists $Q_i \in \mathcal{Q}$ such that $u, v \in Q_i$. The *clique graph* of G is the graph $\text{CL}(G)$ on $V(\text{CL}(G)) = \mathcal{Q}$ for which vertices Q_i and Q_j are adjacent if and only if $Q_i \cap Q_j \neq \emptyset$. We obtain the following result.

Theorem 2.4 *For a graph G with no isolated vertices, $G \cup \text{CL}(G)$ is a strict semibound graph.*

Proof. Let $\mathcal{Q} = \{Q_1, Q_2, ..., Q_l\}$ be the family of all cliques of G. We make a poset P such that $V(P) = V(G) \cup \{\alpha_1, \alpha_2, ..., \alpha_l\}$, $x \leq_P \alpha_i$ for $x \in Q_i$ and $w \leq_P w$ for all $w \in V(P)$. Then $\max(P) = \{\alpha_1, \alpha_2, ..., \alpha_l\}$, $\min(P) = V(G)$, $\text{mid}(P) = \emptyset$ and $\text{ssb}(P)$ is $\langle \max(P) \rangle_{\text{ssb}(P)} \cup \langle \min(P) \rangle_{\text{ssb}(P)}$.

For $x, y \in V(G)$, x is adjacent to y in G if and only if x is adjacent to y in $\text{ssb}(P)$, because there exists a clique $Q_i \in \mathcal{Q}$ such that $x, y \in Q_i$ if and only if there exists an element $\alpha_i \in \max(P)$ such that $x, y \leq_P \alpha_i$. So the graph G is isomorphic to the induced subgraph $\langle \min(P) \rangle_{\text{ssb}(P)}$ of $\text{ssb}(P)$.

For vertices $\alpha_i, \alpha_j \in \{\alpha_1, \alpha_2, ..., \alpha_l\}$ and corresponding cliques $Q_i, Q_j \in \mathcal{Q}$, α_i is adjacent to α_j in $\text{ssb}(P)$ if and only if $Q_i \cap Q_j \neq \emptyset$, because there exists a minimal element x such that $x \leq_P \alpha_i, \alpha_j$ if and only if $x \in Q_i \cap Q_j$. So the induced subgraph $\langle \max(P) \rangle_{\text{ssb}(P)}$ of $\text{ssb}(P)$ is isomorphic to $\text{CL}(G)$.

Therefore $\text{ssb}(P)$ is isomorphic to $G \cup \text{CL}(G)$, and $G \cup \text{CL}(G)$ is a strict semibound graph. □

3 Triangle-free strict semibound graphs

Next we consider triangle-free strict semibound graphs. In this section we use the following definition.

Definition 3.1

(a) Let T_n be a total ordered set with $n \geq 1$ vertices such that $V(T_n) = \{x_i : 1 \leq i \leq n\}$ and $x_i \leq_{T_n} x_j$ for $1 \leq i \leq j \leq n$.

(b) Let $Z_{l,m}$ $(1 \leq l \leq m \leq l+1)$ be a poset such that $V(Z_{l,m}) = \{\beta_i : 1 \leq i \leq l\} \cup \{\alpha_j : 1 \leq j \leq m\}$,
$\beta_i \leq_{Z_{l,m}} \alpha_i, \alpha_{i+1}$ for $1 \leq i \leq l-1$,
$\beta_l \leq_{Z_{l,m}} \alpha_l$ if $m = l$,
$\beta_l \leq_{Z_{l,m}} \alpha_l, \alpha_{l+1}$ if $m = l+1$, and
$w \leq_{Z_{l,m}} w$ for all $w \in V(Z_{l,m})$.

(c) Let $Z^+_{l,l}$ $(l \geq 1)$ be a poset $Z_{l,l}$ adding a relation $\beta_l \leq_{Z^+_{l,l}} \alpha_1$. Then $Z^+_{1,1} = Z_{1,1}$.

For triangle-free strict semibound graphs, we obtain a following result. Then let C_l be a cycle with l vertices and P_r be a path with r vertices.

Theorem 3.2 *Let $P = (V(P), \leq_P)$ be a connected finite poset, where $|V(P)| \geq 2$. Then the following are equivalent:*

1. *The strict semibound graph $\text{ssb}(P)$ is a triangle-free graph,*
2. *The poset P is isomorphic to T_3, $Z_{l,l}$, $Z_{l,l+1}$, the dual poset $Z^d_{l,l+1}$ for $l \geq 1$, or $Z^+_{r,r}$ for $r \geq 2$ and $r \neq 3$,*
3. *The strict semibound graph $\text{ssb}(P)$ is isomorphic to P_3, $P_l \cup P_l$, $P_l \cup P_{l+1}$ for $l \geq 1$ or $C_r \cup C_r$ for $r \geq 4$.*

Proof. First we show that the condition 1 implies the condition 2. We assume that $\text{ssb}(P)$ is a triangle-free graph.
Case (a): $\text{mid}(P) \neq \emptyset$.

For $x \in \text{mid}(P)$, there exist $\beta \in \min(P)$ and $\alpha \in \max(P)$ such that $\beta \leq_P x \leq_P \alpha$. If $U^-_P(\beta) - \{x, \alpha\} \neq \emptyset$, then for $w \in U^-_P(\beta) - \{x, \alpha\}$, x, α and w have a common lower bound β and $\langle\{x, \alpha, w\}\rangle_{\text{ssb}(P)}$ is a triangle, which is a contradiction. So $U^-_P(\beta) - \{x, \alpha\} = \emptyset$. Similarly $L^-_P(\alpha) - \{x, \beta\} = \emptyset$. Since

P is connected, $\mathrm{mid}(P) = \{x\}, \min(P) = \{\beta\}$ and $\max(P) = \{\alpha\}$. Thus P is isomorphic to T_3.

Case (b): $\mathrm{mid}(P) = \emptyset$.

Since $\mathrm{ssb}(P)$ is triangle-free, $|L_P^-(\alpha)| \leq 2$ for $\alpha \in \max(P)$ and $|U_P^-(\beta)| \leq 2$ for $\beta \in \min(P)$. Since P is connected, the comparability graph $\mathrm{com}(P)$ of P is a connected graph. For $\alpha \in \max(P)$, $N_{\mathrm{com}(P)}(\alpha) = L_P^-(\alpha)$ and $\deg_{\mathrm{com}(P)}(\alpha) = |L_P^-(\alpha)|$. For $\beta \in \min(P)$, $N_{\mathrm{com}(P)}(\beta) = U_P^-(\beta)$ and $\deg_{\mathrm{com}(P)}(\beta) = |U_P^-(\beta)|$. So for all $x \in V(\mathrm{com}(P))$, $\deg_{\mathrm{com}(P)}(x) \leq 2$.

Case (b)-1: $\deg_{\mathrm{com}(P)}(x) = 2$ for all $x \in V(\mathrm{com}(P))$.

Then $\mathrm{com}(P)$ is isomorphic to a cycle with even vertices and $\mathrm{com}(P)$ is a bipartite graph with partite sets $\max(P)$ and $\min(P)$. Then $|\max(P)| \geq 2$, and $|\min(P)| \geq 2$. Let $x_1, x_2, ..., x_{2r}, x_1$ be the cycle of $\mathrm{com}(P)$ and $x_1 \in \max(P)$. Since $x_1 \in \max(P)$ and x_2 is adjacent to x_1, $x_2 \in \min(P)$ and $x_2 \leq_P x_1$. Since $x_2 \in \min(P)$ and x_3 is adjacent to x_2, $x_3 \in \max(P)$ and $x_2 \leq_P x_3$ and so on. Thus $x_1, x_3, ..., x_{2r-1} \in \max(P)$, $x_2, x_4, ..., x_{2r} \in \min(P)$, $x_{2i} \leq_P x_{2i-1}, x_{2i+1}$ for $1 \leq i \leq r-1$ and $x_{2r} \leq_P x_{2r-1}, x_1$. Then P is isomorphic to $Z_{r,r}^+$. Since $\mathrm{ssb}(Z_{3,3}^+)$ is $C_3 \cup C_3$ and $\mathrm{ssb}(P)$ is a triangle-free graph, $r \neq 3$. And $r \neq 1$, because $\mathrm{com}(Z_{1,1}^+)$ is P_1 which has degree one vertices. Therefore $r \geq 2$ and $r \neq 3$.

Case (b)-2: There exists a vertex whose degree is zero.

Then P is disconnected or $|V(P)| = 1$, which is a contradiction.

Case (b)-3: There exists a degree one vertex.

Since for all $x \in V(\mathrm{com}(P))$, $\deg_{\mathrm{com}(P)}(x) \leq 2$ and $\mathrm{com}(P)$ is connected, then there exist exactly two vertices whose degrees are one. Thus $\mathrm{com}(P)$ is isomorphic to a path $x_1, x_2, ..., x_t$.

Case (b)-3-1: $x_1 \in \max(P)$.

Then $x_i \in \max(P)$ for every odd i and $x_i \in \min(P)$ for every even i. In the case t is even, say $t = 2l$, $x_t \in \min(P)$ and P is isomorphic to $Z_{l,l}$. In the case t is odd, say $t = 2l+1$, $x_t \in \max(P)$ and P is isomorphic to $Z_{l,l+1}$.

Case (b)-3-2: $x_1 \in \min(P)$.

Then $x_i \in \min(P)$ for every odd i and $x_i \in \max(P)$ for every even i. In the case t is even, say $t = 2l$, $x_t \in \max(P)$ and P is isomorphic to $Z_{l,l}$. In the case t is odd, say $t = 2l+1$, $x_t \in \min(P)$ and P is isomorphic to the dual of $Z_{l,l+1}$.

Therefore the condition 1 implies the condition 2.

The strict semibound graph $\mathrm{ssb}(T_3)$ of T_3 is isomorphic to P_3, $\mathrm{ssb}(Z_{l,l}) \cong P_l \cup P_l$, $\mathrm{ssb}(Z_{l,l+1}) \cong \mathrm{ssb}(Z_{l,l+1}^d) \cong P_l \cup P_{l+1}$, and $\mathrm{ssb}(Z_{2,2}^+) \cong P_2 \cup P_2$. For $r \geq 4$, $\mathrm{ssb}(Z_{r,r}^+) \cong C_r \cup C_r$. So the condition 2 implies the condition 3.

For $r \geq 4$ and $l \geq 1$, P_3, $P_l \cup P_l$, $P_l \cup P_{l+1}$ and $C_r \cup C_r$ have no triangles. So the condition 3 implies the condition 1. □

4 Complete graphs without some edges

Since maximal elements of a poset are not adjacent to minimal elements of a poset in strict semibound graphs, a complete graph $K_n (n \geq 2)$ is not an strict semibound graph. Given a finite poset $P = (V(P), \leq_P)$ with n elements and $|\max(P)| = |\min(P)| = 1$, ssb$(P)$ is isomorphic to $K_n - e$, which is a complete graph without one edge. We consider nearly complete graphs and obtain the following results. Given a graph G and a subgraph H of G, the graph $G - E(H)$ is the graph with the vertex set $V(G - E(H)) = V(G)$ and the edge set $E(G - E(H)) = E(G) - E(H)$.

Proposition 4.1 *For $l, m \geq 1$ and $n \geq 0$, $K_{l+m+n} - E(K_{l,m})$ is an strict semibound graph.*

Proof. Let P be a poset such that $V(P) = \{\alpha_i : 1 \leq i \leq l\} \cup \{\beta_j : 1 \leq j \leq m\} \cup \{w_k : 1 \leq k \leq n\}$, $\beta_j \leq_P w_k \leq_P \alpha_i$ for $1 \leq i \leq l$, $1 \leq j \leq m$ and $1 \leq k \leq n$, and for all $x \in V(P)$, $x \leq_P x$. For each maximal element α_i, $L_P^-(\alpha_i) = \min(P) \cup \mathrm{mid}(P)$. For each minimal element β_j, $U_P^-(\beta_j) = \max(P) \cup \mathrm{mid}(P)$. In ssb$(P)$, α_i is adjacent to $\alpha_{i'}$ for $\alpha_i, \alpha_{i'} \in \max(P)$, β_j is adjacent to $\beta_{j'}$ for $\beta_j, \beta_{j'} \in \min(P)$, w_k is adjacent to $w_{k'}$ for $w_k, w_{k'} \in \mathrm{mid}(P)$, α_i is adjacent to w_k for $\alpha_i \in \max(P)$ and $w_k \in \mathrm{mid}(P)$, and β_j is adjacent to w_k for $\beta_j \in \min(P)$ and $w_k \in \mathrm{mid}(P)$. For any pair of a maximal element α_i and a minimal element β_j, α_i is not adjacent to β_j, because $U_P^-(\alpha_i) = \emptyset$ and $L_P^-(\beta_i) = \emptyset$. Therefore ssb$(P)$ is isomorphic to $K_{l+m+n} - E(K_{l,m})$. □

Corollary 4.2 *Graphs P_3, $K_n - e$ $(n \geq 2)$ and $K_l \cup K_m$ $(l, m \geq 1)$ are strict semibound graphs.* □

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