Uniform i-spotty-byte error control codes*

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Abstract. Irregular-spotty-byte error control codes were devised by the author in [2] and their properties were further studied in [3] and [4]. These codes are suitable for semi-conductor memories where an I/O word is divided into irregular bytes not necessarily of the same length. The i-spotty-byte errors are defined as t_i or fewer bit errors in an i-byte of length n_i where $1 \leq t_i \leq n_i$ and $1 \leq i \leq s$. However, an important and practical situation is when i-spotty-byte errors caused by the hit of high energetic particles are confined to i-bytes of the same size only which are aligned together or in words errors occur usually in adjacent RAM chips at a particular time. Keeping this view, in this paper, we propose a new model of i-spotty-byte errors viz. uniform i-spotty-byte errors and present a new class of codes viz. uniform i-spotty-byte error control codes which are capable of correcting all uniform i-spotty-byte errors of i-spotty measure μ (or less). The study made in this paper will be helpful in designing modified semi-conductor memories consisting of irregular RAM chips with those of equal length aligned together.

AMS Subject Classification (2000): 94B05

Keywords: i-spotty-byte errors, i-spotty-byte distance, error control codes

1. Introduction

The i-spotty-byte error control codes devised by the author [2-4] generalizes the usual notion of spotty-byte error control codes [1, 5-7]. In i-spotty-byte error control codes, an I/O word is divided into irregular bytes not necessarily of the same length in contrast to the spotty-byte error control codes where an I/O word is divided into regular bytes of the same length "b". Also, in i-spotty-byte error control codes, a RAM chip corresponds to

*This research is supported by the National Board for Higher Mathematics (NBHM), Department of Atomic Energy, Government of India, under a Research Project vide ref. no. 2/48(17)/2012 NBHM(R.P.)/R&D II/9134 dated 22/07/2013.

an i-byte and all RAM chips are physically independent. However, all RAM chips of the same size are aligned together and constitute a "sector".

A practical situation is when semi-conductor memories with i-byte arrangement are exposed to strong electromagnetic waves, radioactive particles or energetic cosmic particles, then the errors caused due to a single hit are confined to i-bytes of the same size constituting a "sector". Considering this situation, this paper proposes a new model of i-spotty-byte errors viz. uniform i-spotty-byte errors and present codes for the correction of the same followed by a decoding algorithm.

2. Definitions and Notations

Let $q = p^m$ be a power of prime number p and $\mathbf{F}_q$ be the finite field with q elements. A partition, P, of a positive integer N is defined as

$$P : N = m_1 + m_2 + \cdots + m_g, 1 \le m_1 \le m_2 \cdots \le m_g \quad g \ge 1.$$

and is denoted as

$$P = [m_1][m_2]\cdots[m_g] = [n_1]^{\lambda_1}[n_2]^{\lambda_2}\cdots[n_s]^{\lambda_s},$$

if

$$\begin{aligned}
&m_1 = m_2 = \cdots = m_{\lambda_1} = n_1, \\
&m_{\lambda_1+1} = m_{\lambda_1+2} = \cdots = m_{\lambda_1+\lambda_2} = n_2, \\
&\vdots \\
&\vdots \\
&\vdots \\
&m_{\lambda_1+\lambda_2+\cdots+\lambda_{s-1}+1} = m_{\lambda_1+\lambda_2\cdots\lambda_{s-1}+2} \\
&= \cdots = m_{\lambda_1+\lambda_2+\cdots+\lambda_s} = n_s.
\end{aligned}$$

Then we can write the field $\mathbf{F}_q^N$ as

$$\begin{aligned}
\mathbf{F}_q^N &= \mathbf{F}_q^{m_1} \oplus \mathbf{F}_q^{m_2} \oplus \cdots \oplus \mathbf{F}_q^{m_g} \\
&= \bigoplus_{i=1}^{s}\Big(\bigoplus_{\lambda_i - copies} \mathbf{F}_q^{n_i}\Big).
\end{aligned}$$

Each vector $v \in \mathbf{F}_q^N = \bigoplus_{i=1}^{s}(\bigoplus_{\lambda_i-copies} \mathbf{F}_q^{n_i})$ can be uniquely written as $v = (v_1, v_2, \cdots, v_s)$ where $v_j \in (\mathbf{F}_q^{n_j})^{\lambda_j}$ for all $1 \leq j \leq s$ and is represented as

$$v_j = (v_j^1, v_2^2, \cdots, v_j^{\lambda_j}), \quad v_j^a \in \mathbf{F}_q^{n_j} \text{ for all } 1 \leq a \leq \lambda_j, \tag{1}$$

or equivalently

$$\begin{aligned} v_j &= (v_j^{(1,1)}, v_j^{(1,2)}, \cdots, v_j^{(1,n_j)}, (v_j^{(2,1)}, v_j^{(2,2)}, \cdots, v_j^{(2,n_j)}, \cdots \\ &\quad \cdots, v_j^{(\lambda_j,1)}, v_j^{(\lambda_j,2)}, \cdots, v_j^{(\lambda_j,n_j)}), \end{aligned}$$

where $v_j^a = (v_j^{(a,1)}, v_j^{(a,2)}, \cdots, v_j^{(a,n_j)})$, $v_j^{(a,b)} \in \mathbf{F}_q$ for all $i \leq a \leq \lambda_j$ and $1 \leq b \leq n_j$.

Here $v_j (1 \leq j \leq s)$ is called the "j^{th} sector of v" consisting of λ_j i-bytes viz. $v_j^1, v_j^2, \cdots, v_j^{\lambda_j}$ each of length n_j. Thus the length of the j^{th} sector v_j is $\lambda_j n_j$. The partition P is named as *primary partition* or *irregular-byte partition*. Further, let $1 \leq T \leq N$ be a positive integer such that $P' : T = [t_1]^{\lambda_1}[t_2]^{\lambda_2} \cdots [t_s]^{\lambda_s}$ be a partition of T where $1 \leq t_i \leq n_i$ for all $1 \leq i \leq s$ and also $1 \leq t_1 \leq t_2 \leq \cdots \leq t_s$. Then P' is called as the *"secondary partition"* or *"error partition"*. Note that the secondary partition depends upon the primary partition. The number N is called the *primary number* and the number T is called the *secondary number.*

Clearly,

$$N = \lambda_1 n_1 + \lambda_2 n_2 + \cdots + \lambda_s n_s$$

and

$$T = \lambda_1 t_1 + \lambda_2 t_2 + \cdots + \lambda_s t_s.$$

We give below few definitions given in [2] with slight modifications.

Definition 2.1 [2]. Let N and T be the primary and secondary numbers respectively as discussed in the preceeding paragraph corresponding to the partitions P and P' resp. given by

$$\begin{aligned} P : N &= [n_1]^{\lambda_1}[n_2]^{\lambda_2} \cdots [n_s]^{\lambda_s}, \\ &\text{and} \\ P' : T &= [t_1]^{\lambda_1}[t_2]^{\lambda_2} \cdots [t_s]^{\lambda_s}, \end{aligned}$$

where $1 \leq t_i \leq n_i$ for all $1 \leq i \leq s$.

Let $v = (v_1, v_2, \cdots, v_s)$ be a vector in $\mathbf{F}_q^N = \bigoplus_{i=1}^{s} \Big(\bigoplus_{\lambda_i - copies} \mathbf{F}_q^{n_i} \Big)$ as given in (1). The *irregular-spotty-byte weight* (or simply *i-spotty-byte weight*) $w_\beta^{(P,P')}(v)$ corresponding to the primary partition P and secondary partition P' is given by

$$w_\beta^{(P,P')}(v) = \sum_{i=1}^{s} \sum_{a=1}^{\lambda_i} \left\lceil \frac{\sum_{b=1}^{n_i} w_H(v_i^{(a,b)})}{t_i} \right\rceil, \tag{2}$$

where $\sum_{b=1}^{n_i} w_H(v_i^{(a,b)})$ is the Hamming weight of the a^{th} i-byte in the i^{th} sector v_i and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

Definition 2.2 [2]. The *irregular-spotty distance* (or simply *i-spotty distance*) between two vectors $u, v \in \mathbf{F}_q^N$ corresponding to the primary partition P and secondary partition P' is given by

$$\begin{aligned} d_\beta^{(P,P')}(u,v) = w_\beta^{(P,P')}(u-v) &= \sum_{i=1}^{s} \sum_{a=1}^{\lambda_i} \left\lceil \frac{\sum_{b=1}^{n_i} w_H(u_i^{(a,b)} - v_i^{(a,b)})}{t_i} \right\rceil \\ &= \sum_{i=1}^{s} \sum_{a=1}^{\lambda_i} \left\lceil \frac{\sum_{b=1}^{n_i} d_H(u_i^{(a,b)}, v_i^{(a,b)})}{t_i} \right\rceil, \end{aligned} \tag{3}$$

where $\sum_{b=1}^{n_i} d_H(u_i^{(a,b)}, v_i^{(a,b)})$ is the Hamming distance between the a^{th} i-bytes of the i^{th} sectors u_i and v_i of u and v respectively. Then i-spotty-byte distance is a metric function.

Note. We also call the i-spotty weight and i-spotty distance as "t_i/n_i-weight" and "t_i/n_i-distance" respectively. Moreover, we simply denote the i-spotty weight $w_\beta^{(P,P')}$ and i-spotty distance $d_\beta^{(P,P')}$ by w_β and d_β respectively when the primary partition P and secondary partition P' are clear from the context.

Definition 2.3 [2]. Let T and N be the primary and secondary numbers corresponding to the primary and secondary partitions P and P' resp. where P and P' are given by

$$\begin{aligned} P : N &= [n_1]^{\lambda_1}[n_2]^{\lambda_2}\cdots[n_s]^{\lambda_s}, \\ P' : T &= [t_1]^{\lambda_1}[t_2]^{\lambda_2}\cdots[t_s]^{\lambda_s}, \end{aligned}$$

and $1 \le t_i \le n_i$ for all $1 \le i \le s$.

Let $V \subseteq \mathbf{F}_q^N = \bigoplus_{i=1}^{s}\Big(\bigoplus_{\lambda_i-copies} \mathbf{F}_q^{n_i}\Big)$ be an $\mathbf{F}_q$ subspace of $\mathbf{F}_q^N$ equipped with the i-spotty-byte metric d_β. Then V is called an *"irregular-spotty-byte"* (or simply *"i-spotty-byte"*) error control code and is denoted by $[N, k, d_\beta; P, P']$ where

$$\begin{aligned} N &= n_1\lambda_1 + n_2\lambda_2 + \cdots + n_s\lambda_s \\ &= \text{length of the code}, \\ k &= \dim_{\mathbf{F}_q}(V), \text{ and} \\ d_\beta &= \min_{\substack{x,y\in V \\ x\neq y}} d_\beta(x,y). \end{aligned}$$

3. Uniform i-spotty-byte error control codes

In this section, we define uniform i-spotty-byte errors and then design codes to control these type of errors. We begin with the definition of vectors of i-spotty weight or i-spotty measure $\mu(\mu \ge 1)$ in relation to Definition 2.1.

Definition 3.1. Let $v = (v_1, v_2, \cdots, v_s) \in \mathbf{F}_q^N = \bigoplus_{i=1}^{s}(\bigoplus_{\lambda_i-copies} \mathbf{F}_q^{n_i})$. If $w_\beta(v) = w_\beta^{(P,P')}(v) = \mu$, where $w_\beta^{(P,P')}(v)$ is given by (2), then we may say that i-spotty-weight or i-spotty measure of v is $\mu(\mu \ge 1)$ or equivalently we say that t_i/n_i-measure of v is μ.

Definition 3.2. A *"uniform i-spotty-byte error"* of i-spotty measure μ is an error vector of i-spotty measure μ in which all the erroneous digits are confined to i-bytes of the same sector.

Example 3.3. Let $N = 13, T = 9$ and

$$P := N = 13 = [1]^3[2]^2[3]^2,$$
$$P' : T = 9 = [1]^3[1]^2[2]^2,$$

be the primary and secondary partitions corresponding to $N = 13$ and $T = 9$ respectively. Then

$$u = (000\vdots 00 \;\; 00\vdots 110 \;\; 011) \in \mathbf{F}_2^{13}$$

is a uniform i-spotty-byte error of measure 2. But $v = (010\vdots 01 \;\; 00\vdots 000 \;\; 000) \in \mathbf{F}_2^{13}$ is not a uniform i-spotty-byte error of measure 2.

Note. (i) It is to be noted that $b_j = \left\lceil \frac{n_j}{t_j} \right\rceil, 1 \leq j \leq s$ is the maximum number of t_j/n_j-errors that can occur in any i-byte of th j^{th} sector and $\lambda_j b_j$ is the maximum number of t_j/n_j-errors that can occur in the j^{th} sector of length $\lambda_j n_j$ of a received word.

(ii) Let θ_{z_j} be the number of (erroneous) i-bytes in the j^{th} sector $(1 \leq j \leq s)$ having z_j number of i-spotty-byte errors where $z_j = 1, 2, \cdots b_j$.

Let

$$\begin{aligned} \sigma_j &= \theta_1 + \theta_2 + \cdots + \theta_{b_j} \\ &= \text{total number of erroneous i-bytes in the } j^{th} \text{ sector.} \end{aligned}$$

Then the total number of i-byte in the j^{th} sector of a word is expressed as

$$\begin{aligned} \sigma_j &= \sigma_j + \theta_0 \\ &= \theta_0 + \theta_1 + \theta_2 + \cdots + \theta_{b_j}. \end{aligned}$$

Definition 3.4 [5]**.** Given a monic primitive polynomial $g(x)$ of degree r over $\mathbf{F}_q$, the $r \times r$ companion matrix M corresponding to $g(x)$ is defined as follows:

$$g(x) = g_0 + g_1 x + g_2 x^2 + \cdots\cdots + g_{r-2}x^{r-2} + g_{r-1}x^{r-1} + x^r,$$

$$M = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -g_0 \\ 1 & 0 & \cdots & 0 & 0 & -g_1 \\ 0 & 1 & \cdots & 0 & 0 & -g_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -g_{r-2} \\ 0 & 0 & \cdots & 0 & 1 & -g_{r-1} \end{pmatrix}_{r \times r}$$

Observations.

(i) Let α be a primitive element of $\mathbf{F}_q^r$ and a root of $g(x)$. Its companion matrix M has its columns $\begin{pmatrix} \vdots \\ \vdots \\ \alpha^i \\ \vdots \\ \vdots \end{pmatrix}$ for $i = 1$ to r where $\begin{pmatrix} \vdots \\ \vdots \\ \alpha^i \\ \vdots \\ \vdots \end{pmatrix}$ is the coefficient vector of $x^i (mod\ g(x))$.

The companion matrix of α^j is M^j and its column vectors are expressed as follows:

$$M^j = \begin{pmatrix} \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ \alpha^j & \alpha^{j+1} & \cdots & \alpha^{j+r-1} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \end{pmatrix}_{r \times r} .$$

Let e be the exponent of $g(x)$, that is, $y = e$ is the least positive solution of $x^y \equiv (mod\ g(x))$. The companion matrix M has the following properties [5]:

(a) M is non singular.

(b) $M^0 = M^e = I_r$.

(c) $M^i = M^j$ if and only if $i \equiv j (mod\ e)$.

Now, we present the code construction method of uniform i-spotty-byte error control codes. Using the following definition:

Definition 3.5. Let $\mu, n_1 \leq n_2 \leq \cdots \leq n_s$ and $t_1 \leq t_2 \leq \cdots \leq t_s$ be positive integers with $1 \leq t_i \leq n_i$ for all $1 \leq i \leq s$. Let l and r be the positive integers such that

$$l \geq \max_{i=1}^{s}\{2\mu t_i\} \quad \text{and} \quad r \geq \max_{i=1}^{s}\{\mu t_i\}.$$

Further, for $i = 1$ to s, let

(i) $H'_i = [h'_{i,1}, h'_{i,2} \cdots h'_{i,n_i}], h'_{i,k} \in \mathbf{F}_q^l$ be $l \times n_i$ matrices over $\mathbf{F}_q$ satisfying the following two properties:

  (a) Every set of $2\mu t_i$ (or fewer) columns of H'_i are linearly independent over $\mathbf{F}_q$.

  (b) Every set of $\mu(t_i + t_j)$ (or fewer) columns with μt_i(or fewer) columns taken from H'_i and μt_j(or fewer) columns taken from $H'_j (1 \leq i, j, , \leq s)$ are linearly independent over $\mathbf{F}_q$.

(ii) $H''_i = [h''_{i,1}, h''_{i,2} \cdots h''_{i,n_i}], h''_{i,j} \in \mathbf{F}_q^r$ for all $1 \leq j \leq n_i$, be $r \times n_i$ matrices over $\mathbf{F}_q$ such that every set of μt_i (or fewer) columns of H''_i are linearly independent over $\mathbf{F}_q$.

Theorem 3.6. *Using the notations as given in Definitions 3.5, let M be an $r \times r$ companion matrix over $\mathbf{F}_q$. Let $m = q^r - 1$. For each $1 = 1$ to s, let λ_i be the positive integers satisfying $1 \leq \lambda_i \leq m$ for all i. Then the null space of $H = [H_1, H_2, \cdots, H_s]$, where each $H_i (1 \leq i \leq s)$ is a $(l + (2\mu - 1)r) \times \lambda_i n_i$ submatrix given by*

$$H_i = \begin{pmatrix} H'_i & H'_i & H'_i & \cdots & H'_i \\ M^0 H''_i & M^1 H''_i & M^2 H''_i & \cdots & M^{(\lambda_i - 1)} H''_i \\ M^0 H''_i & M^2 H''_i & M^4 H''_i & \cdots & M^{2(\lambda_i - 1)} H''_i \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ M^0 H''_i & M^{(2\mu-1)} H''_i & M^{2(2\mu-1)} H''_i & \cdots & M^{(2\mu-1)(\lambda_i - 1)} H''_i \end{pmatrix}_{(l+(2\mu-1)r) \times \lambda_i n_i}$$

is a uniform i-spotty-byte error control code V correcting all uniform i-spotty-byte errors of measure μ (or less) and having check but length $R = l + (2\mu - 1)r$ and code length $N = \lambda_1 n_1 + \lambda_1 n_2 + \cdots + \lambda_1 n_s$. The parameters of the resulting code will be

$$[N, N - R, d; P, P'],$$

where $P : N = [n_1]^{\lambda_1}[n_2]^{\lambda_2}\cdots[n_s]^{\lambda_s}$, $P' : T = [t_1]^{\lambda_1}[t_2]^{\lambda_2}\cdots\cdots[t_s]^{\lambda_s}$ and $d \leq 2\mu + 1$.

Proof. It suffices to prove that the code V which is the null space of H detects all i-spotty-byte errors of measure 2μ or less with errors confined to at most two sectors meaning thereby that the code corrects all uniform i-spotty-byte errors of measure μ or less.

Let $e \in \mathbf{F}_q^N = \bigoplus_{j=1}^{s}\Big(\bigoplus_{\lambda_j - copies} \mathbf{F}_q^{n_j}\Big)$.

Then e is of the form

$$\begin{aligned} e &= (e_1 \cdots e_s) \\ &= (e_1^0, e_1^1, \cdots, e_1^{\lambda_1 - 1}, \cdots, e_2^0, e_2^1, \cdots e_2^{\lambda_2 - 1}, \cdots, e_s^0, e_s^1, \cdots, e_s^{\lambda_s - 1}), \end{aligned}$$

where $e_j^{u_j} \in \mathbf{F}_q^{n_j}$ for all $1 \leq j \leq s$ and $0 \leq u_j \leq \lambda_j - 1$.

Suppose $w_\beta(e) \leq 2\mu$ with erroneous i-bytes confined to at most two sectors. We claim that $eH^T \neq 0$.

There are two cases to consider:

Case 1. When there is only one erroneous sector, say j^{th} sector with erroneous i-bytes say $e_j^{u_1}, e_j^{u_2}, \cdots, e_j^{u_{j^*}}$ with

$$\sum_{k=1}^{j^*} \left\lceil \frac{w_H(e_j^{u_k})}{t_j} \right\rceil \leq 2\mu.$$

Then the Hamming weight of the j^{th} sector $e_j = (e_j^0, e_j^1, \cdots e_j^{\lambda_j - 1})$ in e is less than or equal to $2\mu t_j$. Since H_j' is an $l \times n_j$ q-ary matrix whose every set of $2\mu t_j$ (or fewer) columns are linearly independent over $\mathbf{F}_q$, therefore, we must have $eH^T \neq 0$.

Case 2. When the number of erroneous sectors in e is equal to 2.

Let e_j and e_k be the erroneous sectors in e such that $e_j^{u_1}, e_j^{u_2}, \cdots, e_j^{u_{j^*}}$ be the erroneous i-bytes in e_j; $e_k^{v_1}, e_k^{v_2}, \cdots, e_k^{v_{k^*}}$ be the erroneous i-bytes in e_k, where

$$\sum_{\pi = j,k} \sum_{\eta = u_1 \cdots u_{j^*}, v_1 \cdots v_{k^*}} \left\lceil \frac{w_H(e_\pi^\eta)}{t_\pi} \right\rceil \leq 2\mu,$$

and

$$0 \le u_1, u_2, \cdots, u_{j^*} \le \lambda_j - 1,$$
$$0 \le v_1, \cdots, v_{k^*} \le \lambda_k - 1.$$

Then $eH^T = 0$ gives the following relation:

$$\begin{aligned}
& e_j^{u_1}\left[H_j'^T \quad (M^{u_1}H_j'')^T \quad (M^{2u_1}H_j'')^T \cdots (M^{(2\mu-1)u_1}H_j'')^T\right] \\
& +e_j^{u_2}\left[H_j'^T \quad (M^{u_2}H_j'')^T \quad (M^{2u_2}H_j'')^T \cdots (M^{(2\mu-1)u_2}H_j'')^T\right] \\
& +\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& +e_j^{u_{j^*}}\left[H_j'^T (M^{u_{j^*}}H_j'')^T \quad (M^{2u_{j^*}}H_j'')^T \cdots (M^{(2\mu-1)u_{j^*}}H_j'')^T\right] \qquad (4) \\
& +e_k^{v_1}\left[H_k'^T \quad (M^{v_1}H_k'')^T \quad (M^{2v_1}H_k'')^T \cdots (M^{(2\mu-1)v_1}H_k'')^T\right] \\
& +\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& +e_k^{v_{k^*}}\left[H_k'^T \quad (M^{v_{k^*}}H_k'')^T \quad (M^{(2v_{k^*}}H_k'')^T \cdots (M^{(2\mu-1)v_{k^*}}H_k'')^T\right] \\
= \ & [O_l \quad O_r \quad O_r \cdots\cdots O_r],
\end{aligned}$$

where O_l and O_r are the $1 \times l$ and $1 \times r$ null matrices over $\mathbf{F}_q$ respectively.
The relation

$$\left(\sum_{\rho=u_1}^{u_{j^*}} e_j^\rho\right) H_j'^T + \left(\sum_{w=v_1}^{v_{k^*}} e_k^w\right) H_k'^T = O_l$$

leads to

$$\sum_{\rho=u_1}^{u_{j^*}} e_j^\rho = O_{n_j} \quad \text{and} \quad \sum_{w=v_1}^{v_{k^*}} e_k^w = O_{n_k},$$

because of property (i) (b) of Matrix H_i' given in Definition 3.5.

Multiplying the equation $\sum_{\rho=u_1}^{u_{j^*}} e_j^\rho = O_{n_j}$ by $(H_j'')^T$, $\sum_{w=v_1}^{v_{k^*}} e_k^w = O_{n_k}$ by $(H_k'')^T$ from right gives

$$\left(\sum_{\rho=u_1}^{u_{j^*}} e_j^\rho\right) H_j''^T = O_r, \qquad \text{and}$$

$$\left(\sum_{w=v_1}^{v_{k*}} e_k^w\right) H_k''^T = O_r.$$

The following equation from (4) is obtained:

$$\begin{aligned}
&\left[(e_j^{u_1} H_j''^T)(M^{u_1})^T \cdots\cdots (e_j^{u_1} H_j''^T)(M^{(2\mu-1)u_1})^T\right] \\
&+\left[(e_j^{u_2} H_j''^T)(M^{u_2})^T \cdots\cdots (e_j^{u_2} H_j''^T)(M^{(2\mu-1)u_2})^T\right] \\
&+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&+\left[(e_j^{u_{j*}} H_j''^T)(M^{u_{j*}})^T \cdots\cdots (e_j^{u_{j*}} H_j''^T)(M^{(2\mu-1)u_{j*}})^T\right] \\
&+\left[(e_k^{v_1} H_k''^T)(M^{v_1})^T \cdots\cdots (e_k^{v_1} H_k''^T)(M^{(2\mu-1)v_1})^T\right] \\
&+\left[(e_k^{v_2} H_k''^T)(M^{v_2})^T \cdots\cdots (e_k^{v_2} H_k''^T)(M^{(2\mu-1)v_2})^T\right] \\
&+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&+\left[(e_k^{v_{k*}} H_k''^T)(M^{v_{k*}})^T \cdots\cdots (e_k^{v_{k*}} H_k''^T)(M^{(2\mu-1)v_{k*}})^T\right] \\
&= [O_r \quad O_r \quad \cdots O_r].
\end{aligned} \tag{5}$$

Let $e_j^{u_1} H_j''^T, e_j^{u_2} H_j''^T, \cdots, e_j^{u_{j*}} H_j''^T$ be denoted by $r_{u_1}, r_{u_2}, \cdots, r_{u_{j*}}$ and $e_k^{v_1} H_k''^T$, $e_k^{v_2} H_k''^T, \cdots, e_k^{v_{k*}} H_k''^T$ be denoted by $r_{v_1}, r_{v_2}, \cdots, r_{v_{k*}}$ resp. Then (5) can be rewritten as

$$\begin{aligned}
&r_{u_1} + \cdots + r_{u_{j*}} + r_{v_1} + \cdots + r_{v_{k*}} = O_r \\
&r_{u_1}(M^{u_1})^T + \cdots + r_{u_{j*}}(M^{u_{j*}})^T + r_{v_1}(M^{v_1})^T + \cdots\cdots \\
&+ r_{v_{k*}}(M^{v_{k*}})^T = O_r \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&r_{u_1}(M^{(2\mu-1)u_1})^T + \cdots + r_{u_{j*}}(M^{(2\mu-1)u_{j*}})^T + r_{v_1}(M^{(2\mu-1)v_1})^T \\
&+ \cdots + r_{v_{k*}}(M^{(2\mu-1)v_{k*}})^T = O_r.
\end{aligned} \tag{6}$$

Writing the above equation in the matrix form gives

$$(r_{u_1}, \cdots, r_{u_{j*}}, r_{v_1}, \cdots, r_{v_{k*}}) \times$$

$$\times \begin{pmatrix} 1 & (M^{u_1})^T & \dots & (M^{(2\mu-1)u_1})^T \\ \vdots & \vdots & \vdots & \vdots \\ 1 & (M^{u_{j^*}})^T & \dots & (M^{(2\mu-1)u_{j^*}})^T \\ 1 & (M^{v_1})^T & \dots & (M^{(2\mu-1)v_1})^T \\ \vdots & \vdots & \vdots & \vdots \\ 1 & (M^{v_{k^*}})^T & \dots & (M^{(2\mu-1)v_{k^*}})^T \end{pmatrix}$$

$$= (O_r \quad O_r \quad \cdots O_r),$$

or equivalently

$$(r_{u_1}, \cdots, r_{u_{j^*}}, r_{v_1}, \cdots, r_{v_{k^*}}) \times$$

$$\times \begin{pmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ M^{u_1} & \dots & M^{u_{j^*}} & M^{v_1} & \dots & M_{v_{k^*}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ M^{(2\mu-1)u_1} & \dots & M^{(2\mu-1)u_{j^*}} & M^{(2\mu-1)v_1} & \dots & M^{(2\mu-1)v_{k^*}} \end{pmatrix}^T$$

$$= (O_r \quad O_r \quad \cdots O_r).$$

Since the total numbers of erroneous i-bytes in the two erroneous sectors is $j^* + k^* = p + 1$ (say) which is less than or equal to 2μ, therfore, writing the above matrix equation for the top $p + 1 (\leq 2\mu)$ relations, we get

$$(r_{u_1}, \cdots, r_{u_{j^*}}, r_{v_1}, \cdots, r_{v_{k^*}}) \times$$

$$\times \begin{pmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ M^{u_1} & \dots & M^{u_{j^*}} & M^{v_1} & \dots & M_{v_{k^*}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ M^{pu_1} & \dots & M^{pu_{j^*}} & M^{pv_1} & \dots & M^{pv_{k^*}} \end{pmatrix}^T$$

$$= (O_r \quad O_r \quad \cdots O_r).$$

The coefficient matrix in the above equation being Vandermonde's matrix is non-singular. Therefore, relations (6) have a solution given by $r_{u_1} = \cdots = r_{u_{j^*}} = r_{v_1} = \cdots = r_{v_{k^*}} = O_r$.

This implies that

$$e_j^{u_1} H_j''^T = \cdots\cdots = e_j^{u_{j^*}} H_j''^T = e_k^{v_1} H_k''^T = \cdots = e_k^{v_{k^*}} H_k''^T = O_r$$

which further gives

$$e_j^{u_1} = \cdots\cdots = e_j^{u_{j^*}} = O_{n_j} \text{ and } e_k^{v_1} = \cdots = e_k^{v_{k^*}} = O_{n_k},$$

as every set of μt_j (or fewer) columns of H_j'' and every set of μt_k (or fewer) columns of H_k'' are linearly independent over $\mathbf{F}_q$. A contradiction. Hence $eH^T \neq 0$. □

Example 3.7. Let $q = 2$, $n_1 = 4, n_2 = 2, t_1 = 2, t_2 = 1, \lambda_1 = 3, \lambda_2 = 2$, $\mu = 1$, $l = 4$ and $r = 3$. Further, let

$$H_1' = I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{4\times 4}, \quad H_2' = I_4 \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}_{4\times 2},$$

$$H_1'' = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}_{3\times 4}, \quad H_2'' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}_{3\times 2}.$$

Then

(i) Every set of 4 (or fewer) columns of H_1' are linearly independent over $\mathbf{F}_2$;

(ii) Every set of 1 column of H_2' is linearly independent over $\mathbf{F}_2$;

(iii) Every set of columns with 2 (or fewer) columns taken from H_1' and 1 (or fewer) column taken from H_2' are linearly independent over $\mathbf{F}_2$;

(iv) Every set of 2 (or fewer) columns of H_1'' is linearly independent over $\mathbf{F}_2$;

(v) Every single column of H_2'' is linearly independent over $\mathbf{F}_2$.

Let $\alpha \in \mathbf{F}_2^3$ be a root of the primitive polynomial $g(x) = x^3 + x + 1 \in \mathbf{F}_2[x]$. Then the null space of H where

$$H = \left[\begin{array}{ccc:cc} H_1' & H_1' & H_1' & H_2' & H_2' \\ \alpha^0 H_1'' & \alpha^1 H_1'' & \alpha^2 H_1'' & \alpha^0 H_2'' & \alpha^1 H_2'' \end{array} \right],$$

$$\alpha^i H_1'' = \left[\begin{array}{cccc} \alpha^i & \alpha^{i+1} & \alpha^{i+2} & \alpha^{i+3} \end{array} \right], \quad 0 \le i \le 2,$$

and

$$\alpha^j H_2'' = \left[\begin{array}{cc} \alpha^j & \alpha^{j+3} \end{array} \right], \quad 0 \le j \le 1,$$

is a uniform i-spotty-byte error control code that corrects all uniform i-spotty-byte errors of measure 1 and having check bit length $R = 7$ and code length $N = 16$.

Example 3.8. Let $q = 2$, $n_1 = 4, n_2 = 3, t_1 = t_2 = 1, \lambda_1 = 4, \lambda_2 = 5$ and $\mu = 2$. Let $l = 6$ and $r = 3$. Let $\alpha \in \mathbf{F}_2^3$ be a primitive element defined by $g(x) = x^3 + x + 1 \in \mathbf{F}_2[x]$. The 3×3 companion matrix for $g(x)$ is given as

$$M = \left[\begin{array}{ccc} \alpha & \alpha^2 & \alpha^3 \end{array} \right] = \left[\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{array} \right]_{3\times 3},$$

Let

$$H_1' = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]_{6\times 4}, \quad H_2' = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{array} \right]_{6\times 3},$$

$$\begin{aligned} H_1'' &= \left[\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right]_{3\times 4} = \left[\begin{array}{cccc} 1 & \alpha & \alpha^2 & \alpha^3 \end{array} \right], \\ H_2'' &= I_3 = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]_{3\times 3}. \end{aligned}$$

Then

(i) All the 4 columns of H_1' are linearly independent over $\mathbf{F}_2$;

(ii) All the 3 columns of H_2' are linearly independent over $\mathbf{F}_2$;

(iii) Every set of columns such that 2 (or fewer) columns taken from H_1' and 2 (or fewer) column taken from H_2' is linearly independent over $\mathbf{F}_2$;

(iv) Every set of 2 (or fewer) columns of $H_i''(i = 1, 2)$ is linearly independent over $\mathbf{F}_2$.

Then the null space of H where

$$H = \begin{pmatrix} H_1' & H_1' & H_1' & H_1' & \vdots & H_2' & H_2' & H_2' & H_2' & H_2' \\ M^0 H_1'' & M^1 H_1'' & M^2 H_1'' & M^3 H_1'' & \vdots & M^0 H_2'' & M^1 H_2'' & M^2 H_2'' & M^3 H_2'' & M^4 H_2'' \\ M^0 H_1'' & M^2 H_1'' & M^4 H_1'' & M^6 H_1'' & \vdots & M^0 H_2'' & M^2 H_2'' & M^4 H_2'' & M^6 H_2'' & M^8 H_2'' \\ M^0 H_1'' & M^3 H_1'' & M^6 H_1'' & M^9 H_1'' & \vdots & M^0 H_2'' & M^3 H_2'' & M^6 H_2'' & M^9 H_2'' & M^{12} H_2'' \end{pmatrix}$$

is a uniform i-spotty-byte error correcting code that corrects all uniform i-spotty-byte errors of measure 2 (or less) and having check bit length $R = 15$ and code length $N = 31$.

4. Decoding of uniform i-spotty-byte error correcting codes

Let V be a uniform i-spotty-byte error correcting code that corrects all uniform i-spotty-byte errors of measure μ or less.Let c, v and e be a codeword of V, a received word and an error vector respectively. The syndrome S is calculated as

$$\begin{aligned} S &= [S_0 \quad S_1 \quad S_2 \cdots S_{2\mu-1}] \\ &= vH^T = (c+e)H^T = eH^T, \end{aligned}$$

where $S_0 \in \mathbf{F}_q^l$ is an l-bit q-ary row vector and $S_p \in \mathbf{F}_q^r, 1 \leq p \leq 2\mu - 1$ is an r-bit q-ary row vector. If μ or fewer uniform i-spotty-byte errors occur in the j^{th} sector e_j with erroneous i-bytes $e_j^{u_1}, e_j^{u_2}, \cdots, e_j^{u_{j^*}}$ $(j^* \leq \mu)$ such that

$$w_\beta(e_j^{u_1}) + w_\beta(e_j^{u_2}) + \cdots + w_\beta(e_j^{u_{j^*}}) \leq \mu,$$

then the syndrome S is given by:

$$S = \begin{bmatrix} S_0 \\ S_1 \\ \vdots \\ \vdots \\ S_{2\mu-1} \end{bmatrix}^T$$

$$= \begin{bmatrix} e_j^{u_1} H_j'^T + e_j^{u_2} H_j'^T + \cdots + e_j^{u_{j^*}} H_j''^T \\ (e_j^{u_1} H_j''^T)(M^{u_1})^T + (e_j^{u_2} H_j''^T)(M^{u_2})^T \cdots + \\ \cdots + (e_j^{u_{j^*}} H_j''^T)(M^{u_{j^*}})^T \\ \vdots \\ \vdots \\ (e_j^{u_1} H_j''^T)(M^{(2\mu-1)u_1})^T + (e_j^{u_2} H_j''^T)(M^{(2\mu-1)u_2})^T \cdots + \\ \cdots + (e_j^{u_{j^*}} H_j''^T)(M^{(2\mu-1)u_{j^*}})^T \end{bmatrix}^T$$

Let

$$e_j^* = e_j^{u_1} + e_j^{u_2} + \cdots + e_j^{u_{j^*}}.$$

Then the relation

$$S_0 = \sum_{\rho=u_1}^{u_{j^*}} e_j^\rho H_j'^T = e_j^* H_j'^T$$

can determine the sector number j and sum of erroneous i-byte e_j^* uniquely because of the fact that the matrices $H_i'(1 \leq i \leq s)$ satisfy the conditions (i)(a) and (i)(b) of Definition 3.5. Now multiply e_j^* by $n_j \times r$ q-ary matrix $H_j''^T$ from right gives

$$e_j^* H_j''^T \in \mathbf{F}_q^r.$$

Let us denote $e_j^{u_1} H_j''^T, e_j^{u_2} H_j''^T, \cdots, e_j^{u_{j^*}} H_j''^T$ by $r_{u_1}, \cdots, r_{u_{j^*}}$ respectively where $r_{u_1}, r_{u_2} \cdots, r_{u_{j^*}} \in \mathbf{F}_q^r$.

Let α be a root of $g(x)$ which defines the companion matrix M. The operation $r_\rho (M^\rho)^T, (u_1 \leq \rho \leq u_{j^*})$ is equivalent to the product of r_ρ and α^ρ over $\mathbf{F}_q^r$. We write the new syndromes S' as given below:

$$S' = \begin{bmatrix} S_0' \\ S_1 \\ S_2 \\ \vdots \\ \vdots \\ \vdots \\ S_{2\mu-1} \end{bmatrix}^T$$

$$
= \begin{bmatrix} r_{u_1} + r_{u_2} + \cdots + r^{u_{j^*}} \\ r_{u_1}\alpha^{u_1} + r_{u_2}\alpha^{u_2} + \cdots + r_{u_{j^*}}\alpha^{u_{j^*}} \\ r_{u_2}\alpha^{2u_1} + r_{u_2}\alpha^{2u_2} + \cdots + r_{u_{j^*}}\alpha^{2u_{j^*}} \\ \vdots \\ \vdots \\ \vdots \\ r_{u_1}\alpha^{(2\mu-1)u_1} + r_{u_2}\alpha^{(2\mu-1)u_2} + \cdots + r_{u_{j^*}}\alpha^{(2\mu-1)u_{j^*}} \end{bmatrix}^T . \quad (7)
$$

The syndrome S' given in (7) is identical to that of RS code with minimum Hamming distance $(2\mu + 1)$ over $\mathbf{F}_q^r$. The error patters over $\mathbf{F}_q^r$ and error locations are determined by using the existing decoding algorithms of RS codes such as Berlekemp-Massey algorithm.

In the final step of decoding, the error patterns $\hat{e}_j^\rho \in \mathbf{F}_q^{n_j}$ where $\rho = u_1, u_2, \cdots, u_{j^*}$ are transformed from the corresponding r-bit error patterns $r_\rho \in \mathbf{F}_q^r$ according to one-to-one mapping from r_ρ to $\hat{e}_j^\rho$ for $\rho = u_1, u_2, \cdots, u_{j^*}$. This mapping is implemented by the table as discussed in [1, 5]. Here, at most, one of the $\hat{e}_j^\rho$ may be miscorrected, that is, $\hat{e}_j^\rho \neq e_j^\rho$. The following relation proven in [1, 5] determines whether or not $\hat{e}_j^\rho = e_j^\rho$. That is, if $\hat{e}_j^\rho$ satisfies the relation (8), then $\hat{e}_j^\rho = e_j^\rho$, otherwise not.

$$
w_\beta(\hat{e}_j^\rho + e_j^*) \leq \mu - w_\beta(\hat{e}_j^\rho). \quad (8)
$$

Summarizing the above discussion, the decoding is performed according to the following algorithm:

Step 1. The erroneous sector number j and the sum of erroneous i-bytes e_j^* is obtained by the relation $S_0 = e_j^* H_j'^T$ which is satisfied only for a unique $j (1 \leq j \leq s)$.

Step 2. The first element S_0 in S is transformed to $S_0' \in \mathbf{F}_q^r$ by the operation $S_0' = e_j^* H_j''^T$.

Step 3. Error locations $u_1, u_2, \cdots, u_{j^*}$ and error patterns $r_{u_1}, r_{u_2}, \cdots, r_{u_{j^*}}$ are determined from the syndrome S' by the decoding algorithm of the RS code over $\mathbf{F}_q^r$.

Step 4. The error pattern $\hat{e}_j^\rho$ is obtained from $r_\rho (u_1 \leq \rho \leq u_{j^*})$ according to the mapping table discussed in [1, 5].

Step 5. The error patterns $\hat{e}_j^\rho, \rho = u_1, u_2, \cdots, u_{j^*}$, obtained in the previous step are checked whether or not they satisfy the relation (8). If satisfied, then $\hat{e}_j^\rho = e_j^\rho$.

Step 6. If $\hat{e}_j^\sigma$, for some σ, does not satisfy the relation (8) or cannot be transformed from r_σ in the mapping table, the error pattern e_j^σ is recovered from the other error patterns obtained in Step 5 as follows:

$$e_j^\sigma = e_j^* - \sum_{\substack{\rho=u_1 \\ \rho\neq\sigma}}^{u_{j^*}} e_j^\rho.$$

5. Conclusion.

In this paper, we have presented a new class of i-spotty-byte-codes viz. Uniform i-spotty-byte error control codes and discussed their design method and decoding algorithm.

Acknowledgment. The author would like to thank her spouse Dr. Arihant Jain for his constant support and encouragement for pursuing research.

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