A Contribution to Upper Domination, Irredundance and Distance-2 Domination in Graphs

Gerd H. Fricke
Chris Schroeder
Department of Mathematics, Computer Science, and Physics
Morehead State University
Morehead, KY 40351
and
Sandra M. Hedetniemi
Stephen T. Hedetniemi, Professor Emeritus
School of Computing
Renu C. Laskar, Professor Emerita
Department of Mathematical Sciences
Clemson University
Clemson, SC 29634

Abstract

Let $G = (V, E)$ be a graph. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u|uv \in V\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *open neighborhood* of set S of vertices is the set $N(S) = \bigcup_{v \in S} N(v)$, while the *closed neighborhood* of a set S is the set $N[S] = \bigcup_{v \in S} N[v]$. A set $S \subset V$ *dominates* a set $T \subset V$ if $T \subseteq N[S]$, written $S \rightarrow T$. A set $S \subset V$ is a *dominating set* if $N[S] = V$; and is a *minimal dominating set* if it is a dominating set, but no proper subset of S is also a dominating set; and is a γ*-set* if it is a dominating set of minimum cardinality. In this paper we consider the family $\mathcal{D}$ of all dominating sets of a graph G, the family $\mathcal{MD}$ of all minimal dominating sets of a graph G, and the family γ of all γ-sets of a graph G. The study of these three families of sets provides new characterizations of the *distance-2 domination number*, the *upper domination number* and the *upper irredundance number* in graphs.

JCMCC **100** (2017), pp. 97-112

1 Introduction

Let $G = (V, E)$ be a graph. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u | uv \in V\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *open neighborhood* of set S of vertices is the set $N(S) = \bigcup_{v \in S} N(v)$, while the *closed neighborhood* of a set S is the set $N[S] = \bigcup_{v \in S} N[v]$. A set $S \subset V$ is a *dominating set* if $N[S] = V$. The minimum cardinality of a dominating set in a graph G is called the *domination number* of G, and is denoted $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is called a γ*-set* of G.

A dominating set S is *minimal* if no proper subset $S' \subset S$ is also a dominating set. A dominating set is 1*-minimal* if for every vertex $u \in S$, the set $S - \{u\}$ is not a dominating set. The property of being a dominating set is *super-hereditary* in the sense that every superset of a dominating set is also a dominating set. Because of this, it can be shown [2] that a dominating set is *minimal* if and only if it is 1*-minimal*.

An equivalent definition of a dominating set is a set S having the property that every vertex $v \in V - S$ is adjacent to, or within distance-1 of, some vertex $u \in S$. A set S is, therefore, a *distance-*2 *dominating set* if every vertex $v \in V - S$ is within distance-2 of some vertex $u \in S$. The minimum cardinality of a distance-2 dominating set in a graph G is denoted $\gamma_{\leq 2}(G)$. For a thorough discussion of distance domination in graphs the reader is refereed to the chapter on this subject written by Henning in [4].

A set $S \subset V$ is said to be *irredundant* if for every vertex $u \in S$, $N[u] - N[S - \{u\}] \neq \emptyset$. If $pn[u, S] = N[u] - N[S - \{u\}] \neq \emptyset$ for a vertex $u \in S$ then we say that u has a *private neighbor with respect to* S, and every vertex in $pn[u, S]$ is said to be a *private neighbor of* u. The condition that for every vertex $u \in S$, $N[u] - N[S - \{u\}] \neq \emptyset$ is equivalent to the condition that for every vertex $u \in S$, $(N[u] \cap V) - (N[S - \{u\}] \cap V) \neq \emptyset$. Thus, one could say that every irredundant set $S \subset V$ is *irredundant with respect to* V. Let $IR(G)$ denote the maximum cardinality of an irredundant set in a graph G.

A set $S \subset V$ is *independent* if no two vertices in S are adjacent. Notice that according to the definition of an irredundant set, every independent set is irredundant since $u \in pn[u, S] \neq \emptyset$, for every vertex $u \in S$, that is, every vertex in S is its own private neighbor.

In this paper we consider the family $\mathcal{D}$ of all dominating sets of a graph G, the subfamily $\mathcal{MD} \subset \mathcal{D}$ consisting of all minimal dominating sets of a graph G, and the subfamily $\gamma \subset \mathcal{MD}$ consisting of all γ-sets of a graph G.

The study of these three families of sets provides new characterizations of *distance-2 domination* and *irredundance* in graphs.

2 Dominators of Sets of Vertices

A set $S \subset V$ *dominates* a set $T \subset V$ if $T \subseteq N[S]$, written $S \rightarrow T$. In this case we also say that S is a *dominator* of T. The *domination number of a set* T, denoted $\gamma(T)$, equals the minimum cardinality of a dominator S of T. The following results provide some insights into the nature of dominators in graphs.

Proposition 1 *Every set S is a dominator of itself, that is, $S \rightarrow S$, although S might not be a minimal dominator of itself.*

Proposition 2 *A set S is a minimal dominator of itself if and only if S is an independent set.*

Proposition 3 *Every set $S \subseteq V$ in a graph G has a minimal dominator S' such that $S' \subseteq S$.*

Proposition 4 *Every set $S \subset V$ in a graph G has a minimal independent dominator S' such that $S' \subseteq S$.*

Proposition 5 *If $S' \rightarrow S$ and S' is a minimal dominator of S, then $|S'| \leq |S|$.*

From these propositions we are led to ask: when does a set S have a disjoint dominator S', that is $S' \rightarrow S$ and $S' \cap S = \emptyset$?

A partial answer to this question is given by Ore's classical theorem [7].

Theorem 1 *[Ore] The complement $V - S$ of a minimal dominating set S of a graph G without isolated vertices is a dominating set.*

Thus, every minimal dominating set S in a graph without isolated vertices has a disjoint dominator. But the complete answer to this question is given by the following. Given a set $S \subset V$, an *enclave* is a vertex $u \in S$ such that $N[u] \subseteq S$; thus a vertex $u \in S$ is an enclave if every neighbor of u is also a vertex in S. Notice that if a vertex $u \in S$ is an isolated vertex in G, then u is an enclave in S.

Proposition 6 *A set $S \subset V$ has a disjoint dominator if and only if S does not contain an enclave.*

Proposition 7 *If a set $S \subset V$ has a disjoint dominator, then it has a minimal disjoint dominator.*

We say that a set $S \subset V$ is *irredundant with respect to a set* T if for every vertex $u \in S$, $(N[u] \cap T) - (N[S - \{u\} \cap T) \neq \emptyset$. In this case we say that every vertex $u \in S$ has a *private neighbor with respect to* T.

Proposition 8 *If S is a minimal dominating set of a set T, then S is irredundant with respect to T and irredundant with respect to V.*

Proof. Assume that S is a minimal dominating set of a set T in a graph $G = (V, E)$. If S minimally dominates T then for every vertex $u \in S$, $S - \{u\}$ does not dominate T.

Case 1. If $S \cap T \neq \emptyset$ then this can happen in three ways:
(i) $u \in S - T$ and has a private neighbor in T.
(ii) $u \in S \cap T$ and is isolated in S and in T, and is therefore its own private neighbor with respect to T,
(iii) $u \in S \cap T$, is not isolated in T and has a private neighbor in T other than itself.

Case 2. If $S \cap T = \emptyset$, then S minimally dominates T if and only if every vertex $u \in S$ has a private neighbor in T, that is, it dominates a vertex in T that no other vertex in S dominates. □

3 Edge Covers

An *edge cover* of a set $S \subset V$ in a graph $G = (V, E)$ is a set $M \subset E$ of edges such that every vertex in S is contained in an edge in M. The *edge cover number* of a set S (or of a graph G) denoted $\alpha_1(S)$ (or $\alpha_1(G)$) is the minimum cardinality of an edge cover of S (or of V). Define the following three parameters:

$\alpha_{1\mathcal{D}}(G) = min\{\alpha_1(D),\ D \in \mathcal{D}\}$.
$\alpha_{1\mathcal{MD}}(G) = min\{\alpha_1(D),\ D \in \mathcal{MD}\}$.
$\alpha_{1\gamma}(G) = min\{\alpha_1(D),\ D \in \gamma\}$.

A set $S \subset V$ is called a *paired dominating set* if S is a dominating set and the subgraph $G[S]$ induced by S has a *perfect matching*, that is, a set M of independent edges (no two edges have a vertex in common) and $V(M) = S$, that is the set of vertices contained in an edge in M is precisely S. The *paired domination number*, denoted $\gamma_{pr}(G)$, equals the minimum cardinality of a paired dominating set in G.

We say that an edge $e = uv$ *ev-dominates a vertex* $x \in V$ if either $x = u$, or $x = v$, or x is adjacent to either u or v. The *ev-domination number*, denoted $\gamma_{ev}(G)$, equals the minimum cardinality of a set of edges that ev-dominates every vertex in V. This concept was studied by Lewis in his Ph.D. thesis [6].

The following theorem was discovered by Haynes, Hedetniemi and Hedetniemi [3].

Theorem 2 *For any graph* $G = (V, E)$, $\alpha_{1\mathcal{D}}(G) = \gamma_{pr}(G)/2 = \gamma_{ev}(G)$.

This theorem, in its full generality, is the basis for this paper.

4 Distance-2 Domination

Motivated by the definitions of $\alpha_{1\mathcal{D}}(G)$, $\alpha_{1\mathcal{MD}}(G)$, and $\alpha_{1\gamma}(G)$, we define the following three new parameters, where the *domination number of a set* $D \subseteq V$ in a graph $G = (V, E)$, denoted $\gamma(D)$, equals the minimum cardinality of a set $S \subseteq V$ such that $S \rightarrow D$. We make the following three definitions, as follows:

$\gamma_{\mathcal{D}}(G) = min\{\gamma(D),\ D \in \mathcal{D}\}$.
$\gamma_{\mathcal{MD}}(G) = min\{\gamma(D),\ D \in \mathcal{MD}\}$.
$\gamma_{\gamma}(G) = min\{\gamma(D),\ D \in \gamma\}$.

Proposition 9 *Any dominator S of a dominating set D is a distance-2 dominating set of G.*

Theorem 3 *For any graph* G, $\gamma_{\leq 2}(G) \leq \gamma_{\mathcal{D}}(G) \leq \gamma_{\mathcal{MD}}(G) \leq \gamma_{\gamma}(G) \leq \gamma(G)$.

Proof. The first inequality follows immediately from Proposition 9. The next two inequalities follow from the definitions. The fourth inequality, that $\gamma_{\gamma}(G) \leq \gamma(G)$, follows from Proposition 5. □

Theorem 4 *For any graph G without isolated vertices,* $\gamma_{\leq 2}(G) = \gamma_{\mathcal{D}}(G)$.

Proof. The fact that $\gamma_{\leq 2}(G) \leq \gamma_{\mathcal{D}}(G)$, again, follows from Proposition 9, that is, any dominating set of a dominating set is a distance-2 dominating set. It only remains to show that $\gamma_{\leq 2}(G) \geq \gamma_{\mathcal{D}}(G)$. Let S be a $\gamma_{\leq 2}$-set of a graph G, that is, a distance-2 dominating set of minimum cardinality. Let $S_1 = N[S] - S$ and let $S_2 = V - S - S_1$. It follows therefore that $V = S \cup S_1 \cup S_2$ and $S \to S_1$.

It also follows that S_1 is a dominating set of G. Suppose not. Clearly, since S is a distance-2 dominating set of G, it must be the case that $S_1 \to S_2$. Thus, if S_1 is not a dominating set of G, then there must be a vertex $u \in S$ that is not adjacent to any vertices in S_1. Since we have assumed that G has no isolated vertices, it follows that u must be adjacent only to vertices in S. But in this case, if follows that $S - \{u\}$ is also a distance-2 dominating set: a contradiction.

It follows, therefore, that S is a dominator of a dominating set, S_1, of G, and therefore, $|S| \geq \gamma_{\mathcal{D}}(G)$. □

Corollary 1 *For any graph G,* $\gamma_{\leq 2}(G) = \gamma_{\mathcal{D}}(G)$.

Proof. The proof of the preceding theorem assumes that the graph G has no isolated vertices. But a similar proof can be constructed where the graph G has isolated vertices. If a graph G has k isolated vertices, and G' is the graph obtained from G by deleting these k isolated vertices, then it is easy to see that $\gamma_{\leq 2}(G) = k + \gamma_{\leq 2}(G')$. Similarly, it is easy to see that $\gamma_{\mathcal{D}}(G) = k + \gamma_{\mathcal{D}}(G')$. □

Corollary 2 *The distance-2 domination number of a graph G equals the minimum cardinality of a dominator of a dominating set of G.*

Corollary 3 *If a set S is any minimal distance-2 dominating set of a graph G without isolated vertices, then the set* $S_1 = N[S] - S$ *is a dominating set of G.*

An even stronger result than Theorem 4 exists.

Theorem 5 *For any graph G,* $\gamma_{\mathcal{D}}(G) = \gamma_{\mathcal{MD}}(G)$.

Proof. By definition it follows that $\gamma_{\mathcal{D}}(G) \leq \gamma_{\mathcal{MD}}(G)$. Thus, we must show that $\gamma_{\mathcal{D}}(G) \geq \gamma_{\mathcal{MD}}(G)$. Let $S \in \mathcal{D}$ be a dominating set for which $\gamma(S) = \gamma_{\mathcal{D}}(G)$. If S is a minimal dominating set, then clearly $\gamma_{\mathcal{MD}}(G) \leq \gamma_{\mathcal{D}}(G)$.

Thus, assume that no minimal dominating set S achieves $\gamma(S) = \gamma_{\mathcal{D}}(G)$. Let S be a non-minimal dominating set for which $\gamma(S) = \gamma_{\mathcal{D}}(G)$, and let S^* be a γ-set of S where $|S^*| = \gamma_{\mathcal{D}}(G)$. Now let S' be a minimal dominating set that is a subset of S. Note that S^* also dominates $S' \subset S$. But S^* may not be a γ-set of S'. So let S" be a γ-set of S'. Clearly,

$$|S"| \leq| S^*| = \gamma_{\mathcal{D}}(G).$$

Thus, $\gamma_{\mathcal{MD}}(G) \leq \gamma_{\mathcal{D}}(G)$. □

Corollary 4 *For any graph G, $\gamma_{\leq 2}(G) = \gamma_{\mathcal{D}}(G) = \gamma_{\mathcal{MD}}(G) \leq \gamma_{\gamma}(G) \leq \gamma(G)$.*

The path P_{19} of order $n = 19$ is an example of a graph for which

$$\gamma_{\leq 2}(P_{19}) = \gamma_{\mathcal{D}}(P_{19}) = \gamma_{\mathcal{MD}}(P_{19}) = 4 < \gamma_{\gamma}(P_{19}) = 6 < \gamma(P_{19}) = 7.$$

It is natural to ask when is $\gamma_{\gamma}(G) = \gamma(G)$? The answer is straightforward. A set $S \subseteq V$ is a *2-packing* if for every $u, v \in S$, $N[u] \cap N[v] = \emptyset$. An equivalent definition of a 2-packing is that for every vertex $v \in V$, $|N[v] \cap S| \leq 1$. Still a third definition of a 2-packing is that for every $u, v \in S$, $d(u, v) > 2$, where $d(u, v)$ equals the minimum length of a path between u and v.

Theorem 6 *For any graph G, $\gamma_{\gamma}(G) = \gamma(G)$ if and only if every γ-set of G is a 2-packing.*

Proof. If every γ-set of a graph G is a 2-packing, then the minimum cardinality of any dominator of a γ-set must be $\gamma(G)$, since no one vertex can dominate two or more vertices of a γ-set. Therefore, $\gamma_{\gamma}(G) = \gamma(G)$.

Conversely, if $\gamma_{\gamma}(G) = \gamma(G)$ then there cannot exist a γ-set S in which two vertices of S can be dominated by one vertex. This means that for any two vertices $u, v \in S$, $N[u] \cap N[v] = \emptyset$, i.e. S is a 2-packing. □

A dominating set S is *efficient* if for any vertex $v \in V$, $|N[v] \cap S| = 1$. This means that every vertex in $V - S$ is adjacent to exactly one vertex in S, and no two vertices in S are adjacent.

Corollary 5 *For any graph G, $\gamma_\gamma(G) = \gamma(G)$ if and only if every γ-set is efficient.*

The class of trees for which $\gamma_\gamma(T) = \gamma(T)$ can be characterized as follows. A *star* is a tree, denoted $K_{1,r}$, having r leaves, each of which is adjacent to a central vertex x of degree r. Let $\mathcal{T}(\gamma_\gamma)$ denote the class of trees that can be constructed recursively as follows. Let $T_0 = K_{1,r}$, for any $r \geq 2$. Let $T_{i+1} = T_i \circ K_{1,s}$, where T_{i+1} is constructed from a previously constructed tree T_i by adding a star $K_{1,s}$, and an edge joining a leaf of $K_{1,s}$ to a non-central vertex of T_i.

Theorem 7 *For a tree T, $\gamma_\gamma(T) = \gamma(T)$ if and only if $T \in \mathcal{T}(\gamma_\gamma)$.*

Proof. Note that, by construction, every γ-set of any tree $T \in \mathcal{T}$ must contain the central vertex in every star, and therefore is an efficient dominating set. Therefore, by Corollary 5, $\gamma_\gamma(T) = \gamma(T)$. Conversely, let $S = \{u_1, u_2, \ldots, u_k\}$ be any γ_γ-set of a tree T. Since $\gamma_\gamma(T) = \gamma(T)$, we know from Corollary 5 that S is an efficient dominating set of T. It follows that $\{N[u_1], N[u_2], \ldots, N[u_k]\}$ is a partition of $V(T)$ into stars, and therefore that T is a member of $\mathcal{T}$. □

Recall that $\alpha_{1\gamma}(G)$ equals the minimum cardinality of an edge cover of a γ-set of G.

Proposition 10 *For any graph G without isolated vertices, $\gamma_\gamma(G) \leq \alpha_{1\gamma}(G) \leq \gamma(G)$.*

Proof. Let S be any γ-set of a graph G without isolated vertices. Assume that $\gamma(G) = k$. By selecting k edges, one adjacent to each vertex in S, you will form an edge cover of S of cardinality k. Thus, $\alpha_{1\gamma}(G) \leq \gamma(G)$. Similarly, let $F \subseteq E$ be a minimum cardinality edge cover of a γ-set, say S, of G. Thus, $|F| = \alpha_{1\gamma}(G)$. By selecting any one vertex from each edge in F, you will form a dominating set of a γ-set S of G of cardinality at most $\alpha_{1\gamma}(G)$. Thus, $\gamma_\gamma(G) \leq \alpha_{1\gamma}(G)$. □

Question 1 *Is it true that $\gamma_{\leq 2}(T) = \gamma(T)$ if and only if $T \in \mathcal{T}(\gamma_\gamma)$?*

5 Independent domination

A set $S \subset V$ is an *independent dominating set* if it is both an independent set and a dominating set. The *independent domination number* $i(G)$ equals the minimum cardinality of an independent dominating set in G, while the *vertex independence number* $\beta_0(G)$ equals the maximum cardinality of an independent set in G.

Let $i(D)$ denote the minimum cardinality of an independent dominator of a set $D \subseteq V$.

Let $\Gamma(G)$ equal the maximum cardinality of a minimal dominating set in G. It is well known that for any graph G,

$$\gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G).$$

We have previously shown that for any graph G:

$$\gamma_{\leq 2}(G) = \gamma_{\mathcal{D}}(G) = \gamma_{\mathcal{MD}}(G) \leq \gamma_{\gamma}(G) \leq \gamma(G).$$

To these inequalities we can add several more, having to do with independent domination.

$i_{\mathcal{D}}(G) = min\{i(D),\, D \in \mathcal{D}\}$.
$i_{\mathcal{MD}}(G) = min\{i(D),\, D \in \mathcal{MD}\}$.
$i_{\gamma}(G) = min\{i(D),\, D \in \gamma\}$.
$i_{\leq 2}(G)$ = minimum cardinality of an independent distance-2 dominating set in G.

Proposition 11 *Any independent dominator S of a dominating set D is an independent distance-2 dominating set of G.*

Since the proof of the following theorem is virtually identical to that of Theorems 4 and 5, we omit the details.

Theorem 8 *For any graph G, $i_{\leq 2}(G) = i_{\mathcal{D}}(G) = i_{\mathcal{MD}}(G) \leq i_{\gamma}(G) \leq i(G)$.*

It can also be observed, as above, that for the path P_{19},

$$i_{\mathcal{MD}}(P_{19}) = 4 < i_{\gamma}(P_{19}) = 6 < i(P_{19}) = 7.$$

Proposition 12 *For any graph G, $\gamma_{\gamma}(G) \leq i_{\gamma}(G) \leq \gamma(G) \leq i(G)$.*

Proof. It is immediate, from the definitions, that $\gamma_{\gamma}(G) \leq i_{\gamma}(G)$ and $\gamma(G) \leq i(G)$. It only remains to show that $i_{\gamma}(G) \leq \gamma(G)$, but this follows immediately from Proposition 4.

6 Upper Domination

As we did for independent domination, we can define several parameters involving what is called *upper domination*, namely, the maximum cardinality of a minimal dominating set. Let $\Gamma(D)$ denote the maximum cardinality of a minimal dominator of a set D.

$\Gamma(G)$ = maximum cardinality of a minimal dominating set in G.
$\Gamma_{\mathcal{D}}(G) = max\{\Gamma(D),\ D \in \mathcal{D}\}$.
$\Gamma_{\mathcal{MD}}(G) = max\{\Gamma(D),\ D \in \mathcal{MD}\}$.
$\Gamma_{\gamma}(G) = max\{\Gamma(D),\ D \in \gamma\}$.
$\Gamma_{\Gamma}(G) = max\{\Gamma(D),\ D \in \Gamma\}$, of where $D \in \Gamma$ refers to the set of all minimal dominating sets D of cardinality $\Gamma(G)$.
$\Gamma_{\leq 2}(G)$ = maximum cardinality of a minimal distance-2 dominating set in G.

Proposition 13 *For any graph G, $\Gamma_{\Gamma}(G) \leq \Gamma_{\mathcal{MD}}(G) \leq \Gamma_{\mathcal{D}}(G) \leq |V|$.*

Proof. This follows from the definitions, since the classes $\Gamma \subseteq \mathcal{MD} \subseteq \mathcal{D}$, and the fact that $V \in \mathcal{D}$, and every minimal dominator of a set S has cardinality no greater than the cardinality of S. □

Proposition 14 *For every graph G, $\beta_0(G) \leq \Gamma(G) \leq \Gamma_{\Gamma}(G)$.*

Proof. Let S be any β_0-set of G. Then either $\beta_0(G) = \Gamma(G)$ or $\beta_0(G) < \Gamma(G)$.

If $\beta_0(G) = \Gamma(G)$, then S is both a β_0-set and a Γ-set. But then S minimally dominates itself. Therefore, $\Gamma_{\Gamma}(G) \geq \beta_0(G) = \Gamma(G)$.

If $\beta_0(G) < \Gamma(G)$, then let S be any Γ-set. Since S is a minimal dominating set, it is a maximal irredundant set. Therefore, every vertex $u \in S$ has a private neighbor, say u'. Either $u = u'$, i.e. u is its own private neighbor, or $u' \in V - S$ is an external private neighbor of u. Let $S = \{u_1, u_2, \ldots, u_k\}$ and let $S' = \{u'_1, u'_2, \ldots, u'_k\}$, where for $1 \leq i \leq k$, u'_i is a private neighbor of u_i. Then it is easy to see that S' is a minimal dominating set of S. Therefore, $\Gamma_{\Gamma}(G) \geq |S'| = |S| = \Gamma(G) > \beta_0(G)$. □.

Proposition 15 *For any graph G, $\Gamma_{\Gamma}(G) = \Gamma_{\mathcal{MD}}(G)$.*

Proof. By definition, and Proposition 13, we know that $\Gamma_{\Gamma}(G) \leq \Gamma_{\mathcal{MD}}(G)$. Therefore, we must show that $\Gamma_{\Gamma}(G) \geq \Gamma_{\mathcal{MD}}(G)$. For any minimal dominating set $S \in \mathcal{MD}$, it follows that $\Gamma(S) \leq |S|$. From this it follows that $\Gamma_{\mathcal{MD}}(G) \leq \Gamma(G)$. But from Proposition 14, it follows that $\Gamma_\Gamma(G) \geq \Gamma(G)$, from which it follows that $\Gamma_\Gamma(G) \geq \Gamma(G) \geq \Gamma_{\mathcal{MD}}(G)$. □

Proposition 16 *For any graph G, $\Gamma_{\mathcal{MD}}(G) = \Gamma(G)$.*

Proof. In the proof of Proposition 15 we show that $\Gamma_{\mathcal{MD}}(G) \leq \Gamma(G)$. It only remains to show that $\Gamma_{\mathcal{MD}}(G) \geq \Gamma_\Gamma(G) \geq \Gamma(G)$. But this is shown in the proof of Proposition 14. □

Proposition 17 *For any graph G having no isolated vertices, $\Gamma_{\mathcal{D}}(G) = IR(G)$.*

Proof. Let S_0 be any IR-set. Then S_0 is a distance-2 dominating set. Let $S_1 = N[S_0] - S_0$. It follows that S_1 is a dominating set of G, and therefore $S_1 \in \mathcal{D}$. It also follows that $S_0 \cup S_1$ is a dominating set of G, and therefore $S_0 \cup S_1 \in \mathcal{D}$. But since S_0 is a maximal irredundant set in G, S_0 minimally dominates $S_0 \cup S_1$, since each vertex of S_0 has a private neighbor in $S_0 \cup S_1$. Therefore, $\Gamma_{\mathcal{D}}(G) \geq IR(G)$.

Conversely, we must show that $\Gamma_{\mathcal{D}}(G) \leq IR(G)$. But for every minimal dominating set S of a dominating set S', $\Gamma(S) \leq |S'|$. But every minimal dominator S of a dominating set S' is irredundant. By definition, therefore, $\Gamma_{\mathcal{D}}(G) \leq IR(G)$. □

Proposition 18 *For any graph G without isolated vertices, $\Gamma_\gamma(G) = \gamma(G)$.*

Proof. Bollobas and Cockayne [1] have show that every graph G without isolated vertices has a γ-set S in which every vertex $u \in S$ has an external private neighbor, that is there exists a vertex $v \in V - S$ for which $N(v) \cap S = \{u\}$. Therefore, let $S' = \{u'_1, u'_2, \ldots, u'_k\}$ be a set of external private neighbors of the vertices $S = \{u_1, u_2, \ldots, u_k\}$, where u'_i is a private neighbor in $V - S$ of vertex $u_i \in S$. It then follows that S' is a minimal dominating set of a γ-set S in G, where $|S'| = |S| = \gamma(G)$. Therefore, $\Gamma_\gamma(G) \geq \gamma(G)$. But from Proposition 5 we know that every minimal dominating set S' of a set S has cardinality no greater than than of S. Therefore, $\Gamma_\gamma(G) \leq \gamma(G)$..
□

Proposition 19 *For any graph G,*

$$\gamma_\gamma(G) \leq i_\gamma(G) \leq \Gamma_\gamma(G) = \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) = \Gamma_\Gamma(G)$$
$$= \Gamma_{\mathcal{MD}}(G) \leq \Gamma_{\mathcal{D}}(G) = IR(G).$$

Proof. These inequalities follow from the definitions, Proposition 3 and Propositions 14, 15, 16, 17 and 18. □

7 Upper Distance-2 Domination

While distance-k domination, including distance-2 domination, has received some study, [4], very little is known about $\Gamma_{\leq 2}(G)$, which equals the maximum cardinality of a minimal distance-2 domination set in a graph G. As an illustration, consider the *subdivided star* $S(K_{1,n})$, which is the tree having one central vertex, say x, of degree n, n vertices of degree 2, each of which is adjacent to vertex x, and n leaves, each of which is adjacent to a vertex of degree 2. For this tree, $\gamma_{\leq 2}(S(K_{1,n})) = 1$. On the other hand, $\Gamma_{\leq 2}(S(K_{1,n})) = n$, since the set of n leaves is a minimal distance-2 dominating set. In addition, the set of n vertices of degree 2 is also a minimal distance-2 dominating set.

We ask the general question: which inequalities, if any, exist between the parameter $\Gamma_{\leq 2}(G)$ and the other parameters discussed in this paper?

In [5] the following characterization of minimal distance-k dominating sets is given.

Proposition 20 (Henning, Oellermann, Swart) *For $k \geq 1$, let S be a distance-k dominating set of a graph G. Then S is a minimal distance-k dominating set if and only if each vertex $u \in S$ satisfies at least one of the following two conditions:*

(i) there exists a vertex $v \in V - S$ such that the only vertex in S within distance-k of v is the vertex u, or

(ii) the vertex u is at distance at least $k+1$ from every other vertex in S.

The upper distance-2 domination number $\Gamma_{\leq 2}(G)$ is not comparable with either the vertex independence number $\beta_0(G)$, the independent domination number $i(G)$ or the domination number $\gamma(G)$. Consider the Cartesian product of the form $G = K_n \square P_3$, for $n \geq 4$. The columns of this graph consist of three copies of the complete graph K_n. The rows of this graph consist of n paths of length two. It is easy to see that the domination number, the independent domination number and the independence number of this graph all equal three. However, the set of n vertices in the complete graph in the first column forms a minimal distance-2 dominating set. Thus, it can be seen that $\Gamma_{\leq 2}(G) = n$.

On the other hand, for the star $G = K_{1,n}$, $\beta_0(G) = n > \Gamma_{\leq 2}(G) = 1$. Therefore, $\beta_0(G)$ and $\Gamma_{\leq 2}(G)$ are incomparable.

For the cycle $G = C_4$ or the cycle $G = C_5$, however, $\Gamma_{\leq 2}(G) = 1 < \gamma(G) = i(G) = 2$. Therefore, $\Gamma_{\leq 2}(G)$ is not comparable with either $\gamma(G)$ or $i(G)$.

On the other hand, a good upper bound for $\Gamma_{\leq 2}(G)$ exists.

Theorem 9 *For any graph G having no isolated vertices,* $\Gamma_{\leq 2}(G) \leq \Gamma_{\mathcal{D}}(G) = IR(G)$.

Proof. Let S be a $\Gamma_{\leq 2}$-set of a graph G. Obviously, S is a dominating set of $N[S]$. Let $S^* \subseteq S$ be a minimal dominating set of $N[S]$, that is, $N[S^*] = N[S]$. Then, $N[N[S^*]] = N[N[S]] = V$ and S^* is a distance-2 dominating set of G. By minimality of S, we have $|S^*| = |S|$, and thus, $S^* = S$. Hence, S is a minimal dominating set of $N[S]$. Any minimal dominating set of a subset of V is an irredundant set in G. Hence, $|S| \leq IR(G)$. □

This theorem raises the question about the relationship between $\Gamma_{\leq 2}(G)$ and $\Gamma(G)$.

We close with the following theorem.

Theorem 10 *For any graph G having no isolated vertices,* $\Gamma_{\leq 2}(G) \leq \Gamma(G)$.

Proof. Let G be a graph of order $n \geq 2$ such that G has no isolated vertices. Let S be a $\Gamma_{\leq 2}$ set of G. Since S is a minimum cardinality, distance-2 dominating set, it partitions V into three disjoint sets, $V = S \cup V_1 \cup V_2$, where V_1 is the set of vertices in $V - S$ having a neighbor in S, and V_2 is the set of vertices at distance-2 from S.

From Proposition 20 we know that every $x \in S$ has a distance-2 private neighbor (*d2pn*). Let $S_0 = \{x \in S | x \text{ is its own } d2pn\}$. Then, for every $x \in S_0$, and every $y \in S - \{x\}$, $d(x, y) \geq 3$.

Let $S_1 = \{x \in S - S_0 | x \text{ is adjacent to one of its distance-2 private neighbors}\}$. If $x \in S_1$, then there exists $v \in N(x)$ such that v is at least distance-3 away from $S - \{x\}$. That is, if $x \in S_1$, then there exists $y \in N(x)$ such that for every $z \in S - \{x\}$, $d(y, z) \geq 3$.

Finally, let $S_2 = S - (S_0 \cup S_1)$. Note that if $x \in S_2$, then every *d2pn* of x, say y, is distance-2 from x, and for every $z \in S - \{x\}$, $d(y, z) \geq 3$.

For every $x \in S$, let $P_x = N(x) - S = N(x) \cap V_1$ be the neighbors of $x \in V_1$. Now define:

- $P_0 = \bigcup_{x \in S_0} P_x = \{v | v \in N(x) \text{ for some } x \in S_0\} = N(S_0) \cap V_1$
- $P_1 = \bigcup_{x \in S_1} P_x = \{v | v \in N(x), x \in S_1\} = N(S_1) \cap V_1$
- $P_2 = \bigcup_{x \in S_2} P_x = \{v | v \in N(x) - S, x \in S_2\} = N(S_2) \cap V_1$

Notice that $V_1 = P_0 \cup P_1 \cup P_2$, since every vertex in V_1 has a neighbor in S and $S = S_0 \cup S_1 \cup S_2$. Notice also the following:

- V_1 is a dominating set of G.
- $P_x = N(x)$ if $x \in S_0 \cup S_1$, since $S_0 \cup S_1$ is an independent set.
- $P_x \neq \emptyset$ for any $x \in S$, that is, every vertex in S has a neighbor in V_1, since G has no isolated vertices.
- P_x dominates x for all $x \in S$.

Notice that V_1 dominates S. All vertices $v \notin S$ that are distance-1 away from some $x \in S$ are in V_1, and all vertices $v \notin S$ that are distance-2 from S, i.e. all vertices in V_2, are dominated by V_1, since every vertex in V_2 must have a neighbor in V_1.

Now, let $\hat{V} \subseteq V_1$ be a minimal dominating set of G. Then $\hat{V} - \{x\}$ is not a dominating set of G, for any $x \in \hat{V}_1$.

Now define the following:

- $\hat{P}_2 = P_2 \cap \hat{V}$
- $\hat{P}_1 = \left(\hat{V} - \hat{P}_2\right) \cap P_1 = \left(P_1 \cap \hat{V}\right) - \hat{P}_2$
- $\hat{P}_0 = \hat{V} - \left(\hat{P}_1 \cup \hat{P}_2\right) = P_0 \cap \hat{V}$

Since $\hat{V}$ is a dominating set, for each $x \in S_0$ there exists $v \in \hat{V}$ that dominates x. By the definition of $S_0, v \in \hat{V}$ cannot dominate two or more vertices in S. Thus, if v dominates $x \in S_0$, then $v \notin \hat{P}_1 \cup \hat{P}_2$; $v \in \hat{P}_0$, and $|\hat{P}_0| \geq |S_0|$.

Every $x \in S_1$ has a *d2pn* $v \in N(x) \cap V_1$. But $v \in V_1$ cannot be in $\hat{P}_0 \cup \hat{P}_2$ nor adjacent to a vertex in $\hat{P}_0 \cup \hat{P}_2$. Otherwise, v would not be a *d2pn*. Since $\hat{V}$ is a dominating set, v must be dominated by a vertex $z \in \hat{P}_1$, where either $z = v$ or z is adjacent to v. However, $z \in \hat{P}_1$ cannot dominate the *d2pn* v of x and also be adjacent to another vertex $w \in S_1$, where $w \neq x$, else v cannot be a private *d2pn* of $x \in S_1$. Hence the number of vertices in $\hat{P}_1$ that dominate a *d2pn* of S_1 is greater than or equal to $|S_1|$. Thus, $|\hat{P}_1| \geq |S_1|$.

Finally, let $x \in S_2$. Then there exists a *d2pn* $v \in V_2$ of x that is distance-2 from S_2. This vertex v is dominated by a vertex in $\hat{P}$. Since v is a *d2pn*, v is not dominated by any vertex in $\hat{P}_0 \cup \hat{P}_1$. Thus, v is dominated by a vertex $y \in \hat{P}_2$. Now y cannot dominate a *d2pn* w of any vertex in S_2 other than x. Thus, $|\hat{P}_2| \geq |S_2|$.

Thus, $\Gamma(G) \geq |\hat{P}| = |\hat{P}_0| + |\hat{P}_1| + |\hat{P}_2| \geq |S_0| + |S_1| + |S_2| = |S| = \Gamma_{\leq 2}(G)$.
□

8 Summary

In this paper we have provided a new characterization of minimum distance-2 domination, i.e. for any graph G,

$$\gamma_{\leq 2}(G) = \gamma_{\mathcal{D}}(G) = \gamma_{\mathcal{MD}}(G).$$

We have provided a new characterization for the upper domination number, i.e. for any graph G,

$$\Gamma(G) = \Gamma_{\Gamma}(G) = \Gamma_{\mathcal{MD}}(G).$$

We have also provided a new characterization of the upper irredundance number, i.e. for any graph G,

$$\Gamma_{\mathcal{D}}(G) = IR(G).$$

In addition we have provided the following new inequality chains:

$$\gamma_{\leq 2}(G) = \gamma_{\mathcal{D}}(G) = \gamma_{\mathcal{MD}}(G) \leq \gamma_{\gamma}(G) \leq \alpha_{1\gamma}(G) \leq \gamma(G), \text{ and}$$

$$\gamma_\gamma(G) \le i_\gamma(G) \le \Gamma_\gamma(G) = \gamma(G) \le i(G) \le \beta_0(G) \le \Gamma(G) = \Gamma_\Gamma(G)$$
$$= \Gamma_{\mathcal{MD}}(G) \le \Gamma_{\mathcal{D}}(G) = IR(G).$$

Finally, we showed that for any graph G, $\Gamma_{\le 2}(G) \le \Gamma(G)$, but $\Gamma_{\le 2}(G)$ is not comparable with either $\beta_0(G)$, $i(G)$, or $\gamma(G)$.

Acknowledgments

The authors would like to thank the referee who gave the paper a very thorough reading and made several good suggestions for improving the paper.

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