On Signless Laplacian Spectra of H-joins

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Abstract

We determine the signless Laplacian spectrum for the H-join of regular graphs $G_1, \ldots, G_p$. We also find an expression and upper bounds for the signless Laplacian spread of the H-join of regular graphs $G_1, \ldots, G_p$.

Keywords: signless Laplacian eigenvalues, signless Laplacian spread, graph, regular graph, H-join

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1 Introduction

We consider simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $V(G) = [n]$, let d_i be the degree of vertex i for $i \in [n]$. Let $D(G)$ be the degree diagonal matrix of G, where the (i,i)-entry of $D(G)$ is equal to the degree of vertex i, and $A(G) = (a_{ij})$ be the adjacency matrix of G, where $a_{ij} = 1$ if vertices i and j are adjacent in G and $a_{ij} = 0$ otherwise. The signless Laplacian matrix of G is defined as $Q(G) = D(G) + A(G)$.

The signless Laplacian spectrum (multiset of the eigenvalues) of G is the spectrum of $Q(G)$, denoted by

$$\sigma(G) = (\sigma_1(G), \sigma_2(G), \ldots, \sigma_n(G)),$$

where $\sigma_1(G) \geq \sigma_2(G) \geq \cdots \geq \sigma_n(G)$ are the signless Laplacian eigenvalues of G. The signless Laplacian spread of G is defined as $s(G) = \sigma_1(G) - \sigma_n(G)$. Recall that $\sigma_n(G) \geq 0$ with equality if and only if G has a bipartite component [6]. The signless Laplacian spectrum has received much attention, especially in recent years, see, e.g., [3–7, 11, 12].

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Consider a family of p graphs $\mathbb{F} = \{G_1, \ldots, G_p\}$, where each graph G_i has order n_i for $i = 1, \ldots, p$, and a graph H such that $V(H) = \{1, \ldots, p\}$. Each vertex $i \in V(H)$ is assigned to the graph $G_i \in \mathbb{F}$. The H-join of $G_1, \ldots, G_p$ is the graph $G = \bigvee_H \{G_i : i \in V(H)\}$ (or $G = H[G_1, \ldots, G_p]$) such that $V(G) = \cup_{i=1}^{p} V(G_i)$ and

$$E(G) = \left(\cup_{i=1}^{p} E(G_i)\right) \cup \left(\cup_{rs \in E(H)} \{uv : u \in V(G_r), v \in V(G_s)\}\right).$$

Then $|V(G)| = \sum_{i=1}^{p} n_i = n$, $|E(G)| = \sum_{i=1}^{p} |E(G_i)| + \sum_{ij \in E(H)} n_i n_j$. This graph operation was introduced in [10] under the name generalized composition and in [2] under the name H-join. The adjacency spectrum of $\bigvee_H \{G_i : i \in V(H)\}$ when G_i for $i = 1, \ldots, p$ are all regular was determined in [10]. The adjacency and Laplacian spectra of graphs produced by a generalized H-join operation on families of graphs (regular in the case of adjacency spectra and arbitrary in the case of Laplacian spectra) were recently determined in [2].

The rest of the paper is organized as follows. In Section 2, we give preliminary concepts and notions. In Section 3, we give the signless Laplacian spectrum, the expression and upper bounds for the signless Laplacian spread of the H-join of regular graphs $G_1, \ldots, G_p$.

2 Preliminaries

Let e_n be the column vector of size n with all entries equal to one, let $J_{i,j}$ be the $i \times j$ matrix whose entries are all ones, and let 0_n be a column zero vector of size n.

For $i, j \in V(H)$, let

$$\delta_{i,j}(H) = \begin{cases} 1 & \text{if } ij \in E(H) \\ 0 & \text{otherwise} \end{cases},$$

$N_H(i) = \{j : ij \in E(H)\}$ (the set of neighbors of vertex i in H), $N_i = \sum_{j \in N_H(i)} n_j$ and

$$M' = \begin{pmatrix} 2d_1 + N_1 & \delta_{1,2}\sqrt{n_1 n_2} & \cdots & \delta_{1,p}\sqrt{n_1 n_p} \\ \delta_{2,1}\sqrt{n_2 n_1} & 2d_2 + N_2 & \cdots & \delta_{2,p}\sqrt{n_2 n_p} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{p,1}\sqrt{n_p n_1} & \delta_{p,2}\sqrt{n_p n_2} & \cdots & 2d_p + N_p \end{pmatrix}.$$

For $G = \bigvee_H \mathbb{F}$, we have

$$Q(G) = \begin{pmatrix} Q(G_1) + N_1 I & \delta_{1,2} J_{n_1,n_2} & \cdots & \delta_{1,p} J_{n_1,n_p} \\ \delta_{2,1} J_{n_2,n_1} & Q(G_2) + N_2 I & \cdots & \delta_{2,p} J_{n_2,n_p} \\ \vdots & \vdots & \ddots & \\ \delta_{p,1} J_{n_p,n_1} & \delta_{p,2} J_{n_p,n_1} & \cdots & Q(G_p) + N_p I \end{pmatrix}.$$

For a symmetric nonnegative matrix A of order n, let $\lambda_1(A), \ldots, \lambda_n(A)$ be the eigenvalues of A arranged in a non-increasing order. For a multiset S with $a \in S$ and a real number b , let

$$S - \{a\} + b = \{s + b : s \in S - \{a\}\},$$

where $S - \{a\}$ denotes the multiset obtained by deleting one a from S.

3 Results

First, we give the signless Laplacian spectrum of the H-join of regular graphs $G_1, \ldots, G_p$.

Theorem 1. *Let H be a graph of order p and $\mathbb{F} = \{G_1, \ldots, G_p\}$ be a family of regular graphs, where G_i has degree d_i and order n_i for $i = 1, \ldots, p$. For $G = \bigvee_H \mathbb{F}$, we have*

$$\sigma(G) = \cup_{i=1}^{p} (\sigma(G_i) \setminus \{2d_i\} + N_i) \cup \sigma(M').$$

Proof. For $i = 1, \ldots, p$, let $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_{n_i}}$ be the signless Laplacian eigenvalues of G_i, arranged in a non-increasing order. Since G_i is d_i-regular, $\lambda_{i_1} = 2d_i$. Let $u_{i_1}, u_{i_2}, \ldots, u_{i_{n_i}}$ be orthogonal eigenvectors of $Q(G_i)$ corresponding to the eigenvalues $\lambda_{i_1} = 2d_i, \lambda_{i_2}, \ldots, \lambda_{i_{n_i}}$, respectively.

Since G_i is regular, u_{i_1} is a multiple of e_{n_i}, and thus $J_{k,n_i} u_k = 0$, for $k = i_2, \ldots, i_{n_i}$.

For $j = 2, \ldots, n_i$ and $i = 1, \ldots, p$, we have

$$Q(G) \begin{pmatrix} 0_{n_1} \\ \vdots \\ 0_{n_{i-1}} \\ u_{i_j} \\ 0_{n_{i+1}} \\ \vdots \\ 0_{n_p} \end{pmatrix} = \begin{pmatrix} \delta_{1,i} J_{n_1,n_i} u_{i_j} \\ \vdots \\ \delta_{i-1,i} J_{n_{i-1},n_i} u_{i_j} \\ (Q(G_i) + N_i I) u_{i_j} \\ \delta_{i+1,i} J_{n_{i+1},n_i} u_{i_j} \\ \vdots \\ \delta_{p,i} J_{n_p,n_i} u_{i_j} \end{pmatrix}$$

$$
= \begin{pmatrix} 0_{n_1} \\ \vdots \\ 0_{n_{i-1}} \\ Q(G_i)u_{i_j} + N_i I u_{i_j} \\ 0_{n_{i+1}} \\ \vdots \\ 0_{n_p} \end{pmatrix}
$$

$$
= (\lambda_{i_j} + N_i) \begin{pmatrix} 0_{n_1} \\ \vdots \\ 0_{n_{i-1}} \\ u_{i_j} \\ 0_{n_{i+1}} \\ \vdots \\ 0_{n_p} \end{pmatrix},
$$

and thus $\lambda_{i_j} + N_i$ is a signless Laplacian eigenvalue of G with eigenvectors $(0_{n_1}^{\mathsf{T}}, \ldots, 0_{n_{i-1}}^{\mathsf{T}}, u_{i_j}^{\mathsf{T}}, 0_{n_{i+1}}^{\mathsf{T}}, \ldots, 0_{n_p}^{\mathsf{T}})^{\mathsf{T}}$. Thus we obtain $\sum_{i=1}^{p}(n_i - 1) = n - p$ signless Laplacian eigenvalues λ_{ik} of G and corresponding orthogonal eigenvectors $x_{ik} = (0_{n_1}^{\mathsf{T}}, \ldots, 0_{n_{i-1}}^{\mathsf{T}}, u_{ik}^{\mathsf{T}}, 0_{n_{i+1}}^{\mathsf{T}}, \ldots, 0_{n_p}^{\mathsf{T}})^{\mathsf{T}}$ for $i = 1, \ldots, p$ and $k = 2, \ldots, n_i$.

Let $D = diag(2d_1 + N_1, \ldots, 2d_p + N_p)$, $N = diag(n_1, \ldots, n_p)$,

$$
M = A(H)N + D = \begin{pmatrix} 2d_1 + N_1 & \delta_{1,2}n_2 & \ldots & \delta_{1,p}n_p \\ \delta_{2,1}n_1 & 2d_2 + N_2 & \ldots & \delta_{2,p}n_p \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{p,1}n_1 & \delta_{p,2}n_2 & \ldots & 2d_p + N_p \end{pmatrix},
$$

and $K = diag(\sqrt{n_1}, \ldots, \sqrt{n_p})$. Then $M' = KMK^{-1}$, i.e., M is similar to M', and thus $\sigma(M) = \sigma(M')$. Note that M' is a real symmetric matrix. Let $\lambda_1, \ldots, \lambda_p$ be the eigenvalues of M' with corresponding orthogonal eigenvectors $\theta_1, \ldots, \theta_p$. Then $\lambda_1, \ldots, \lambda_p$ are the eigenvalues of M with corresponding orthogonal eigenvectors $K^{-1}\theta_1, \ldots, K^{-1}\theta_p$. Let $K^{-1}\theta_j = c_j = (c_{j_1}, \ldots, c_{j_p})^{\mathsf{T}}$ for $j = 1, \ldots, p$. Then

$$
Mc_j = \lambda_j c_j \text{ for } j = 1, \ldots, p,
$$

and thus

$$
\lambda_j \begin{pmatrix} c_{j_1}e_{n_1} \\ c_{j_2}e_{n_2} \\ \vdots \\ c_{j_p}e_{n_p} \end{pmatrix} = \begin{pmatrix} (2d_1c_{j_1} + N_1c_{j_1} + \sum_{k\neq 1}\delta_{1,k}c_{j_k}n_k)e_{n_1} \\ (2d_2c_{j_2} + N_2c_{j_2} + \sum_{k\neq 2}\delta_{2,k}c_{j_k}n_k)e_{n_2} \\ \vdots \\ (2d_pc_{j_p} + N_pc_{j_p} + \sum_{k\neq p}\delta_{p,k}c_{j_k}n_k)e_{n_p} \end{pmatrix},
$$

i.e.,

$$\lambda_j \begin{pmatrix} w_{j_1} \\ w_{j_2} \\ \vdots \\ w_{J_p} \end{pmatrix} = \begin{pmatrix} (Q(G_1)+N_1I)w_{j_1} + \sum_{k\neq 1}\delta_{1,k}J_{n_1,n_k}w_{j_k} \\ (Q(G_2)+N_2I)w_{j_2} + \sum_{k\neq 2}\delta_{2,k}J_{n_2,n_k}w_{j_k} \\ \vdots \\ (Q(G_p)+N_pI)w_{j_p} + \sum_{k\neq p}\delta_{p,k}J_{n_p,n_k}w_{j_k} \end{pmatrix}$$

$$= Q(G) \begin{pmatrix} w_{j_1} \\ w_{j_2} \\ \vdots \\ w_{j_p} \end{pmatrix},$$

where $w_{j_i} = c_{j_i}e_{n_i}$ for $i = 1,\ldots,p$. Thus we obtain p signless Laplacian eigenvalues λ_k of G and corresponding eigenvectors $y_k = (w_{k_1}^{\mathsf{T}},\ldots,w_{k_p}^{\mathsf{T}})^{\mathsf{T}}$ for $k = 1,\ldots,p$.

Suppose that there are real a_{ik} for $i = 1,\ldots,p$ and $k = 2,\ldots,n_i$, and b_j for $j = 1,\ldots,p$ such that

$$\sum_{i=1}^{p}\sum_{k=2}^{n_i} a_{ik}x_{ik} + \sum_{j=1}^{p} b_jy_j = 0,$$

i.e.,

$$\sum_{i=1}^{p}\sum_{k=2}^{n_i} a_{ik}x_{ik} = -\sum_{j=1}^{p} b_jy_j.$$

Since $u_{ik}e_{n_i} = 0$ for $i = 1,\ldots,p$ and $k = 2,\ldots,n_i$, we have

$$\sum_{k=2}^{n_i} a_{ik}u_{ik}(\sum_{j=1}^{p} b_jc_{j_i})e_{n_i} = 0 \text{ for } i = 1,\ldots,p.$$

Thus

$$\left(\sum_{i=1}^{p}\sum_{k=2}^{n_i} a_{ik}x_{ik}\right)^{\mathsf{T}} \sum_{j=1}^{p} b_jy_j = 0,$$

i.e.,

$$\left(\sum_{i=1}^{p}\sum_{k=2}^{n_i} a_{ik}x_{ik}\right)^{\mathsf{T}} \left(\sum_{i=1}^{p}\sum_{k=2}^{n_i} a_{ik}x_{ik}\right) = 0.$$

It is easy to see that the signless Laplacian eigenvectors x_{ik} for $i = 1,\ldots,p$ and $k = 2,\ldots,n_i$ are linearly independent. Then $a_{ik} = 0$ for $i = 1,\ldots,p$ and $k = 2,\ldots,n_i$, and thus $b_j = 0$ for $j = 1,\ldots,p$ (since $y_1,\ldots,y_p$ are linearly independent). It follows that the signless Laplacian eigenvectors

x_{ik} for $k = 2, \ldots, n_i$ and $i = 1, \ldots, p$ and y_k for $k = 1, \ldots, p$ are linearly independent. Thus we have obtained all the signless Laplacian eigenvalues of G, as desired. □

For disjoint graphs G_1 and G_2, the join $G_1 \vee G_2$ is obtained from G_1 and G_2 by adding all possible edges between vertices of G_1 and G_2, that is $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and

$$E(G) = \left(\cup_{i=1}^{2} E(G_i)\right) \cup \left(\{uv : u \in V(G_1), v \in V(G_2)\}\right).$$

Obviously, $G_1 \vee G_2 = \bigvee_H \{G_i : i \in V(H)\}$ with $H = P_2$. We have the following corollary from [1] using a different reasoning.

Corollary 1. *[1] Let G_i be a d_i-regular graph of order n_i, where $i = 1, 2$. Then*

$$\sigma(G_1 \vee G_2) = \cup_{i=1}^{2} \left(\sigma(G_i) \setminus \{2d_i\} + n_{3-i}\right) \cup \{\beta_1, \beta_2\},$$

where

$$\beta_1 = \frac{2d_1 + 2d_2 + n + \sqrt{(2d_1 + n_2 - 2d_2 - n_1)^2 + 4n_1n_2}}{2},$$

$$\beta_2 = \frac{2d_1 + 2d_2 + n - \sqrt{(2d_1 + n_2 - 2d_2 - n_1)^2 + 4n_1n_2}}{2}.$$

Proof. Obviously, $N_1 = n_2$ and $N_2 = n_1$. By Theorem 1 we have

$$\sigma(G_1 \vee G_2) = \cup_{i=1}^{i=2} (\sigma(Q(G_i) \setminus \{2d_i\} + N_i) \cup \{\beta_1, \beta_2\},$$

where β_i for $i = 1, 2$ are eigenvalues of the matrix

$$M' = \begin{pmatrix} 2d_1 + n_2 & \sqrt{n_1n_2} \\ \sqrt{n_1n_2} & 2d_2 + n_1 \end{pmatrix}.$$

By direct calculation, we have the expressions for β_1 and β_2, as desired. □

Now we give an expression for the signless Laplacian spread of the join of two regular graphs.

Corollary 2. *Let G_i be a d_i-regular graph of order n_i, where $i = 1, 2$. Let $G = G_1 \vee G_2$ and $n = n_1 + n_2$. Then*

$$s(G) = \begin{cases} \sqrt{(2d_1 + n_2 - 2d_2 - n_1)^2 + 4n_1n_2} & \text{if } \lambda_n(G) = \beta_2, \\ \frac{2d_2 - 2d_1 + n + \sqrt{(2d_1 + n_2 - 2d_2 - n_1)^2 + 4n_1n_2}}{2} + s(G_1) - n_2 & \text{if } \lambda_n(G) = \lambda_{n_1}(G_1) + n_2, \\ \frac{2d_1 - 2d_2 + n + \sqrt{(2d_1 + n_2 - 2d_2 - n_1)^2 + 4n_1n_2}}{2} + s(G_2) - n_1 & \text{if } \lambda_n(G) = \lambda_{n_2}(G_2) + n_1. \end{cases}$$

Proof. Recall that $\sigma(G_1 \vee G_2)$ is given in Corollary 1. If $2d_2+n_1 \leq 2d_1+n_2$, then

$$\begin{aligned}
& \beta_1 - (2d_1+n_2) \\
= & \frac{2d_1+2d_2+n+\sqrt{(2d_1+n_2-2d_2-n_1)^2+4n_1n_2}}{2} \\
& -(2d_1+n_2) \\
> & \frac{2d_1+2d_2+n_1+n_2+2d_1+n_2-2d_2-n_1}{2} \\
& -(2d_1+n_2) \\
= & (2d_1+n_2)-(2d_1+n_2) \\
= & 0,
\end{aligned}$$

and if $2d_2+n_1 > 2d_1+n_2$, then

$$\begin{aligned}
& \beta_1 - (2d_2+n_1) \\
= & \frac{2d_1+2d_2+n+\sqrt{(2d_1+n_2-2d_2-n_1)^2+4n_1n_2}}{2} \\
& -(2d_2+n_1) \\
> & \frac{2d_1+2d_2+n_1+n_2+2d_2+n_1-2d_1-n_2}{2} \\
& -(2d_2+n_1) \\
= & (2d_2+n_1)-(2d_2+n_1) \\
= & 0.
\end{aligned}$$

It follows that $\beta_1 > 2d_i + N_i$ for $i = 1,2$, and thus $\lambda_1(G) = \beta_1$. On the other hand, $\lambda_{n_i} = 2d_i + N_i - s(G_i)$, for $i = 1,2$. □

Corollary 3. *Let $G = G_1 \vee G_2$, where G_i is a d_i-regular graph of order n_i for $i = 1,2$. If $|d_1 - d_2| > |n_1 - n_2|$, then $s(G) > n_1 + n_2$.*

Proof. By Corollary 1, β_1, $\beta_2 \in \sigma(G)$. Thus

$$\begin{aligned}
s(G) & \geq \beta_1 - \beta_2 \\
& = \sqrt{(2d_1+n_2-2d_2-n_1)^2+4n_1n_2} \\
& = \sqrt{(2(d_1-d_2)-(n_2-n_1))^2+4n_1n_2} \\
& > \sqrt{(n_2-n_1)^2+4n_1n_2} \\
& = n_1+n_2,
\end{aligned}$$

as desired. □

Next, we give an expression for the signless Laplacian spread of the H-join of regular graphs $G_1, \ldots, G_p$.

Theorem 2. *Let H be a graph of order p and $\mathbb{F} = \{G_1, \ldots, G_p\}$ be a family of regular graphs, where G_i has degree d_i and order n_i for $i = 1, \ldots, p$. If $G = \bigvee_H \mathbb{F}$, then*

$$s(G) = s(M') + \max_{1 \le i \le p} \{\lambda_p(M') + s(G_i) - 2d_i - N_i, 0\}.$$

Proof. By Theorem 1,

$$\sigma(G) = \cup_{i=1}^{p} (\sigma(G_i) \setminus \{2d_i\} + N_i) \cup \sigma(M').$$

For any $i \in 1, \ldots, p$, $\lambda_{n_i}(G_i) = 2d_i - s(G_i)$, and then

$$\lambda_n(G) \in \cup_{i=1}^{p} \{2d_i + N_i - s(G_i)\} \cup \{\lambda_p(M')\}.$$

Note that $\lambda_1(M') \ge \max_{1 \le i \le p}\{2d_i + N_i\} > t$ for any $t \in \cup_{i=1}^{p} \sigma(G_i) \setminus \{2d_i\} + N_i$. We have $\lambda_1(G) = \lambda_1(M')$, and then

$$\begin{aligned} s(G) &= \lambda_1(G) - \lambda_n(G) \\ &= \lambda_1(M') - \lambda_p(M') - \min_{1 \le i \le p} \{2d_i + N_i - s(G_i) - \lambda_p(M'), 0\} \\ &= \lambda_1(M') - \lambda_p(M') + \max_{1 \le i \le p} \{\lambda_p(M') + s(G_i) - 2d_i - N_i, 0\} \\ &= s(M) + \max_{1 \le i \le p} \{\lambda_p(M') + s(G_i) - 2d_i - N_i, 0\}, \end{aligned}$$

as desired. □

Now we give an upper bound for the signless Laplacian spread of the H-join of regular graphs $G_1, \ldots, G_p$.

Theorem 3. *Let H be a graph of order p and $\mathbb{F} = \{G_1, \ldots, G_p\}$ be a family of regular graphs, where G_i has degree d_i and order n_i for $i = 1, \ldots, p$. Let H be a graph of order p and $\mathbb{F} = \{G_1, \ldots, G_p\}$ be a family of regular graphs, where G_i has degree d_i and order n_i for $i = 1, \ldots, p$. Let*

$$P = \begin{pmatrix} 0 & \sqrt{n_1 n_2} & \cdots & \sqrt{n_1 n_p} \\ \sqrt{n_2 n_1} & 0 & \cdots & \sqrt{n_2 n_p} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{n_p n_1} & \sqrt{n_p n_2} & \cdots & 0 \end{pmatrix}.$$

If $G = \bigvee_H \mathbb{F}$, then

$$s(G) \le \max_{1 \le i \le p} \{2d_i + N_i\} + \lambda_1(H)\lambda_1(P) - \min_{1 \le i \le p} \{2d_i + N_i - s(G_i), \lambda_p(M')\}.$$

Proof. By Theorem 1,

$$\sigma(G) = \cup_{i=1}^{p} (\sigma(G_i) \setminus \{2d_i\} + N_i) \cup \sigma(M'),$$

where $M' = D + Q(H) \circ P$, $D = diag(2d_1 + N_1, \ldots, 2d_p + N_p)$, and $\circ$ denotes the Hadamard product. Note that for two symmetric nonnegative matrices of order p, A and B, we have $\lambda_1(A + B) \leq \lambda_1(A) + \lambda_1(B)$ [8, Theorem 4.3.1] and $\lambda_1(A \circ B) \leq \lambda_1(A)\lambda_1(B)$ [9, Observation 5.7.3]. Thus

$$\begin{aligned} \lambda_1(M') &\leq \lambda_1(D) + \lambda_1(Q(H) \circ P) \leq \lambda_1(D) + \lambda_1(H)\lambda_1(P) \\ &= \max_{1 \leq i \leq p} \{2d_i + N_i\} + \lambda_1(H)\lambda_1(P) \end{aligned}$$

Since $\lambda_n(G) = \min_{1 \leq i \leq p} \{2d_i + N_i - s(G_i), \lambda_p(M')\}$ and $\lambda_1(G) = \lambda_1(M')$, we have

$$\begin{aligned} s(G) &= \lambda_1(G) - \lambda_n(G) \\ &\leq \max_{1 \leq i \leq p} \{2d_i + N_i\} + \lambda_1(H)\lambda_1(P) \\ &\quad - \min_{1 \leq i \leq p} \{2d_i + N_i - s(G_i), \lambda_p(M')\}, \end{aligned}$$

as desired. □

Finally, we give an example for which the upper bound for the signless Laplacian spread may be attained.

Theorem 4. *For positive integers $p, q \geq 3$ and $n \in N$, such that $n \geq p+q+3$. Let $H = P_3$ and let $\mathbb{F} = \{G_1, G_2, G_3\}$ be a family of graphs, where $G_1 = C_p$, $G_2 = C_q$, $G_3 = C_{n-p-q}$. Let $G = \bigvee_H \mathbb{F}$. Then $s(G) \leq n+1$ with equality if and only if $q = 3$ and there is an even cycle in G_1, G_3.*

Proof. By Theorem 1 we have

$$\sigma(G) = \cup_{i=1}^{3} (\sigma(G_i) \setminus \{4\} + N_i) \cup \{\beta_1, \beta_2, \beta_3\},$$

where $N_1 = q$, $N_2 = n - q$, $N_3 = q$, and β_i for $i \in \{1, 2, 3\}$ ($\beta_1 \leq \beta_2 \leq \beta_3$) are the roots of the characteristic polynomial of the matrix

$$M = \begin{pmatrix} 4+q & q & 0 \\ p & 4+n-q & n-p-q \\ 0 & q & 4+q \end{pmatrix}.$$

By direct calculation, $\beta_1 = 4$, $\beta_2 = q + 4$, and $\beta_3 = n + 4$. Then $\lambda_1(G) = \beta_3 = n + 4$,

$$\lambda_n(G) = \min\{\lambda_p(G_1) + N_1, \lambda_q(G_2) + N_2, \lambda_{n-p-q}(G_3) + N_3, 4\}$$

$$
\begin{aligned}
&\geq \min\{q, n-q, 4\} \\
&\geq 3
\end{aligned}
$$

with equalities if and only if $q = 3$ (since $n - q > 3$), and $\lambda_p(G_1) = 0$ or $\lambda_{n-p-q}(G_3) = 0$, that is at least one of G_1 and G_3 is even, and thus

$$s(G) = n + 4 - \lambda_n(G) \leq n + 4 - 3 = n + 1$$

with equality if and only if $q = 3$ and one of G_1 and G_3 is even. □

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