Preservers of Upper Ideals of Matrices: Tournaments; Primitivity*†

LeRoy B. Beasley
Department of Mathematics and Statistics, Utah State University
Logan, Utah 84322-3900, USA
email: leroy.b.beasley@aggiemail.usu.edu

Abstract

Let $\mathcal{M}$ denote the set of matrices over some semiring. An upper ideal of matrices in $\mathcal{M}$ is a set $\mathcal{U}$ such that if $A \in \mathcal{U}$ and B is any matrix in $\mathcal{M}$, then $A + B \in \mathcal{U}$. We investigate linear operators that strongly preserve certain upper ideals (that is, linear operators on $\mathcal{M}$ with the property that $X \in \mathcal{U}$ if and only if $T(X) \in \mathcal{U}$). We then characterize linear operators that strongly preserve sets of tournament matrices and sets of primitive matrices. Specifically we show that if T strongly preserves the set of regular tournaments when n is odd or nearly regular tournaments when n is even, then for some permutation matrix, P, $T(X) = P^tXP$ for all matrices X with zero main diagonal, or $T(X) = P^tX^tP$ for all matrices X with zero main diagonal. Similar results are shown for linear operators that strongly preserve the set of primitive matrices whose exponent is k for some values of k, and for those that strongly preserve the set of nearly reducible primitive matrices.

1 Introduction

The study of the invariants of maps has been an ongoing topic of research for centuries. The eigenvalue-eigenvector problem is one of the most basic and fundamental of these. The study of invariants of linear transformations on matrix spaces began over a century ago. This research involved two

*Keywords: Semiring, Boolean matrix, Linear operator, Tournament, Primitive matrix.
†AMS Classicication: 15A33.

basic types of questions. First: what are the invariant sets given a linear transformation (the eigenvalue-eigenvector problem)? And second: given a set or function what are the linear transformations that leave that set or function invariant? When the transformations are between sets of matrices, that study was begun by Fröbenius in 1897 [14] when he classified linear operators that preserve the determinant function. Since that time much research has been published on preserver problems. See [20, 16] for an excellent survey.

Recently, the study of linear operators on matrix spaces over sets which are not fields has become established. In particular, the study of linear transformations on spaces of $(0,1)$-matrices. This study is related to maps on directed or undirected graphs and bipartite graphs, and so has importance in combinatorics, computing, etc. The underlying set of scalars in this case is usually Boolean in that addition acts much like union of sets and multiplication much like intersection.

Unlike the investigation of preservers of sets of matrices over fields, preservers of sets of Boolean matrices usually require more hypothesis than to just assume that a transformation preserves the set. Defining a transformation to be O at O and the image of all other matrices to be a fixed element in the specified set produces a transformation that preserves that set. Clearly this transformation is not very interesting. Thus, an additional hypothesis is needed. This additional hypothesis is commonly that the transformation is bijective. Another condition also used, and the one we are using in this article, is that the transformation "strongly" preserves the set, that is the image of an element in the set is in the set, while the image of an element not in the set is not in the set. This condition appears not only in research on matrices over discrete semirings, but also in research on preservers over real and complex matrices, See for example [18] and [23].

Examples of research on strong preservers of matrices over antinegative semirings are many. In addition to the chapter in [20] on miscellaneous problems, recent examples include strong preservers of term rank k [2, 3], regular matrices [5, 12], idempotent matrices [7], nilpotent matrices [17, 22], etc. A search in MathSciNet with an input of ANYWHERE="strongly pres*" and MSC= "15* or 05*" produces 45 articles appearing in the last 25 years.

For basic facts and definitions of linear algebraic concepts we refer the reader to Horn and Johnson, [15].

2 Semirings, Semimodules, Upper Ideals and Preservers

In this section, the specific definitions and sets that we investigate are given.

2.1 Semirings and Semimodules

A *semiring* is a system, $(\mathbb{S}, +, \times)$, where $\mathbb{S}$ is a nonempty set, $(\mathbb{S}, +)$ is an Abelian monoid (identity 0), $(\mathbb{S}, \times)$ is a monoid (identity 1), $\times$ distributes over $+$, and $0 \times s = s \times 0 = 0$ for all $s \in \mathbb{S}$. Usually $\mathbb{S}$ denotes the system and $\times$ is denoted by juxtaposition. If $(\mathbb{S}, \times)$ is Abelian then we say $\mathbb{S}$ is *commutative.* If 0 is the only element of $\mathbb{S}$ that has an additive inverse then $\mathbb{S}$ is *antinegative.* Note that all rings with unity are semirings, but none are antinegative. Algebraic terms like *unit* and *zero divisor* are defined for semirings as if $\mathbb{S}$ were a ring.

In this paper, unless specified differently, we will assume that $\mathbb{S}$ is commutative, antinegative and with no zero divisors. These semirings occur frequently in combinatorics. They include the nonnegative members of any real ring containing 1, including the nonnegative reals, the nonnegative integers, etc., the fuzzy scalars $([0, 1], + = \max, \times = \min))$, and Boolean semirings, families of sets with $+ =$ union and $\times =$ intersection. including especially the binary Boolean semiring $\{\{\emptyset, \mathcal{U}\}, + = \cup, \times = \cap\}$, equivalent to $\mathbb{B} = (\{0, 1\}, + = \max, \times = \min)$. Note that $\mathbb{B}$ has arithmetic the same as real arithmetic except that $1 + 1 = 1$.

A *semimodule over* $\mathbb{S}$ is a triple $(\mathcal{K}, +, \bullet)$, where $\mathcal{K}$ is a nonempty set, $(\mathcal{K}, +)$ is an Abelian monoid (identity O) and $\bullet$ is a scalar product. Recall that $\bullet$ is a scalar product if for all $\alpha, \beta \in \mathbb{S}$ and $K, L \in \mathcal{K}$: $\alpha \bullet K \in \mathcal{K}$; $\alpha \bullet (\beta \bullet K) = (\alpha\beta) \bullet K$; $0 \bullet K = O$; $1 \bullet K = K$; $(\alpha + \beta) \bullet K = \alpha \bullet K + \beta \bullet K$; and $\alpha \bullet (K + L) = \alpha \bullet K + \alpha \bullet L$. Thus, a semimodule satisfies all the properties of a vector space that do not require additive inverses. As with semirings, $\mathcal{K}$ will denote the system and $\bullet$ is denoted by juxtaposition.

Let $\mathcal{M}_{m,n}(\mathbb{S})$ denote the set of all $m \times n$ matrices with entries in $\mathbb{S}$. Clearly $\mathcal{M}_{m,n}(\mathbb{S})$ is a semimodule over $\mathbb{S}$ where all operations are defined as if $\mathbb{S}$ were a field. If $m = n$ we use the notation $\mathcal{M}_n(\mathbb{S})$. Some semimodules of interest also include: $\mathcal{M}_n^{(0)}(\mathbb{S})$ the set of matrices in $\mathcal{M}_n(\mathbb{S})$ with all diagonal entries zero; and $\mathcal{UT}_n(\mathbb{S})$, the matrices in $\mathcal{M}_n(\mathbb{S})$ which are upper triangular, $\mathcal{S}_n(\mathbb{S})$, the set of all $n \times n$ symmetric matrices over $\mathbb{S}$; and $\mathcal{S}_n^{(0)}(\mathbb{S})$, the set of all $n \times n$ symmetric matrices over $\mathbb{S}$ with zero main diagonal.

We let $J_{m.n}$ denote the $m \times n$ matrix of all ones, $O_{m,n}$ the $m \times n$ matrix of all zeros, and I_n the $n \times n$ identity matrix. The subscripts are suppressed unless there is possibility of confusion and write J, O, I respectively.. The matrix $E_{i,j}$ in $\mathcal{M}_{m,n}(\mathbb{S})$ is the $m \times n$ matrix with exactly one nonzero entry, that being a one in the (i, j) position. We call $E_{i,j}$ a *cell.* The *weight* of matrix A, $|A|$, is the number of nonzero entries in A. So the weight of the matrix A is the minimum number of cells whose algebraic sum is A.

Let $A, B \in \mathcal{M}_n(\mathbb{S})$. We say that A *dominates* B if $a_{i,j} = 0$ implies that $b_{i,j} = 0$. This is denoted by $A \sqsupseteq B$ or $B \sqsubseteq A$.

Let A be a matrix. A *line* of A is a row or column of A. A *line matrix* is a matrix all of whose entries lie on a single line. A *full line matrix* is a matrix whose nonzero entries are all on one line and no other line matrix dominates it. Two cells are *collinear* if they are dominated by a line matrix. The *term rank* of A, $tr(A)$, is the least number of lines that contain all the nonzero entries of A.

If R_i denotes the full line matrix in $\mathcal{M}_n(\mathbb{B})$ or $\mathcal{M}_n^{(0)}(\mathbb{B})$ consisting all the cells in row i and C_i denotes the full line matrix in $\mathcal{M}_n(\mathbb{B})$ or $\mathcal{M}_n^{(0)}(\mathbb{B})$ consisting all the cells in column i, then the *double star* centered on i is the matrix in $\mathcal{S}_n(\mathbb{S})$ or $\mathcal{S}_n^{(0)}(\mathbb{S})$ which is the sum $R_i + C_i$.

Let $S \subset \mathcal{M}_{m,n}(\mathbb{S})$. The span of S, $\langle S \rangle$, is the set of all linear combinations (algebraic sums) of elements of S.

Let $\mathcal{K}$ be a subsemimodule of $\mathcal{M}_{m,n}(\mathbb{S})$. A *base element* of $\mathcal{K}$ is an element v of $\mathcal{K}$ such that $v \notin \langle \mathcal{K} \setminus \{v\} \rangle$. Note that the set of base elements define a basis of $\mathcal{K}$.

Example 1 *The following table gives the structure of the base elements of several semimodules of interest.*

$\mathcal{K}$	*base elements*
$\mathcal{M}_{m,n}(\mathbb{S})$	$E_{i,j}$
$\mathcal{M}_n^{(0)}(\mathbb{S})$	$E_{i,j}, i \neq j$
$\mathcal{S}_n(\mathbb{S})$	$E_{i,i}$ *or* $D_{i,j} = E_{i,j} + E_{j,i}, i \neq j$
$\mathcal{S}_n^{(0)}(\mathbb{S})$	$D_{i,j} = E_{i,j} + E_{j,i}, i \neq j$
$\mathcal{UT}_n(\mathbb{S})$	$E_{i,j}; i \leq j$

Primarily, we are interested only in subsemimodules that have base elements whose entries are only zeros and ones. However, there are semimodules that do not, for example, if $\mathcal{K} = \langle \mathcal{S}_n(\mathbb{S}) \cup \alpha \bullet \mathcal{UT}_n(\mathbb{S}) \rangle$ then the base elements are either digons, $D_{i,j}$, or weighted upper triangular cells, $\alpha \bullet E_{i,j}$. So there are subsemimodules whose base elements are not matrices with only zero or one entries.

Let S be a subset of semimodule $\mathcal{K}$. We say that S *separates base elements* if, given any two distinct base elements, E and F, there is some $N \in \mathcal{K}$ such that $N + E \in S$ and $N + F \notin S$. In this case we say that S separates E from F.

2.2 Upper Ideals

Preservers of upper ideals of matrices were first investigated by Beasley and Pullman in [8]. They considered only upper ideals in $\mathcal{M}_{m,n}(\mathbb{B})$. Here we consider the generalization to any of the above mentioned semimodules.

Definition 2 Let $\mathcal{K}$ be a semimodule over $\mathbb{S}$ and let $\mathcal{U}$ be a subset of $\mathcal{K}$. $\mathcal{U}$ is said to be an *upper ideal* of $\mathcal{K}$ if for every $A \in \mathcal{U}$ and $X \in \mathcal{K}$, $A + X \in \mathcal{U}$.

Example 3

a *If $\mathcal{K} = \mathcal{M}_{m,n}(\mathbb{S})$ and $\mathcal{U}$ is the set of all matrices of term rank at least k, then $\mathcal{U}$ is an upper ideal since by adding any matrix to another of fixed term rank does not decrease its term rank.*

b *If $\mathcal{K} = \mathcal{S}_n^{(0)}(\mathbb{S})$ and $\mathcal{U}$ is the set of all symmetric primitive matrices of exponent at most k then $\mathcal{U}$ is an upper ideal.*

c *If $\mathcal{K} = \mathcal{UT}_n(\mathbb{S})$ and $\mathcal{U}$ is the subset of matrices having the first row all nonzero, then $\mathcal{U}$ is an upper ideal.*

Note that it is easy to show that the upper ideal in Example 3a separates base elements (cells). The upper ideal in Example 3c does not separate base elements, and while the one in Example 3b does, it is not so easily shown.

Let S be any subset of $\mathcal{K}$ and define $\mathcal{U}_S$ to be the *upper ideal generated by* S, that is, $\mathcal{U}_S = \{A \in \mathcal{K} : A \not\sqsubseteq B \text{ for all } B \in S\}$, that is., $\mathcal{U}_S$ is the set of all elements of $\mathcal{K}$ not dominated by an element of S The fact that $\mathcal{U}_S$ is indeed an upper ideal in $\mathcal{K}$, unless $J \in S$, is easily proven, see [8, Lemma 3.5a].

2.3 Preservers

Let $\mathcal{K}$ be a semimodule over $\mathbb{S}$. A mapping $\Psi : \mathcal{K} \to \mathcal{K}$ is a *linear operator* if for any $\alpha, \beta \in \mathbb{S}$ and $A, B \in \mathcal{K}$, $\Psi(\alpha A + \beta B) = \alpha\Psi(A) + \beta\Psi(B)$. If S is a subset of $\mathcal{K}$ we say that Ψ *preserves* S whenever $X \in S$ implies $\Psi(X) \in S$. Further, Ψ *strongly preserves* S whenever $X \in S$ if and only if $\Psi(X) \in S$.

The following Theorem is easily proved for any finite semimodule $\mathcal{K}$. (See: http://en.wikipedia.org/wiki/Bijection.)

Theorem 4 *Let $\mathcal{K}$ be a finite semimodule and $\Psi : \mathcal{K} \to \mathcal{K}$ be a linear operator. Then the following conditions are equivalent:*

(a) Ψ is bijective;

(b) Ψ is surjective;

(c) Ψ is injective;

The following lemma (for the Boolean case) can be found in [8, Lemma 3.5]. A similar field type theorem involving groups may be found in [13]. The proof is included here for completeness.

Lemma 5 *Let $\mathcal{K}$ denote a finite semimodule over $\mathbb{S}$, $\Psi : \mathcal{K} \to \mathcal{K}$ be a linear operator and let $S \subseteq \mathcal{K}$. Then, if Ψ strongly preserves S, then Ψ strongly preserves $\mathcal{U}_S$.*

Proof. Let $A \notin \mathcal{U}_S$, so that some element of S dominates A. Then there is some $B \in \mathcal{K}$ such that $A + B \in S$. Since Ψ strongly preserves S, $\Psi(A) + \Psi(B) \in S$. That is $\Psi(A)$ is dominated by some element of S, and consequently, $\Psi(A) \notin \mathcal{U}_S$.

Note that since Ψ strongly preserves S, Ψ^d strongly preserves S for any d. In particular for d such that $\Psi^d = \Theta$ is idempotent (this must happen since $\mathcal{K}$ is finite).

Now suppose that $A \in \mathcal{U}_S$ and $\Psi(A) \notin \mathcal{U}_S$ so that $\Psi(A) + Y \in S$ for some Y. Let $Z = \Psi^{d-1}(Y)$. Then, $\Psi^{d-1}(\Psi(A)+Y) = \Theta(A)+Z \in S$. Now, $\Theta(\Theta(A) + Z) \in S$ since Θ strongly preserves S. That is $\Theta^2(A) + \Theta(Z) = \Theta(A) + \Theta(Z) = \Theta(A + Z) \in S$. Since Θ strongly preserves S, we must have that $A+Z \in S$, and hence $A \notin \mathcal{U}_S$, a contradiction. Thus, Ψ strongly preserves $\mathcal{U}_S$. ∎

Lemma 6 *Let $\mathcal{K}$ denote a semimodule over $\mathbb{S}$, $\Psi : \mathcal{K} \to \mathcal{K}$ be a linear operator and let $\mathcal{U}$ be an upper ideal of $\mathcal{K}$. If Ψ strongly preserves $\mathcal{U}$ and $N \in \mathcal{K}$ separates base elements E from F ($N + E \in \mathcal{U}$ and $N + F \notin \mathcal{U}$) then $\Psi(N + E)$ is not dominated by $\Psi(N + F)$.*

Proof. Since $N+F \notin \mathcal{U}$, $N+F$ does not dominate any member of $\mathcal{U}$. Thus, $\Psi(N + F)$ cannot dominate any member of $\mathcal{U}$ and $\Psi(N + E)$ is a member of $\mathcal{U}$. ∎

The following lemma is a generalization of Lemma 3.3 from [8], which was proven for $\mathcal{K} = \mathcal{M}_{m,n}(\mathbb{B})$.

Lemma 7 *Let $\mathcal{K}$ denote a finite semimodule over $\mathbb{S}$, $\Psi : \mathcal{K} \to \mathcal{K}$ be a linear operator and let $\mathcal{U}$ be an upper ideal of $\mathcal{K}$. If Ψ strongly preserves $\mathcal{U}$ and $\mathcal{U}$ separates base elements, then Ψ is bijective on the set of all base elements.*

Proof. Suppose that $\Psi(X) = O$ for some $X \in \mathcal{K}$. Then, there exists a base element $E \sqsubseteq X$ such that $\Psi(E) = O$. Let F be any other base element, and let $N \in \mathcal{K}$ separate E from F, so that $N + E \in \mathcal{U}$ and $N + F \notin \mathcal{U}$. Then, $\Psi(N+E) = \Psi(N) + \Psi(E) = \Psi(N) \sqsubseteq \Psi(N) + \Psi(F) = \Psi(N+F)$, a contradiction by Lemma 6.

Since $\mathcal{K}$ is finite, there is some power of Ψ which is idempotent. Let $\Psi^d = \Theta$ be idempotent.

Let E and F be base elements with $E \neq F$. Suppose that $E \sqsubseteq \Theta(F)$, so that $\Theta(E) \sqsubseteq \Theta^2(F)$, and let N separate E from F. Then, since Θ is idempotent and linear, $\Theta(N + F) = \Theta(N) + \Theta(F) = \Theta(N) + \Theta^2(F) \sqsupseteq \Theta(N) + \Theta(E) = \Theta(N + E)$. This is a contradiction by Lemma 6 since Ψ, and hence Θ, strongly preserves $\mathcal{U}$. Thus, $\Theta(F) = F$, and similarly $\Theta(E) = E$.

Now suppose that $\Psi(E) = \Psi(F)$, and hence that $\Theta(E) = \Theta(F)$, for base elements E and F. Then $E = \Theta(E) = \Theta(F) = F$, so that Ψ is injective on the set of base elements. By Theorem 4 Ψ is bijective on the set of base elements. ∎

Corollary 7.1 *Let $\mathbb{S}$ be finite, $\mathcal{K}$ be any one of the semimodules $\mathcal{M}_{m,n}(\mathbb{S})$, $\mathcal{M}_n^{(0)}(\mathbb{S}), \mathcal{S}_n(\mathbb{S}), \mathcal{S}_n^{(0)}(\mathbb{S})$, or $\mathcal{UT}_n(\mathbb{S})$, and $\mathcal{U}$ be an upper ideal of $\mathcal{K}$ that separates base elements. If $\Psi : \mathcal{K} \to \mathcal{K}$ strongly preserves $\mathcal{U}$, then Ψ is bijective on $\mathcal{K}$.*

Proof. Since $\mathbb{S}$ is antinegative and in each semimodule, each element of $\mathcal{K}$ is uniquely representable as an algebraic sum of base elements, Ψ is bijective on all of $\mathcal{K}$. ∎

The following lemma appeared in [8] for the case that $\mathcal{K} = \mathcal{M}_{m,n}(\mathbb{B})$.

Lemma 8 *Let $\mathbb{S}$ be any commutative, antinegative semiring without zero divisors and let $\mathcal{K}$ be a semimodule. Let $T : \mathcal{K} \to \mathcal{K}$ be a bijective linear operator.*

1. *If $\mathcal{K} = \mathcal{M}_{m,n}(\mathbb{S})$ and T maps pairs of collinear cells to pairs of collinear cells, then T is a (P,Q)-operator.*

2. *If $\mathcal{K} = \mathcal{M}_n^{(0)}(\mathbb{S})$ and T maps pairs of collinear cells to pairs of collinear cells, then T is a (P,P^t)-operator.*

3. *If $\mathcal{K} = \mathcal{S}_n(\mathbb{S})$ and T maps pairs of base elements dominated by a double star to pairs of base elements dominated by a double star, then T is a (P,P^t)-operator.*

4. *If $\mathcal{K} = \mathcal{S}_n^{(0)}(\mathbb{S})$ and T maps pairs of digons dominated by a double star to pairs of digons dominated by a double star, then T is a (P,P^t)-operator.*

Proof. Since in each case, cells are mapped to cells, so by linearlty, the image of a weighted cell is that same weighting on the cell that is the image of the unweighted cell. Thus, the arguments for arbitrary semirings are exactly the same as for the Boolean case. The proof of (1) is that found in [8, Lemma 3.4] by noting that a "2-claw" used in [8] is a pair of collinear cells in this article. The proof of (2) follows from (1) by the fact that a (P,Q)-operator maps nondiagonal cells to nondiagonal cells requires that $Q = P^t$. We will give a proof of (3). The proof of (4) is parallel.

Any sum of a pair of base elements dominated by a double star is either of the form $E_{i,i} + D_{i,j}$ or $D_{i,j} + D_{i,k}$ for some i, j, k. If T maps pairs of base elements dominated by a double star to pairs of base elements dominated by a double star, the T maps double stars to double stars. Now, if T maps the double star dominating $E_{i.i}$ to the double star dominating $E_{j,j}$, define $\sigma : \{1,2,\ldots,n\} \rightarrow \{1,2,\ldots,n\}$ by $\sigma(i) = j$. For $T(E_{i,i}) = E_{k,k}$ and $T(E_{j,j} = E_{l.l}$, we must have $\sigma(i) = k$ and $\sigma(j) = l$ so that $T(E_{i,j}) = E_{k,l}$. Then, since T is bijective, σ is a permutation. Let P be the permutation matrix corresponding to σ^{-1} and we must have that $T(E_{i,j}) = E_{k,l} = E_{\sigma(i),\sigma(j)} = PE_{i,j}P^t$. Since T is linear we have $T(X) = PXP^t$ for any $X \in \mathcal{K}$. ∎

The above lemma makes characterizations of strong preservers of some ideals easy.

3 Applications

In this section, we only consider the Boolean case, specifically we let $\mathbb{S} = \mathbb{B}$. The reasoning behind this is that the applications we consider do not depend upon the nature of nonzero entries in a matrix, only on the fact that they are zero or nonzero. For example, an irreducible matrix is irreducible even when any nonzero entry is replace by any other nonzero member of $\mathbb{S}$. A basic reference for the topics in this section is Brualdi and Ryser, [11].

3.1 Strong Preservers of sets of Tournament Matrices

A *tournament* on n vertices is a directed graph which is an orientation of the complete simple loopless undirected graph. That is a tournament is a loopless digraph in which any two distinct vertices are connected by exactly one arc. See Moon, [19], for more details on tournaments.

Let $\mathcal{T}_n$ denote the set of all tournament digraphs on n vertices. Let $A(T)$ be the adjacency matrix of the tournament T. Then, $A(T)$ is a $(0,1)$-matrix such that $A(T) + A(T)^t + I = J$, where I denotes the identity matrix, J the matrix of all ones and the arithmetic is real. Now, M is a tournament matrix if M has a zero diagonal and $m_{i,j} \neq 0$ if and only if $m_{j,i} = 0$. So a tournament matrix is the adjacency matrix of a tournament digraph.

The outdegree of a vertex in a directed graph is the number of arcs originating at that vertex. The outdegree sequence is an n-vector such that the i^{th} component is the outdegree of the i^{th} vertex. Given a tournament M, the *score sequence*, $\mathrm{s} = (s_1, s_2, \ldots, s_n)$, of T is the outdegree sequence of a tournament T' equivalent to T (that is, whose only difference from T is a relabeling of the vertices) such that the outdegree sequence of T' is s and $s_1 \geq s_2 \geq \cdots \geq s_n$.

Let $\mathcal{T}_{\mathrm{s}}$ be the set of all tournaments in $\mathcal{T}_n$ whose score sequence is s.

A tournament is *regular* if each vertex has outdegree equal to the indegree. A regular tournament matrix is one in which each row sum and each column sum are equal. Necessarily, the row sum, column sum, indegree and out degree mentioned must be $\frac{n-1}{2}$, and it follows that n must be odd. In the case n is even we say that a tournament is *nearly regular* if every vertex has indegree and out degree differing by one. So a nearly regular tournament matrix is a matrix where each row sum differs from its column sum by one. Necessarily, in a nearly regular tournament there are $\frac{n}{2}$ vertices with out degree $\frac{n}{2}$ and $\frac{n}{2}$ vertices with outdegree $\frac{n-2}{2}$. Similarly, a nearly regular tournament matrix has $\frac{n}{2}$ rows with row sum $\frac{n}{2}$ and $\frac{n}{2}$ rows with row sum $\frac{n-2}{2}$. (The same statements hold for indegrees and column sums.) That is the set of regular tournaments is $\mathcal{T}_{\text{reg}} = \mathcal{T}_{\mathbf{s}}$ where $\mathbf{s} = (\frac{n-1}{2}, \frac{n-1}{2}, \ldots, \frac{n-1}{2})$ and the set of all nearly regular tournaments is $\mathcal{T}_{\text{nreg}} = \mathcal{T}_{\mathbf{s}}$ where $\mathbf{s} = (\frac{n}{2}, \frac{n}{2}, \ldots, \frac{n}{2}, \frac{n-2}{2}, \frac{n-2}{2}, \ldots, \frac{n-2}{2})$ ($\frac{n}{2}$ of the components are $\frac{n}{2}$ and $\frac{n}{2}$ are $\frac{n-2}{2}$).

Let $\mathcal{U}_{\mathbf{s}} = \mathcal{U}_{\mathcal{T}_{\mathbf{s}}}$ for any score sequence s, that is $\mathcal{U}_{\mathbf{s}}$ is the set of matrices not dominated by a tournament with score sequence s.

Lemma 9 *Let $\Psi : \mathcal{M}_n^{(0)}(\mathbb{B}) \to \mathcal{M}_n^{(0)}(\mathbb{B})$. If Ψ strongly preserves $\mathcal{T}_{\mathbf{s}}$ then Ψ is bijective on $\mathcal{M}_n^{(0)}(\mathbb{B})$.*

Proof. By Lemma 5 Ψ strongly preserves $\mathcal{U}_{\mathbf{s}}$. By Corollary 7.1, we only need show that $\mathcal{U}_{\mathbf{s}}$ separates base elements (off diagonal cells).

Suppose that E and F are off diagonal cells. Then there is some element, say C, of $\mathcal{T}_{\mathbf{s}}$ that dominates F but not E. Then, $C + E$ is not even a tournament and is not dominated by any tournament, thus, $C + E \in \mathcal{U}_{\mathbf{s}}$. But, $C + F = C \in \mathcal{T}_{\mathbf{s}}$ and hence, $C + F \notin \mathcal{U}_{\mathbf{s}}$, so C separates E from F. Thus $\mathcal{U}_{\mathbf{s}}$ separates cells. ∎

Let $\mathcal{U}_{\text{reg}}$ denote the set of all matrices that are not dominated by a regular tournament. Let $\mathcal{U}_{\text{nreg}}$ denote the set of all matrices that are not dominated by a nearly regular tournament. That is, $\mathcal{U}_{\text{reg}} = \mathcal{U}_{\mathcal{T}_{\text{reg}}}$ and $\mathcal{U}_{\text{nreg}} = \mathcal{U}_{\mathcal{T}_{\text{nreg}}}$. By Lemma 9, if $\Psi : \mathcal{M}_n^{(0)}(\mathbb{B}) \to \mathcal{M}_n^{(0)}(\mathbb{B})$ strongly preserves $\mathcal{U}_{\text{reg}}$ or $\mathcal{U}_{\text{nreg}}$, T is bijective. In [1, Theorem 3.6], Beasley, Brown and Guterman show that if $n > 3$ and Ψ is a surjective operator that preserves regular tournaments if n is odd, or nearly regular tournaments if n is even, then T is a (P, P^t)-operator.

Theorem 10 *Let $\Psi : \mathcal{M}_n^{(0)}(\mathbb{B}) \to \mathcal{M}_n^{(0)}(\mathbb{B})$ be a linear operator. Then Ψ strongly preserves regular tournaments if n is odd or nearly regular tournaments if n is even if and only if Ψ is a (P, P^t)-operator.*

Proof. By Lemma 9, Ψ is bijective. By [1, Theorem 3.6], Ψ a (P, P^t)-operator.

Suppose that T is a (P, P^t) operator. Then, T, considered as a mapping on the set of directed graphs, is a vertex permutation. Thus, tournaments are mapped to tournaments and the row sums (score sequence) is just the permuted row sums of the preimage. ∎

It is believed by the author that any operator that strongly preserves $\mathcal{T}_s$ for any degree sequence is a (P, P^t)-operator. The approach would be to show that collinear cells are mapped to collinear cells. This would require a case by case investigation.

3.2 Strong Preservers of sets of Primitive Matrices

Let $A \in \mathcal{M}_n(\mathbb{B})$. Then, A is *primitive* if some power of A has all nonzero entries. The *exponent* of a primitive matrix is the smallest power that gives a strictly nonnegative matrix, so if A^k has all nonzero entries and A^l has a zero entry for every $l < k$, then the exponent of A is k, $\exp(A) = k$. For notational convenience, we say that the exponent of a non-primitive matrix is zero. Let $\mathcal{E}_k = \{A \in \mathcal{M}_n(\mathbb{B}) | \exp(A) = k\}$. So $\mathcal{E}_0$ is the set of non-primitive matrices, and further, $\mathcal{E}_1 = \{J\}$.

In [6] strong preservers of the set of all primitive matrices were characterized, in [9], the preservers of the index of imprimitivity were characterized, and in [4], the strong preservers of $\mathcal{E}_2$, $\mathcal{E}_{n^2-2n+1}$ and $\mathcal{E}_{n^2-2n+2}$ were characterized. We now extend those results, investigating the strong preservers of $\mathcal{E}_k$ for any k such that $\mathcal{E}_k \neq \emptyset$.

Example 11 *Consider the matrix*

$$\Xi = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 \end{bmatrix}.$$

Note that the exponent of Ξ is 3, and, if any zero is changed to a 1 the resulting matrix has exponent 2. Let $\mathcal{U}$ be the set of all matrices in $\mathcal{M}_n(\mathbb{B})$ which are not dominated by any element of $\mathcal{E}_3$, $\mathcal{U} = \mathcal{U}_{\mathcal{E}_3}$. Then, $\Xi \in \mathcal{E}_3$ and hence not in $\mathcal{U}$, and given any two base elements, E and F, of $\mathcal{M}_n(\mathbb{B})$, there is some permutation, P, such that $P(\Xi)P^t + F = P(\Xi)P^t \in \mathcal{E}_3$ and hence not in $\mathcal{U}$ and $P(\Xi)P^t + E \sqsupset P(\Xi)P^t$ and hence, has exponent 2, and is not dominated by any member of $\mathcal{E}_3$. That is $P(\Xi)P^t + E \in \mathcal{U}$. Thus $\mathcal{U}$ separates base elements of $\mathcal{M}_n(\mathbb{B})$

Example 12 *Consider the matrix*

$$\Psi = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 1 & \ddots & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Note that the exponent of Ψ is 4, and, if any zero is changed to a 1 the resulting matrix has exponent 2 or 3. Let $\mathcal{U}$ be the set of all matrices in $\mathcal{M}_n(\mathbb{B})$ which are not dominated by any element of $\mathcal{E}_4$, $\mathcal{U} = \mathcal{U}_{\mathcal{E}_4}$. Then, $\Psi \in \mathcal{E}_4$ and hence not in $\mathcal{U}$, and given any two base elements, E and F, of $\mathcal{M}_n(\mathbb{B})$, there is some permutation, P, such that $P(\Psi)P^t + F = P(\Psi)P^t \in \mathcal{E}_4$ and hence not in $\mathcal{U}$ and $P(\Psi)P^t + E \sqsupset P(\Psi)P^t$ and hence, has exponent 2 or 3, and is not dominated by any member of $\mathcal{E}_4$. That is $P(\Psi)P^t + E \in \mathcal{U}$. Thus $\mathcal{U}$ separates base elements of $\mathcal{M}_n(\mathbb{B})$

Lemma 13 *Let $3 \leq k \leq 2n-2$ and let $\mathcal{U} \subseteq \mathcal{M}_n(\mathbb{B})$ be the upper ideal of all matrices which are not dominated by any element of $\mathcal{E}_k$. Then $\mathcal{U}$ separates base elements of $\mathcal{M}_n(\mathbb{B})$*

Proof. By the above examples we may assume that $k \geq 5$. Let E and F be two base elements of $\mathcal{M}_n(\mathbb{B})$ (cells).

Case 1. E and F are both diagonal cells.

Suppose that $n \leq k \leq 2n-2$, so $k = 2n-1-d$. Let A be chosen so that A is primitive, has d nonzero diagonal entries and $A \sqsupseteq F$ while $A \not\sqsupseteq E$. Then, $A + E$ has exponent at most $2n - 1 - (d + 1) < k$, and any matrix that dominates $A + E$ also has exponent at most $k - 1$. Thus, $A + E \in \mathcal{U}$. Since $A + F = A \in \mathcal{E}_k$, $A + F \notin \mathcal{U}$.

Now suppose that $5 \leq k \leq n - 1$ and suppose that $E = E_{n,n}$ and

$F = E_{1,1}$. If k is even, $k = 2l$, let

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

If k is odd, $k = 2l - 1$, let

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

In each case, in the digraph of A, there are $n - l$ digons and an l-cycle meeting at vertex 1, a loop at vertex n and when k is odd, the arc $(l-1, 1)$.

Now, $A + E = A \in \mathcal{E}_k$, whereas the exponent of $A + F$ is $k - 1$, and hence $A + E \in \mathcal{U}$ while $A + F \notin \mathcal{U}$. So, $\mathcal{U}$ separates F from E.

Case 2. F is a diagonal cell and E is not. As in case 1, we can find a primitive matrix with exponent k which dominates E but not F and the exponent of $A + F$ is strictly less than k, so that $\mathcal{U}$ separates E from F.

Case 3. E and F are off diagonal cells. Then, there is some element of $\mathcal{E}_k$, say N, with maximal number of nonzero entries that dominates F but not E. Then, since $N \in \mathcal{E}_k$, and $N + F = N$, $N + F \notin \mathcal{U}$. Since $N + E$ has more nonzero entries that N, no member of $\mathcal{E}_k$ dominates $N + E$ and hence $N + E \in \mathcal{U}$, we have that N separates E from F. ∎

Lemma 14 *Let $k > 2n - 2$ and let $\mathcal{U} \subseteq \mathcal{M}_n^{(0)}(\mathbb{B})$ be the upper ideal of all matrices which are not dominated by any element of $\mathcal{E}_k$. Then $\mathcal{U}$ separates base elements of $\mathcal{M}_n^{(0)}(\mathbb{B})$*

Proof. The proof is parallel to the proof of Lemma 13 noting that two diagonal cells cannot be separated and, thus, the condition that $\mathcal{K} = \mathcal{M}_n^{(0)}(\mathbb{B})$ is necessary. ■

Lemma 15 *Let $3 \leq k \leq 2n-2$ be a positive integer such that $\mathcal{E}_k \neq \emptyset$, and $\Psi : \mathcal{M}_n(\mathbb{B}) \to \mathcal{M}_n(\mathbb{B})$ be a linear operator. If Ψ strongly preserves $\mathcal{E}_k$ then Ψ is bijective on $\mathcal{M}_n(\mathbb{B})$.*

Proof. Let $\mathcal{U}$ be the upper ideal of all matrices which are not dominated by any element of $\mathcal{E}_k$. Then, by Lemma 13, $\mathcal{U}$ separates base elements, and by Corollary 7.1, Ψ is bijective on $\mathcal{M}_n(\mathbb{B})$. ■

Lemma 16 *Let $\Psi : \mathcal{M}_n^{(0)}(\mathbb{B}) \to \mathcal{M}_n^{(0)}(\mathbb{B})$ be a linear operator. Let $k > 2n-2$ be a positive integer such that $\mathcal{E}_k \neq \emptyset$. If Ψ strongly preserves $\mathcal{E}_k$ then Ψ is bijective on $\mathcal{M}_n^{(0)}(\mathbb{B})$.*

Proof. Let $\mathcal{U}$ be the upper ideal of all matrices which are not dominated by any element of $\mathcal{E}_k$. Then, by Lemma 14, $\mathcal{U}$ separates base elements, and by Corollary 7.1, Ψ is bijective on $\mathcal{M}_n^{(0)}(\mathbb{B})$. ■

A matrix, $A \in \mathcal{M}_n(\mathbb{B})$ is *reducible* if there exists a permutation matrix such that $PAP^t = \begin{bmatrix} A_1 & O \\ A_2 & A_3 \end{bmatrix}$ where A_1 and A_3 are nonvacuous square matrices, otherwise, A is irreducible. An easy observation is that all primitive matrices are irreducible. An irreducible matrix is *nearly reducible* if changing any nonzero entry in the matrix to a zero results in a reducible matrix.

Let $\mathcal{NRP}_n$ denote the set of all $n \times n$ nearly reducible, primitive matrices in $\mathcal{M}_n(\mathbb{B})$. Several authors have studied this set [10, 21]. Some easily established facts are: (See [21, Lemma 2.1])

a) if $A \in \mathcal{NRP}_n$ then all diagonal entries of A are zero;

b) no cycle in the digraph of A has a chord;

c) if $n \leq 3$ then $\mathcal{NRP}_n = \emptyset$.

Recall that $|A|$ denotes the number of nonzero entries in A. We use the term *digon* to denote a sum of two symmetric off diagonal cells $(E_{i,j} + E_{j,i})$.

Further, a mapping of the form $X \to PXP^t$ where P is apermutation matrix is called a *permutational similarity.*

Proposition 17 *Let E, F, G be off diagonal cells in $\mathcal{M}_n(\mathbb{B})$ and $n \geq 4$. Then, there exists $A \in \mathcal{N}RP_n$ such that $|A| = n+1$ and $A \sqsupseteq E+F+G$ if and only if E, F, and G are non collinear, and if n is odd, $E+F+G$ does not dominate a digon.*

Proof. Suppose that $A \in \mathcal{N}RP_n$ and $|A| = n+1$. If A has three collinear cells, then some row or column of A has only zero entries, and hence is reducible, a contradiction. Thus E, F, and G are non collinear. Suppose that A dominates a digon and n is odd. Then, since A consists of the sum of two cycles and no cycle has a chord, A is the sum of a digon (2-cycle) and an $(n-1)$-cycle. But since n is odd, the greatest common divisor of 2 and $n-1$ is 2, not 1, and hence A is not primitive, a contradiction.

Now, suppose that E, F, and G are non collinear, and if n is odd, $E+F+G$ does not dominate a digon. Up to permutational similarity and/or transpose, there are nine possibilities for a choice of three off diagonal cells:

1. $E_{1,2}, E_{1,3}, E_{1,4}$; (Collinear)
2. $E_{1,2}, E_{2,1}, E_{1,3}$; (Dominates a digon)
3. $E_{1,2}, E_{2,1}, E_{3,4}$; (Dominates a digon)
4. $E_{1,2}, E_{3,2}, E_{3,4}$;
5. $E_{1,2}, E_{3,2}, E_{4,3}$;
6. $E_{1,2}, E_{3,2}, E_{4,5}$;
7. $E_{1,2}, E_{2,3}, E_{3,4}$;
8. $E_{1,2}, E_{2,3}, E_{4,5}$;
9. $E_{1,2}, E_{3,4}, E_{5,6}$.

If n is even, case 1 does not apply. In n is odd cases 1-3 do not apply. In all other cases a choice of $n-2$ cells can be made so that $E+F+G$ plus the sum of those $n-2$ cells is nearly reducible and primitive. As the demonstration of this is routine and can be easily shown using digraphs, the proof is left to the reader. ■

Lemma 18 *Let $n \geq 4$ and $\mathcal{U} \subseteq \mathcal{M}_n^{(0)}(\mathbb{B})$ be the upper ideal of all matrices which are not dominated by any element of $\mathcal{N}RP_n$. Then $\mathcal{U}$ separates base elements of $\mathcal{M}_n^{(0)}(\mathbb{B})$*

Proof. The proof is parallel to the proof of Lemma 13 noting that two diagonal cells cannot be separated and thus the condition that $\mathcal{K} = \mathcal{M}_n^{(0)}(\mathbb{B})$ is necessary. ∎

Lemma 19 *Let $\Psi : \mathcal{M}_n^{(0)}(\mathbb{B}) \to \mathcal{M}_n^{(0)}(\mathbb{B})$ be a linear operator, and $n \geq 4$. If Ψ strongly preserves $\mathcal{N}RP_n$ then Ψ is bijective on $\mathcal{M}_n^{(0)}(\mathbb{B})$.*

Proof. Let $\mathcal{U}$ be the upper ideal of all matrices which are not dominated by any element of $\mathcal{N}RP_n$. Then, by Lemma 18, $\mathcal{U}$ separates base elements, and by Corollary 7.1, Ψ is bijective on $\mathcal{M}_n^{(0)}(\mathbb{B})$. ∎

In [4] the strong preservers of the sets of primitive matrices of exponent k were characterized for certain values of k. These characterizations require the definition of a transformation from the set of diagonal matrices to the set of all matrices. Let Δ_n denote the subset of $\mathcal{M}_n(\mathbb{S})$ consisting of the diagonal matrices. A *diagonal transformation* is a linear mapping $R : \Delta_n \to \mathcal{M}_n(\mathbb{S})$. Further, if X is a subset of $\mathcal{K}$, let X^c denote the complement of the set X in $\mathcal{K}$. Let $\mathcal{D}_{\mathcal{E}_k}$ donate the set of all matrices dominated by some element of $\mathcal{E}_k$.

Theorem 20 [4] *Let $n \geq 3$.*

- *Let $T : \mathcal{M}_n(\mathbb{B}) \to \mathcal{M}_n(\mathbb{B})$ be a linear operator. Then T strongly preserves $\mathcal{E}_2$ if and only if T is a (P, P^t)-operator.*
- *Let $T : \mathcal{M}_n(\mathbb{B}) \to \mathcal{M}_n(\mathbb{B})$ be a linear operator. Then T strongly preserves $\mathcal{E}_{n^2-2n+1}$ or $\mathcal{E}_{n^2-2n+1}$ when $n \geq 5$ if and only if T is the sum of a (P, P^t)-operator on $\mathcal{M}_n^{(0)}(\mathbb{B})$ plus a diagonal transformation which is nonsingular and maps nonzero matrices to matrices in $\mathcal{D}_{\mathcal{E}_k}$. That is, $T(X) = P(X \circ (J \setminus I))P^t + R(X \circ I)$ for all $X \in \mathcal{M}_n(\mathbb{B})$, or $T(X) = P(X \circ (J \setminus I))^t P^t + R(X \circ I)$ for all $X \in \mathcal{M}_n(\mathbb{B})$, where R is any diagonal transformation on $\mathcal{M}_n(\mathbb{B})$ such that $R((X \circ I) \setminus \{O\}) \in \mathcal{D}_{\mathcal{E}_k}$.*

We now consider other specific exponents, and characterize the linear operators that strongly preserve the set of primitive matrices with that exponent.

Theorem 21 *Let $n \geq 4$ and $T : \mathcal{M}_n(\mathbb{B}) \to \mathcal{M}_n(\mathbb{B})$ be a linear operator. Then T strongly preserves $\mathcal{E}_{2n-2}$ if and only if T is a sum of a (P, P^t)-operator on $\mathcal{M}_n^{(0)}(\mathbb{B})$ and any bijection on Δ_n.*

Proof. By Lemma 15 T is bijective. If $A \in \mathcal{E}_{2n-2}$ and A has precisely $n+1$ nonzero entries, then A is the sum of a diagonal cell and a full cycle permutation matrix. Now, suppose that the image of an off diagonal cell is a diagonal cell, say, without loss of generality, that $T(E_{1,2}) = E_{1,1}$. Let $\mathcal{S}$ be the set of all matrices in $\mathcal{M}_n(\mathbb{B})$ which is the sum of a diagonal cell and a matrix C such that $E_{1,2} + C$ is a full cycle permutation matrix. Then, $|\mathcal{S}| = n \cdot (n-2)!$. Let $\mathcal{T}$ denote the set of all matrices of the form $E_{1,1} + G$ where G is a full cycle permutation matrix. Then $|\mathcal{T}| = (n-1)!$. The image of each member of $\mathcal{S}$ is in $\mathcal{T}$, but since T is bijective we must have that $|\mathcal{S}| \leq |\mathcal{T}|$ or $n(n-2)! \leq (n-1)!$, a contradiction. Thus T maps off diagonal cells to off diagonal cells and diagonal cells to diagonal cells.

By Lemma 8(2) if T maps pairs of off diagonal collinear cells to off diagonal collinear cells then T is a (P, P^t)-operator on $\mathcal{M}_n^{(0)}(\mathbb{B})$. Suppose that T does not map some pair of off diagonal collinear cells (E, F), to off diagonal collinear cells, then without loss of generality, by applying a (P, P^t)-operator, we may assume that (E, F) is $(E_{1,2}, E_{1,3})$. Further, the possible choices for $(T(E), T(F))$ are $(E_{1,2}, E_{2,3})$, $(E_{1,2}, E_{3,4})$, or $(E_{1,2}, E_{2,1})$. In the first two cases there is a full cycle matrix dominating the image of $E + F$ and hence the inverse image of that cycle plus a diagonal cell must be in $\mathcal{E}_{2n-2}$, a contradiction since no element of $\mathcal{E}_{2n-2}$ with $n+1$ nonzero entries dominates both a diagonal cell and two collinear off diagonal cells. Thus, $T(E_{1,2}) = E_{1,2}$ and $T(E_{1,3}) = E_{2,1}$. Let $T(E_{1,4}) = E_{u,v}$. Then, either $E_{1,2} + E_{u,v}$ or $E_{2,1} + E_{u,v}$ is dominated by a full cycle, and hence as above, we arrive at a contradiction. Thus T maps pairs of off diagonal collinear cells to off diagonal collinear cells, and hence, T is a (P, P^t)-operator on $\mathcal{M}_n^{(0)}(\mathbb{B})$. That is T is a sum of a (P, P^t)-operator on $\mathcal{M}_n^{(0)}(\mathbb{B})$ and, since T is bijective, any bijection on Δ_n.

For the converse, note that primitive matrices of exponent $2n-2$ are of two types. The first type has a sero main diagonal. The second type has a nonzero diagonal and each consists of a full cycle matrix with exactly one nonzero entry on the main diagonal. For the first type, a bijection on Δ_n does not change the marix and a (P, P^t)-operator on $\mathcal{M}_n^{(0)}(\mathbb{B})$ preserves primitive matrixces and their exponent. For matrices of the second type, a (P, P^t)-operator on $\mathcal{M}_n^{(0)}(\mathbb{B})$ maps a full cycle matrix to a full cycle matrix and any bijection on Δ_n maps a diagonal with exactly one nonzero entry to a diagonal with exactly one nonzero entry. Thus, a (P, P^t)-operator on $\mathcal{M}_n^{(0)}(\mathbb{B})$ plus any bijection on Δ_n preserves primitive matrices of exponent $2n-2$. ■

Based on the above, we make the following conjecture:

Conjecture 22 *For n and k at least 3, if $T : \mathcal{M}_n(\mathbb{B}) \to \mathcal{M}_n(\mathbb{B})$ is a linear operator that strongly preserves $\mathcal{E}_k$ and $\mathcal{E}_k \neq \emptyset$, then either $k \leq 2n-2$ and T is a (P, P^t)-operator on $\mathcal{M}_n^{(0)}(\mathbb{B})$ plus a bijection on Δ_n, or $k > 2n-2$ and T is the sum of a (P, P^t)-operator on $\mathcal{M}_n^{(0)}(\mathbb{B})$ plus a diagonal transformation, $R : \Delta_n \to \mathcal{M}_n(\mathbb{B})$ such that $R(\Delta_n \setminus O) \subseteq (\mathcal{D}_{\mathcal{E}_k})^c$.*

We return now to considering nearly reducible primitive matrices. Let $K = J \setminus I$.

Theorem 23 *Let $n \geq 4$. Then, $T : \mathcal{M}_n(\mathbb{B}) \to \mathcal{M}_n(\mathbb{B})$ strongly preserves $\mathcal{N}RP_n$ if and only if T is a sum of a (P, P^t)-operator on $\mathcal{M}_n^{(0)}(\mathbb{B})$ and a nonsingular diagonal transformation that maps nonzero diagonal matrices to matrices in $(\mathcal{D}_{\mathcal{N}RP_n})^c$. That is, there exists a permutation matrix P such that $T(X) = P^t(X \circ K) + R(X \circ I)$ where R is any diagonal transformation on $\mathcal{M}_n(\mathbb{B})$ such that $X \circ I = O$ or $R(X \circ I) \in (\mathcal{D}_{\mathcal{N}RP_n})^c$.*

Proof. Suppose $T : \mathcal{M}_n(\mathbb{B}) \to \mathcal{M}_n(\mathbb{B})$ strongly preserves $\mathcal{N}RP_n$. Then T is the sum of two transformations, $T = T_o + T_1$ where $T_o = T(X \circ K)$ and $T_1(X) = T(X \circ I)$. Since any matrix in $\mathcal{N}RP_n$ has all diagonal entries zero, and T strongly preserves $\mathcal{N}RP_n$ we have that T_o maps $\mathcal{M}_n^{(0)}(\mathbb{B})$ to $\mathcal{M}_n^{(0)}(\mathbb{B})$, is bijective by Lemma 19, and strongly preserves $\mathcal{N}RP_n$. By Lemma 8(2) we only need show that T maps off diagonal collinear cells to off diagonal collinear cells in order to show that T_o is a (P, P^t)-operator.

If n is even, then the sum of any three non collinear cells together with a proper choice of $n-2$ other cells is in $\mathcal{N}RP_n$ while no matrix with three collinear cells is dominated by an element of $\mathcal{N}RP_n$. Thus, every strong preserver of $\mathcal{N}RP_n$ maps collinear cells to collinear cells.

If n is odd and T_o maps three collinear cells to non collinear cells, then if the image of the sum of these three cells does not dominate a digon the image of some matrix not in $\mathcal{N}RP_n$ is in $\mathcal{N}RP_n$, a contradiction. Suppose that T_o maps three collinear cells to three cells which dominate a digon. Then there are two cells whose image is a digon. These two collinear cells can be added to $n-1$ off diagonal cells to give an element of $\mathcal{N}RP_n$, but their image dominates a digon and hence, by Proposition 17, cannot be in $\mathcal{N}RP_n$, a contradiction.

Thus, whether n is even or odd, T_o maps off diagonal collinear cells to off diagonal collinear cells, and hence, is a (P, P^t)-operator.

Now, if $X \circ I \neq O$ then X is not nearly reducible, and any matrix that dominates X cannot be in $\mathcal{N}RP_n$. Thus, $T(X)$ is not dominated by any member of $\mathcal{N}RP_n$. It follows that $T_1(X \circ I)$ is not dominated by any member of $\mathcal{N}RP_n$ unless $X \circ I = O$.

Thus $T(X) = T_o(X \circ K) + T_1(X \circ I)$ is a sum of a (P, P^t)-operator on $\mathcal{M}_n^{(0)}(\mathbb{B})$ and a nonsingular diagonal transformation that maps nonzero diagonal matrices to matrices in $(\mathcal{D}_{\mathcal{N}RP_n})^c$. ■

4 Strong preservers of sets of primitive matrices in $\mathcal{M}_n(\mathbb{S})$

The fact that the primitivity of a matrix and its exponent do not depend on the nature of the nonzero entries, only on the fact that they are nonzero, gives that any linear operator $T : \mathcal{M}_n(\mathbb{S}) \to \mathcal{M}_n(\mathbb{S})$ preserves some property of primitive matrices if and only if $\overline{T} : \mathcal{M}_n(\mathbb{B}) \to \mathcal{M}_n(\mathbb{B})$ preserves that property of primitive matrices. Thus we state without proof the following theorems that will be a summary of the results of this paper on primitive matrices:

Theorem 24 *Let $n \geq 4$, $\mathbb{S}$ be an antinegative semiring without zero divisors, and $T : \mathcal{M}_n(\mathbb{S}) \to \mathcal{M}_n(\mathbb{S})$ be a linear operator. Then T strongly preserves $\mathcal{E}_{2n-2}$ if and only if there is a permutation matrix $P \in \mathcal{M}_n(\mathbb{S})$, a matrix $B \in \mathcal{M}_n(\mathbb{S})$ with all nonzero entries, and a bijective operator $R : \Delta_n \to \Delta_n$ such that either $T(X) = P((X \circ K) \circ B)P^t + R((X \circ I) \circ B)$ for all $X \in \mathcal{M}_n(\mathbb{S})$ or $T(X) = P((X \circ K) \circ B)^t P^t + R((X \circ I) \circ B)$ for all $X \in \mathcal{M}_n(\mathbb{S})$*

Theorem 25 *Let $n \geq 4$, $\mathbb{S}$ be an antinegative semiring with no zero divisors, and $T : \mathcal{M}_n(\mathbb{S}) \to \mathcal{M}_n(\mathbb{S})$ be a linear operator. Then, T strongly preserves $\mathcal{N}RP_n$ if and only if there is a permutation matrix $P \in \mathcal{M}_n(\mathbb{S})$, a matrix $B \in \mathcal{M}_n(\mathbb{S})$ with all nonzero entries, and a operator $R : \Delta_n \to \mathcal{M}_n(\mathbb{S})$ with $R(\Delta_n \setminus O) \subset (\mathcal{N}RP_n)^c$ such that either $T(X) = P((X \circ K) \circ B)P^t + R((X \circ I) \circ B)$ for all $X \in \mathcal{M}_n(\mathbb{S})$ or $T(X) = P((X \circ K) \circ B)^t P^t +$*

$R((X \circ I) \circ B)$ *for all* $X \in \mathcal{M}_n(\mathbb{S})$.

References

[1] L. B. Beasley, D. E. Brown and A. E. Guterman, Preserving regular tournaments and term rank-1. *Linear Algebra and its Applications*, 431(2009), 926–936.

[2] L. B. Beasley, K.-T. Kang, and S.-Z.Song. Linear preservers of term ranks of matrices over semirings. *Linear Algebra and its Applications*, 436(2012), 1850–1862.

[3] L. B. Beasley, K.-T. Kang, and S.-Z.Song. Preservers of Term Ranks of Symmetric Matrices. *Linear Algebra and its Applications*, 436(2012),1727–1738.

[4] L. Beasley, K.-T. Kang and S.-Z. Song. Linear operators that preserve sets of primitive matrices. *Journal of the Korean Mathematical Society*, in press.

[5] L. B. Beasley, K.-T. Kang, S.-Z.Song and N.-S. Sze. Regular Matrices and Their Strong Preservers over Semirings. *Linear Algebra and its Applications*, 429(2008), 209–223.

[6] L. B. Beasley and N. J. Pullman. Linear operators that strongly preserve primitivity. *Linear and Multilinear Algebra*, 25(1989), 205–213.

[7] L. B. Beasley and N.J. Pullman. Linear operators strongly preserving idempotent matrices over semirings. *Linear Algebra and its Applications*, 160(1992), 217–229.

[8] L. B. Beasley and N.J. Pullman. Linear operators that strongly preserve upper ideals of matrices. *Congressus Numerantium*, 88(1992), 229–244.

[9] L. B. Beasley and N. J. Pullman. Linear operators that strongly preserve the index of imprimitivity. *Linear and Multilinear Algebra*, 31(1992), 267–283.

[10] R. A. Brualdi and J. A. Ross. On the exponent of aa primitive, nearly reducible matrix. *Mathematics of Operations Research*, 5(1980), 229–241.

[11] R. A. Brualdi and H. Ryser. *Combinatorial Matrix Theory.* (Cambridge University Press, New York, 1991).

[12] Y. Chen and X Zhao. On linear operators strongly preserving invariants of Boolean matrices. *Czechoslovak Mathematical Journal,* 62(187)(2012), :169-186.

[13] J. D. Davis. Rigid embedding of simple groups in the general linear group. *Canad. J. Mathl,* 29(1977), :384-391.

[14] G. Fröbenius. Uber die Darstellung der entlichen Gruppen durch Linear Substitutionen. *S. B. Deutsch. Akad. Wiss. Berlin,* (1897), 994–1015.

[15] R. A. Horn and C. R. Johnson. *Matrix Analysis.* (Cambridge University Press, New York, 1985).

[16] C. K. Li and S. Pierce. Linear preserver problems. *American Mathematical Monthly,* 108(2008), 591–605.

[17] H.-H. Li and Y.-J, Tan. Linear operators that strongly preserve nilpotent matrices over semirings. *Northeast Mathematical Journal,* 23(2007), 71–86.

[18] S.-H. Lin, B.-S. Tam. Strong linear preservers of symmetric doubly stochastic or doubly substochastic matrices. *Linear Algebra and its Applications,* 379(2004), 179-200.

[19] J. W. Moon. *Topics in Tournaments.* (Holt, New York, 1968).

[20] S. Pierce and others. A Survey of Linear Preserver Problems. *Linear and Multilinear Algebra,* 33(1992), 1–119.

[21] J. A. Ross. On the exponent of a primitive, nearly reducible matrix. II. *SIAM Journal on Algebraic and Discrete Methods,* 3(1982), 395–410.

[22] S.-Z. Song, K.-T. Kang and Y.-B. Jun. Linear preservers of Boolean nilpotent matrices. *Journal of the Korean Mathematical Society,* 43(2006), 539–552.

[23] J.-L. Xu, C.-G. Cao and X.-M. Tang. Linear Preservers of Idempotence on Triangular Matrix Spaces over any Field. *International Mathematical Forum,* 2(2007), 2305–2319.