Construction of combinatorial batch codes based on semilattices

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Abstract

This paper obtains new combinatorial batch codes (CBCs) from old ones, studies properties of uniform CBCs, and constructs uniform CBCs using semilattices.

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1 Introduction

Batch codes were introduced by Ishai, Kushilevitz, Ostrovsky and Sahai [13], which were motivated by applications to load balancing in distributed storage, private information retrieval and cryptographic protocols. An (n, N, k, m, t) *batch code* over an alphabet Σ encodes a string $x \in \Sigma^n$ into an m-tuple of strings $y_1, y_2, \ldots, y_m \in \Sigma^*$ (also referred to as servers) of total length N, such that for each k-tuple (batch) of distinct indices $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\}$, the entries $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ from x can be decoded by reading at most t symbols from each server. In this paper, we only consider the case $t = 1$.

The combinatorial batch codes (CBCs) were proposed by Paterson, Stinson and Wei [16] to refer to purely replication based batch codes. Let n, N, k and m be positive integers with $k \leq m \leq n$. An (n, N, k, m)-CBC is a set system $(X, \mathcal{B})$, where X is a finite set of elements called points and $\mathcal{B}$ is a collection of subsets of X called blocks, such that the following properties are satisfied:

(i) $|X| = n, |\mathcal{B}| = m, N = \sum_{B \in \mathcal{B}} |B|$, and

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(ii) for every k-subset $\{x_1, x_2, \ldots, x_k\} \subseteq X$ there exists a collection $\{B_1, B_2, \ldots, B_k\} \subseteq \mathcal{B}$ such that $x_i \in B_i$ for $i = 1, 2, \ldots, k$.

The integer N is called the size of the CBC. Brualdi et al. [4, 5, 6, 17] studied constructions for CBCs. Chen et al. [9, 14, 15] studied optimal CBCs. An $(n, N = cn, k, m)$-CBC is called *uniform* if each point is contained in exactly c blocks [2, 3, 16]. We denote by $n(m, c, k)$ the maximum value of n for which there exists a uniform (n, cn, k, m)-CBC. The following general upper bound on $n(m, c, k)$ was established in [16]:

$$n(m, c, k) \leq \frac{(k-1)\binom{m}{c}}{\binom{k-1}{c}}. \tag{1}$$

A c-uniform (n, cn, k, m)-CBC is called *optimal* if $n = (k-1)\binom{m}{c}/\binom{k-1}{c}$ for given m, c and k. Silberstein and Gál [18] constructed uniform CBCs based on transversal designs and affine planes. Their research stimulates us to consider other constructions.

Let $\mathcal{C} = (X, \mathcal{B})$ be an (n, N, k, m)-CBC with the point set $X = \{x_1, x_2, \ldots, x_n\}$ and the block set $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$. For $1 \leq j \leq n$, define $X_j = \{B_{i_1}, \ldots, B_{i_s}\}$ if x_j is contained in blocks $B_{i_1}, \ldots, B_{i_s}$. Then $\mathcal{C}$ is represented by a dual set system $(\mathcal{B}, \mathcal{X})$ such that $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ and $\mathcal{X} = \{X_1, X_2, \ldots, X_n\}$. The following result provides a very useful characterization of CBCs in terms of the dual set system.

Theorem 1.1 *([16]) A set system $(\mathcal{B}, \mathcal{X})$ is the dual set system of an (n, N, k, m)-CBC if $|\mathcal{B}| = m$, $|\mathcal{X}| = n$, $\sum_{Y \in \mathcal{X}} |Y| = N$ and every set of i blocks contains at least i points for $1 \leq i \leq k$.*

In this paper, we focus on the construction of uniform CBCs from a semilattice. The rest of this paper is structured as follows. In Section 2, we study constructions of new CBCs from old ones and discuss properties of uniform CBCs. In Section 3, we show how to construct uniform CBCs from a semilattice. In Section 4, we give thirteen families of examples of semilattices from sets, vector spaces and maps, and study corresponding CBCs. In Section 5 and Section 6, we give examples of semilattices from affine spaces and distance-regular graphs, and study corresponding problems.

2 Constructions of new CBCs from old ones

In this section, we present three simple methods of constructions of new CBCs from old ones, and discuss simple properties of uniform CBCs.

The first method is called the *sum construction.* Given two CBCs with disjoint point sets. If we put their points together as the collection of points and put their blocks together as the collection of blocks, by [8] we obtain a new CBC as stated in the following theorem.

Theorem 2.1 *([8]) Suppose there exist an (n_1, N_1, k_1, m_1)-CBC and an (n_2, N_2, k_2, m_2)-CBC. Then there exists an $(n_1+n_2, N_1+N_2, k = \min\{k_1, k_2\}, m_1 + m_2)$-CBC.*

The second method is called the *product construction.* Suppose that $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$ are two CBCs. Pick $X = X_1 \times X_2 = \{(x_1, x_2) \mid x_1 \in X_1, x_2 \in X_2\}$ and $\mathcal{B} = \{B_1 \times B_2 \mid B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\}$. One point $(x_1, x_2) \in X$ is placed in one block $B_1 \times B_2 \in \mathcal{B}$ if $(x_1, x_2) \in B_1 \times B_2$. Then we obtain a new CBC as stated in the following theorem.

Theorem 2.2 *Suppose there exist an (n_1, N_1, k_1, m_1)-CBC and an(n_2, N_2, k_2, m_2)-CBC. Then there exists an $(n_1 n_2, N_1 N_2, k = \min\{k_1, k_2\}, m_1 m_2)$-CBC.*

Proof. Suppose that $(X_1, \mathcal{B}_1)$ is an (n_1, N_1, k_1, m_1)-CBC and $(X_2, \mathcal{B}_2)$ is an (n_2, N_2, k_2, m_2)-CBC. We will show that $(X = X_1 \times X_2, \mathcal{B} = \{B_1 \times B_2 \mid B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\})$ is an $(n_1 n_2, N_1 N_2, k, m_1 m_2)$-CBC. Clearly, $(X, \mathcal{B})$ has $n_1 n_2$ points and $m_1 m_2$ blocks. Note that

$$N = \sum_{B \in \mathcal{B}} |B| = \sum_{B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2} |B_1||B_2| = N_1 N_2.$$

For every k-subset $\{(x_{11}, x_{21}), (x_{12}, x_{22}), \ldots, (x_{1k}, x_{2k})\} \subseteq X_1 \times X_2$, there exist blocks $B_{11}, B_{12}, \ldots, B_{1k}$ in $\mathcal{B}_1$ and blocks $B_{21}, B_{22}, \ldots, B_{2k}$ in $\mathcal{B}_2$, such that $x_{1i} \in B_{1i}$ and $x_{2i} \in B_{2i}$ for $i = 1, 2, \ldots, k$. It follows that there exist blocks $B_{11} \times B_{21}, B_{12} \times B_{22}, \ldots, B_{1k} \times B_{2k}$ in $\mathcal{B}$ such that $(x_{1i}, x_{2i}) \in B_{1i} \times B_{2i}$ for $i = 1, 2, \ldots, k$. Therefore, the desired result follows. □

Example 2.1 *Pick $X_1 = \{1, 2, 3\}, X_2 = \{4, 5, 6, 7\}, \mathcal{B}_1 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ and $\mathcal{B}_2 = \{\{4, 5, 6\}, \{4, 6, 7\}, \{5, 6, 7\}, \{4, 5, 7\}\}$. Then $(X_1, \mathcal{B}_1)$ is a $(3, 6, 3, 3)$-CBC and $(X_2, \mathcal{B}_2)$ is a $(4, 12, 4, 4)$-CBC. By Theorem 2.2, we obtain a $(12, 72, 3, 12)$-CBC.*

The third construction is called the *subconstruction.* Suppose that $(X, \mathcal{B})$ is a CBC. Let X' be a nonempty subset of X. Define X' as the point set and $\mathcal{B}' = \{X' \cap B \mid B \in \mathcal{B}\} \setminus \{\emptyset\}$ as the block set. Then $(X', \mathcal{B}')$ is a CBC as stated in the following theorem.

Theorem 2.3 *Suppose that $(X, \mathcal{B})$ is an (n, N, k, m)-CBC. Given a nonempty subset X' of X with size n'. Then there exists an (n', N', k', m')-CBC with $N' = \sum_{B' \in \mathcal{B}'} |B'|, k' = \min\{k, |X'|\}$ and $m' = |\mathcal{B}'|$, where $\mathcal{B}' = \{X' \cap B \mid B \in \mathcal{B}\} \setminus \{\emptyset\}$.*

Proof. Note that $N' = \sum_{B' \in \mathcal{B}'} |B|$ and $m' = |\mathcal{B}'|$. For every k'-subset $\{x_1, x_2, \ldots, x_{k'}\} \subseteq X'$, there exists a collection $\{B_1, B_2, \ldots, B_{k'}\} \subseteq \mathcal{B}$ such that $x_i \in X' \cap B_i$ for $i = 1, 2, \ldots, k'$. Therefore, the desired result follows. □

Example 2.2 *Let $(X_1, \mathcal{B}_1)$ be as in Example 2.1. Pick $X' = \{1,2\}$. By Theorem 2.3, we obtain a $(2,4,2,3)$-CBC with block set $\mathcal{B}' = \{\{1,2\},\{1\},\{2\}\}$.*

Now we discuss properties of uniform CBCs.

Theorem 2.4 *$n(m_1 + m_2, c, k) \geq n(m_1, c, k) + n(m_2, c, k)$.*

Proof. Write $n_1 = n(m_1, c, k)$ and $n_2 = n(m_2, c, k)$. Let $(X_1, \mathcal{B}_1)$ be a uniform (n_1, cn_1, k, m_1)-CBC and $(X_2, \mathcal{B}_2)$ be a uniform (n_2, cn_2, k, m_2)-CBC such that $X_1 \cap X_2 = \emptyset$. By Theorem 2.1 we can obtain a uniform $(n_1 + n_2, c(n_1 + n_2), k, m_1 + m_2)$-CBC. It follows that $n(m_1 + m_2, c, k) \geq n_1 + n_2$. □

Theorem 2.5 *Let $(X, \mathcal{B})$ be a uniform $(n, cn, k, m + 1)$-CBC with $n = n(m + 1, c, k)$. Then $n(m + 1, c, k) \leq n(m, c, k) + \min\{|B| \mid B \in \mathcal{B}\}$.*

Proof. Suppose $\mathcal{B} = \{B_1, B_2, \ldots, B_{m+1}\}$. Let i satisfy $|B_i| = \min\{|B_t| \mid B_t \in \mathcal{B}\}$. Define $B'_j = B_j \setminus (B_i \cap B_j)$ for $j = 1, 2, \ldots, m + 1$. Let $\mathcal{B}' = \{B'_j \mid j = 1, 2, \ldots, m + 1\} \setminus \{\emptyset\}$. Then $(X \setminus B_i, \mathcal{B}')$ is a uniform $(n - |B_i|, c(n - |B_i|), k, m')$-CBC, where $m' \leq m$ is the size of $\mathcal{B}'$, which implies that $n(m', c, k) \geq n - |B_i|$. If $m' < m$, from Theorem 2.4 we deduce that $n(m, c, k) \geq n(m', c, k) + n(m - m', c, k) \geq n(m', c, k)$. Hence we have $n(m, c, k) \geq n - |B_i|$, as desired. □

3 Semilattices

Let $(P, \preceq)$ be a finite partially ordered set (poset) with the least element 0. For $x, y \in P$, if $x \preceq y$, we say that y *contains* x. Moreover, if there does not exist element z such that $x \prec z \prec y$, we say that y *covers* x. The poset P is *ranked* and has *rank function*, if there is a function ℓ from P

to the integer set such that $\ell(0) = 0$ and $\ell(y) = \ell(x) + 1$ if y covers x. The maximum value of $\ell(x)$ is called the *rank* of P, denoted by N. The *fibers* (or *levels*) $P_0, P_1, \ldots, P_N$ of the poset are the subsets of P given by $P_i = \{x \in P \mid \ell(x) = i\}$. Pick any $x, y \in P$ such that $x \preceq y$. By the *interval* $[x, y]$, we mean the subposet $[x, y] := \{z \in X \mid x \preceq z \preceq y\}$ of P. Let $(P, \preceq)$ be a finite poset with the rank function ℓ and fibers $P_0, \ldots, P_N$. We call P a *semilattice*, if any two elements x and y of P have the greatest lower bound, denoted by $x \wedge y$. As usual, we denote by $x \vee y$ the least upper bound of x and y if it exists. Note that if P is a semilattice and $x, y \in P$ have a common upper bound, then $x \vee y$ exists.

Let P denote a semilattice with the rank function ℓ and fibers $P_0, \ldots, P_N$. We call P a *strictly semilattice*, if the following properties are satisfied:

(i) For any $x \in P_r$, the number of elements $z \in P_s$ such that $x \preceq z$ is a constant, where $0 \leq r \leq s \leq N$.

(ii) The number $|[u, z] \cap P_s|$ is a constant $\theta(r, s, t)$ for $u \in P_r$ and $z \in P_t$ with $u \preceq z$, and the function $\theta(r, s, t)$ is strictly increasing about t, where $0 \leq r \leq s \leq t \leq N$, i.e. $1 = \theta(r, s, s) < \theta(r, s, s+1) < \cdots < \theta(r, s, N)$.

Theorem 3.1 *Suppose that P is a strictly semilattice with rank N. For positive integers $1 \leq \alpha < \beta \leq N$ with $|P_\alpha| \geq |P_\beta|$, let P_α be the point set X and P_β be the block set $\mathcal{B}$. One point $x \in P_\alpha$ is placed in one block $y \in P_\beta$ if $x \preceq y$. Assume that $\theta(0, \alpha, \beta)/|P_\alpha| > \max\{\theta(0, \xi, \beta)/|P_\xi| \mid \alpha + 1 \leq \xi \leq N\}$. Then the set system $(X, \mathcal{B})$ is a uniform (n, cn, k, m)-CBC with $n = |P_\alpha|, c = m\theta(0, \alpha, \beta)/n, m = |P_\beta|$ and $k = \max\{2c - \Omega, 3c - 3\Omega\}$, where $\Omega := \max\{m\theta(0, \xi, \beta)/|P_\xi| \mid \alpha + 1 \leq \xi \leq N\}$.*

Proof. Note that $n = |X| = |P_\alpha|$ and $m = |\mathcal{B}| = |P_\beta|$ are obvious. The definition of strictly semilattice tells us that the set system $(X, \mathcal{B})$ is a uniform CBC. Counting pairs $(v, w) \in P_\alpha \times P_\beta$ with $v \preceq w$ in two ways yields $cn = m\theta(0, \alpha, \beta)$, which implies that $c = m\theta(0, \alpha, \beta)/n$. Let $(\mathcal{B}, \mathcal{X})$ be the dual set system of the set system $(X, \mathcal{B})$. For any three distinct block X_1, X_2 and X_3 in $\mathcal{X}$, there exist three distinct points x_1, x_2 and x_3 in P_α such that $X_1 = \{y \in P_\beta \mid x_1 \preceq y\}, X_2 = \{y \in P_\beta \mid x_2 \preceq y\}$ and $X_3 = \{y \in P_\beta \mid x_3 \preceq y\}$. Then there are the following two cases to be considered.

Case 1: $X_1 \cap X_2 \neq \emptyset, X_1 \cap X_3 \neq \emptyset$ and $X_2 \cap X_3 \neq \emptyset$. Since $x_1, x_2 \preceq (x_1 \vee x_2)$ and P is a strictly semilattice, we have $\theta(0, 1, \alpha) < \theta(0, 1, \ell(x_1 \vee x_2))$ and $N \geq \ell(x_1 \vee x_2) \geq \alpha + 1$. Counting pairs $(v, w) \in P_{\ell(x_1 \vee x_2)} \times P_\beta$ with $v \preceq w$ in two ways yields that the number of the elements w in P_β satisfying $(x_1 \vee x_2) \preceq w$ is $m\theta(0, \ell(x_1 \vee x_2), \beta)/|P_{\ell(x_1 \vee x_2)}|$. It follows that

$|X_1 \cap X_2| \le \max\{m\theta(0,\xi,\beta)/|P_\xi| \mid \alpha+1 \le \xi \le N\} = \Omega$. Similarly, we have $|X_1 \cap X_3| \le \Omega$ and $|X_2 \cap X_3| \le \Omega$. Therefore, one obtains that

$$\begin{aligned} |X_1 \cup X_2| &= |X_1| + |X_2| - |X_1 \cap X_2| \\ &\ge 2c - \Omega, \\ |X_1 \cup X_2 \cup X_3| &= |X_1| + |X_2| + |X_3| - |X_1 \cap X_2| \\ &\quad -|X_1 \cap X_3| - |X_2 \cap X_3| + |X_1 \cap X_2 \cap X_3| \\ &\ge 3c - 3\Omega, \end{aligned}$$

which implies that k is at least $\max\{2c-\Omega, 3c-3\Omega\}$.

Case 2: $X_1 \cap X_2 = \emptyset, X_1 \cap X_3 = \emptyset$ or $X_2 \cap X_3 = \emptyset$. Without loss of generality, assume that $X_1 \cap X_2 = \emptyset$. Similar to the Case 1, we have that k is at least $\max\{2c-\Omega, 3c-3\Omega\}$. □

Theorem 3.2 *Suppose that P is a strictly semilattice with rank N. For positive integers $1 \le \alpha < \beta \le N$ with $|P_\alpha| \le |P_\beta|$, let P_β be the point set X and P_α be the block set $\mathcal{B}$. One point $x \in P_\beta$ is placed in one block $y \in P_\alpha$ if $y \preceq x$. Then the set system $(X, \mathcal{B})$ is a uniform (n, cn, k, m)-CBC with $n = |P_\beta|, c = \theta(0,\alpha,\beta), m = |P_\alpha|$ and $k = \max\{2\theta(0,\alpha,\beta) - \theta(0,\alpha,\beta-1), 3\theta(0,\alpha,\beta) - 3\theta(0,\alpha,\beta-1)\}$.*

Proof. Note that $n = |X| = |P_\beta|, m = |\mathcal{B}| = |P_\alpha|$ and $c = \theta(0,\alpha,\beta)$ are obvious. Let $(\mathcal{B}, \mathcal{X})$ be the dual set system of the set system $(X, \mathcal{B})$. For any three distinct block X_1, X_2 and X_3 in $\mathcal{X}$, there exist three distinct points x_1, x_2 and x_3 in P_β such that $X_1 = \{y \in P_\alpha \mid y \preceq x_1\}, X_2 = \{y \in P_\alpha \mid y \preceq x_2\}$ and $X_3 = \{y \in P_\alpha \mid y \preceq x_3\}$. Since $x_1 \wedge x_2 \preceq x_1, x_2$ and P is a strictly semilattice, we have $\theta(0,1,\ell(x_1 \wedge x_2)) < \theta(0,1,\beta)$ and $\beta - 1 \ge \ell(x_1 \wedge x_2) \ge 0$. It follows that $|X_1 \cap X_2| \le \max\{\theta(0,\alpha,\xi) \mid 0 \le \xi \le \beta-1\} = \theta(0,\alpha,\beta-1)$. Similarly, we have $|X_1 \cap X_3| \le \theta(0,\alpha,\beta-1)$ and $|X_2 \cap X_3| \le \theta(0,\alpha,\beta-1)$. Therefore, one obtains that

$$\begin{aligned} |X_1 \cup X_2| &= |X_1| + |X_2| - |X_1 \cap X_2| \\ &\ge 2c - \theta(0,\alpha,\beta-1), \\ |X_1 \cup X_2 \cup X_3| &= |X_1| + |X_2| + |X_3| - |X_1 \cap X_2| \\ &\quad -|X_1 \cap X_3| - |X_2 \cap X_3| + |X_1 \cap X_2 \cap X_3| \\ &\ge 3c - 3\theta(0,\alpha,\beta-1), \end{aligned}$$

which implies that k is at least $\max\{2\theta(0,\alpha,\beta) - \theta(0,\alpha,\beta-1), 3\theta(0,\alpha,\beta) - 3\theta(0,\alpha,\beta-1)\}$, as desired. □

4 Semilattices from sets, vector spaces and maps

In this section we give thirteen families of strictly semilattices with rank N, and give their parameters. By Theorems 3.1 and 3.2, we can construct uniform CBCs from these semilattices.

Let q be a positive integer. Fix a positive integer n. The *Gaussian binomial coefficients with basis q* is defined by

$$\begin{bmatrix} n \\ i \end{bmatrix}_q = \begin{cases} \prod_{j=0}^{i-1} \frac{n-j}{i-j} & \text{if } q = 1, \\ \prod_{j=0}^{i-1} \frac{q^n - q^j}{q^i - q^j} & \text{if } q \neq 1. \end{cases}$$

In the case $q = 1$, for convenience, we write $\binom{n}{i}$ instead of $\begin{bmatrix} n \\ i \end{bmatrix}_1$.

Example 4.1 *(The Boolean Algebra) Let P be the collection of all subsets of $[N] := \{1, 2, \ldots, N\}$. Ordered by inclusion, P is a strictly semilattice with the rank function $\ell(x) = |x|$ and the parameters*

$$|P_r| = \binom{N}{r}, \quad \theta(r, s, t) = \binom{t-r}{s-r}.$$

Pick $M = \lfloor (N+1)/2 \rfloor$. If $1 \leq \alpha < \beta \leq M$, let P_β be the point set and P_α be the block set. Then we obtain uniform (n, cn, k, m)-CBC with $n = \binom{N}{\beta}, c = \binom{\beta}{\alpha}, m = \binom{N}{\alpha}$ and $k = \max\{2\binom{\beta}{\alpha} - \binom{\beta-1}{\alpha}, 3\binom{\beta}{\alpha} - 3\binom{\beta-1}{\alpha}\}$. If $M \leq \alpha < \beta \leq N$, let P_α be the point set and P_β be the block set. Then we obtain uniform (n, cn, k, m)-CBC with $n = \binom{N}{\alpha}, c = \binom{N-\alpha}{\beta-\alpha}, m = \binom{N}{\beta}$ and

$$k = \max\left\{2\binom{N-\alpha}{\beta-\alpha} - \binom{N-\alpha-1}{\beta-\alpha-1}, 3\binom{N-\alpha}{\beta-\alpha} - 3\binom{N-\alpha-1}{\beta-\alpha-1}\right\}.$$

Remark. Pick the collection of all 2-subsets of $\{1, 2, 3, 4\}$ as the point set and the collection of all 3-subsets of $\{1, 2, 3, 4\}$ as the block set. Then we can obtain optimal uniform $(6, 12, 4, 4)$-CBC. It seems interesting to construct optimal uniform CBCs using the Boolean algebra.

Example 4.2 *(The Projective Geometry) Let $\mathbb{F}_q^N$ be the N-dimensional vector space over the finite field $\mathbb{F}_q$ and P be the collection of all subspaces of $\mathbb{F}_q^N$. Ordered by inclusion, P is a strictly semilattice with the rank function*

$\ell(x) = \dim x$ and the parameters

$$|P_r| = \begin{bmatrix} N \\ r \end{bmatrix}_q, \quad \theta(r,s,t) = \begin{bmatrix} t-r \\ s-r \end{bmatrix}_q.$$

Pick $M = \lfloor (N+1)/2 \rfloor$. If $1 \le \alpha < \beta \le M$, let P_β be the point set and P_α be the block set. Then we obtain uniform (n, cn, k, m)-CBC with $n = \begin{bmatrix} N \\ \beta \end{bmatrix}_q, c = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}_q, m = \begin{bmatrix} N \\ \alpha \end{bmatrix}_q$ and

$$k = \max\left\{2\begin{bmatrix} \beta \\ \alpha \end{bmatrix}_q - \begin{bmatrix} \beta-1 \\ \alpha \end{bmatrix}_q, 3\begin{bmatrix} \beta \\ \alpha \end{bmatrix}_q - 3\begin{bmatrix} \beta-1 \\ \alpha \end{bmatrix}_q\right\}.$$

If $M \le \alpha < \beta \le N$, let P_α be the point set and P_β be the block set. Then we obtain uniform (n, cn, k, m)-CBC with $n = \begin{bmatrix} N \\ \alpha \end{bmatrix}_q, c = \begin{bmatrix} N-\alpha \\ \beta-\alpha \end{bmatrix}_q, m = \begin{bmatrix} N \\ \beta \end{bmatrix}_q$ and

$$k = \max\left\{2\begin{bmatrix} N-\alpha \\ \beta-\alpha \end{bmatrix}_q - \begin{bmatrix} N-\alpha-1 \\ \beta-\alpha-1 \end{bmatrix}_q, 3\begin{bmatrix} N-\alpha \\ \beta-\alpha \end{bmatrix}_q - 3\begin{bmatrix} N-\alpha-1 \\ \beta-\alpha-1 \end{bmatrix}_q\right\}.$$

Example 4.3 *(The Attenuated Space) For fixed positive integers n and N, let w be a fixed n-dimensional subspace of $\mathbb{F}_q^{n+N}$. Let P be the collection of all subspaces x of $\mathbb{F}_q^{n+N}$ with $x \cap w = \{0\}$. Ordered by inclusion, P is a strictly semilattice with the rank function $\ell(x) = \dim x$ and the parameters*

$$|P_r| = q^{rn}\begin{bmatrix} N \\ r \end{bmatrix}_q, \quad \theta(r,s,t) = \begin{bmatrix} t-r \\ s-r \end{bmatrix}_q.$$

For $1 \le \alpha < \beta \le N \le n$, let P_β be the point set and P_α be the block set. Then we obtain uniform (n, cn, k, m)-CBC with $n = |P_\beta|, c = \theta(0, \alpha, \beta), m = |P_\alpha|$ and

$$k = \max\{2\theta(0,\alpha,\beta) - \theta(0,\alpha,\beta-1), 3\theta(0,\alpha,\beta) - 3\theta(0,\alpha,\beta-1)\}.$$

For $n < N$, pick $M = (N+n+1)/2$. If $M \le \alpha < \beta \le N$, let P_α be the point set and P_β be the block set. Then we obtain uniform (n, cn, k, m)-CBC with $n = |P_\alpha|, m = |P_\beta|, c = m\theta(0,\alpha,\beta)/n$ and $k = \max\{2c - \Omega, 3c - 3\Omega\}$.

Example 4.4 *([7] The Classical Polar Space) Classical finite polar spaces are incidence structures, consisting of all the totally isotropic subspaces of $\mathbb{F}_q^n$ with respect to a certain non-degenerate sesquilinear or quadratic form f. The rank of the polar space is the algebraic dimension of the maximal totally isotropic subspaces, denoted by N. The summary is given by Table 1.*

Table 1: The Classical Polar Spaces

name	n	*form*	$\lvert X_r\rvert$
$[C_N(q)]$	$2N$	*symplectic*	$\begin{bmatrix} N \\ r \end{bmatrix}_q \prod_{i=0}^{r-1}(q^{N-i}+1)$
$[B_N(q)]$	$2N+1$	*quadratic*	$\begin{bmatrix} N \\ r \end{bmatrix}_q \prod_{i=0}^{r-1}(q^{N-i}+1)$
$[D_N(q)]$	$2N$	*quadratic (with rank N)*	$\begin{bmatrix} N \\ r \end{bmatrix}_q \prod_{i=0}^{r-1}(q^{N-i-1}+1)$
$[^2D_{N+1}(q)]$	$2N+2$	*quadratic (with rank N)*	$\begin{bmatrix} N \\ r \end{bmatrix}_q \prod_{i=0}^{r-1}(q^{N-i+1}+1)$
$[^2A_{2N}(r)]$	$2N+1$	*Hermitian* $(q=r^2)$	$\begin{bmatrix} N \\ r \end{bmatrix}_q \prod_{i=0}^{r-1}(q^{N-i+1/2}+1)$
$[^2A_{2N-1}(r)]$	$2N$	*Hermitian* $(q=r^2)$	$\begin{bmatrix} N \\ r \end{bmatrix}_q \prod_{i=0}^{r-1}(q^{N-i-1/2}+1)$

Let P be the collection of all totally isotropic subspaces of $\mathbb{F}_q^n$. Ordered by inclusion, P is a strictly semilattice with the rank function $\ell(x) = \dim x$ and the parameters

$$|P_r| = \begin{bmatrix} N \\ r \end{bmatrix}_q \prod_{i=0}^{r-1}(q^{N+e-i-1}+1), \quad \theta(r,s,t) = \begin{bmatrix} t-r \\ s-r \end{bmatrix}_q.$$

where $e = 1, 1, 0, 2, \frac{3}{2}, \frac{1}{2}$ according to $[C_N(q)], [B_N(q)], [D_N(q)], [^2D_{N+1}(q)]$, $[^2A_{2N}(r)]$, $[^2A_{2N-1}(r)]$, respectively. Pick $M = (2N+e-2)/3$. If $M \le \alpha < \beta \le N$, let P_α be the point set and P_β be the block set. Then we obtain uniform (n, cn, k, m)-CBC with $n = |P_\alpha|, m = |P_\beta|, c = m\theta(0,\alpha,\beta)/n$ and $k = \max\{2c - \Omega, 3c - 3\Omega\}$.

For fixed positive integers n and m, let w be an l-dimensional subspace of $\mathbb{F}_q^{n+m}$, denote also by w an $l \times (n+m)$ matrix of rank l whose rows span the subspace w and call the matrix w a matrix representation of the subspace w.

Example 4.5 *(The Attenuated Classical Polar Space) For fixed positive integers n and m, let $\mathbb{F}_q^n$ be the classical polar space with rank N as in Example 4.4 and $w = (0^{(m,n)}\ I^{(m)})$. Then the quotient space $\mathbb{F}_q^{n+m}/w$ is isomorphic to $\mathbb{F}_q^n$. Let P be the collection of all subspaces $x = (x_1\ x_2)$ of $\mathbb{F}_q^{n+m}$ with $x \cap w = \{0\}$, where x_1 is a totally isotropic subspace of $\mathbb{F}_q^n$ and x_2 is a matrix. Ordered by inclusion, P is a strictly semilattice with the* *rank function* $\ell(x) = \dim x$ *and the parameters*

$$|P_r| = q^{rm} \begin{bmatrix} N \\ r \end{bmatrix}_q \prod_{i=0}^{r-1} (q^{N+e-i-1} + 1), \quad \theta(r,s,t) = \begin{bmatrix} t-r \\ s-r \end{bmatrix}_q .$$

where e *as in Example 4.4. For* $1 \le \alpha < \beta \le N \le m + e - 2$, *let* P_β *be the point set and* P_α *be the block set. Then we obtain uniform* (n, cn, k, m)*-CBC with* $n = |P_\beta|, c = \theta(0, \alpha, \beta), m = |P_\alpha|$ *and* $k = \max\{2\theta(0,\alpha,\beta) - \theta(0,\alpha,\beta-1), 3\theta(0,\alpha,\beta) - 3\theta(0,\alpha,\beta-1)\}$. *For* $(m + e - 2) < N$, *pick* $M = (2N + m + e - 2)/3$. *If* $M \le \alpha < \beta \le N$, *let* P_α *be the point set and* P_β *be the block set. Then we obtain uniform* (n, cn, k, m)*-CBC with* $n = |P_\alpha|, m = |P_\beta|, c = m\theta(0,\alpha,\beta)/n$ *and* $k = \max\{2c - \Omega, 3c - 3\Omega\}$.

Example 4.6 *(The Map) Let* P *be the collection of all pairs* (w, f), *where* w *is a subset of* $[N] := \{1, 2, \ldots, N\}$ *and* $f : w \to [N]$ *is a map. Ordered by inclusion, that is* $(w, f) \preceq (u, g)$ *if* $w \subseteq u$ *and* $g|_w = f$, P *is a strictly semilattice with the rank function* $\ell(w, f) = |w|$ *and the parameters*

$$|P_r| = N^r \binom{N}{r}, \quad \theta(r,s,t) = \binom{t-r}{s-r}.$$

For $1 \le \alpha < \beta \le N$, *let* P_β *be the point set and* P_α *be the block set. Then we obtain uniform* (n, cn, k, m)*-CBC with* $n = N^\beta \binom{N}{\beta}, c = \binom{\beta}{\alpha}, m = N^\alpha \binom{N}{\alpha}$ *and* $k = \max\{2\binom{\beta}{\alpha} - \binom{\beta-1}{\alpha}, 3\binom{\beta}{\alpha} - 3\binom{\beta-1}{\alpha}\}$.

Example 4.7 *(The Injective Map) Let* P *be the collection of all pairs* (w, f), *where* w *is a subset of* $[N]$ *and* $f : w \to [N]$ *is an injective map. Ordered by inclusion,* P *is a strictly semilattice with the rank function* $\ell(w, f) = |w|$ *and the parameters*

$$|P_r| = \binom{N}{r} N(N-1)\cdots(N-r+1), \quad \theta(r,s,t) = \binom{t-r}{s-r}.$$

Pick $M = N + (1 - \sqrt{4N+5})/2$. *If* $1 \le \alpha < \beta \le M$, *let* P_β *be the point set and* P_α *be the block set. Then we obtain uniform* (n, cn, k, m)*-CBC with* $n = |P_\beta|, c = \binom{\beta}{\alpha}, m = |P_\alpha|$ *and* $k = \max\{2\binom{\beta}{\alpha} - \binom{\beta-1}{\alpha}, 3\binom{\beta}{\alpha} - 3\binom{\beta-1}{\alpha}\}$. *If* $M \le \alpha < \beta \le N$, *let* P_α *be the point set and* P_β *be the block set. Then we obtain uniform* (n, cn, k, m)*-CBC with* $n = |P_\alpha|, m = |P_\beta|, c = m\theta(0,\alpha,\beta)/n$ *and* $k = \max\{2c - \Omega, 3c - 3\Omega\}$.

Example 4.8 *(The Bilinear Form) Let P be the collection of all pair (w, f), where w is a subspace of $\mathbb{F}_q^N$ and $f : w \to \mathbb{F}_q^N$ is a linear map. Ordered by inclusion, P is a strictly semilattice with the rank function $\ell(w, f) = \dim w$ and the parameters*

$$|P_r| = q^{rN} \begin{bmatrix} N \\ r \end{bmatrix}_q, \quad \theta(r, s, t) = \begin{bmatrix} t-r \\ s-r \end{bmatrix}_q.$$

For $1 \le \alpha < \beta \le N$, let P_β be the point set and P_α be the block set. Then we obtain uniform (n, cn, k, m)-CBC with $n = q^{\beta N} \begin{bmatrix} N \\ \beta \end{bmatrix}_q, c = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}_q, m = q^{\alpha N} \begin{bmatrix} N \\ \alpha \end{bmatrix}_q$ and

$$k = \max \left\{ 2 \begin{bmatrix} \beta \\ \alpha \end{bmatrix}_q - \begin{bmatrix} \beta-1 \\ \alpha \end{bmatrix}_q, 3 \begin{bmatrix} \beta \\ \alpha \end{bmatrix}_q - 3 \begin{bmatrix} \beta-1 \\ \alpha \end{bmatrix}_q \right\}.$$

Example 4.9 *(The Injective Linear Map) Let P be the collection of all pair (w, f), where w is a subspace of $\mathbb{F}_q^N$ and $f : w \to \mathbb{F}_q^N$ is an injective linear map. Ordered by inclusion, P is a strictly semilattice with the rank function $\ell(w, f) = \dim w$ and the parameters*

$$|P_r| = q^{r(r-1)/2} \prod_{i=N-r+1}^{N} (q^i - 1) \begin{bmatrix} N \\ r \end{bmatrix}_q, \quad \theta(r, s, t) = \begin{bmatrix} t-r \\ s-r \end{bmatrix}_q.$$

For $1 \le \alpha < \beta \le N$, let P_β be the point set and P_α be the block set. Then we obtain uniform (n, cn, k, m)-CBC with $n = q^{\beta(\beta-1)/2} \prod_{i=N-r+1}^{N} (q^i - 1) \begin{bmatrix} N \\ \beta \end{bmatrix}_q, c = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}_q, m = q^{\alpha(\alpha-1)/2} \prod_{i=N-r}^{N} (q^i - 1) \begin{bmatrix} N \\ \alpha \end{bmatrix}_q$ and

$$k = \max \left\{ 2 \begin{bmatrix} \beta \\ \alpha \end{bmatrix}_q - \begin{bmatrix} \beta-1 \\ \alpha \end{bmatrix}_q, 3 \begin{bmatrix} \beta \\ \alpha \end{bmatrix}_q - 3 \begin{bmatrix} \beta-1 \\ \alpha \end{bmatrix}_q \right\}.$$

Example 4.10 *(The Square Bilinear Form) Let P be the collection of all pair (w, f), where w is a subspace of $\mathbb{F}_q^N$ and $f : w \to w$ is a bilinear form on w. Ordered by inclusion, P is a strictly semilattice with the rank function $\ell(w, f) = \dim w$ and the parameters*

$$|P_r| = q^{r^2} \begin{bmatrix} N \\ r \end{bmatrix}_q, \quad \theta(r, s, t) = \begin{bmatrix} t-r \\ s-r \end{bmatrix}_q.$$

For $1 \le \alpha < \beta \le N$, let P_β be the point set and P_α be the block set. Then we obtain uniform (n, cn, k, m)-CBC with $n = q^{\beta^2} \begin{bmatrix} N \\ \beta \end{bmatrix}_q$, $c = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}_q$, $m = q^{\alpha^2} \begin{bmatrix} N \\ \alpha \end{bmatrix}_q$ and

$$k = \max \left\{ 2\begin{bmatrix} \beta \\ \alpha \end{bmatrix}_q - \begin{bmatrix} \beta - 1 \\ \alpha \end{bmatrix}_q, 3\begin{bmatrix} \beta \\ \alpha \end{bmatrix}_q - 3\begin{bmatrix} \beta - 1 \\ \alpha \end{bmatrix}_q \right\}.$$

Example 4.11 *(The Alternating Form) Let P be the collection of all pair (w, f), where w is a subspace of $\mathbb{F}_q^N$ and $f : w \to w$ is an alternating bilinear form on w. Ordered by inclusion, P is a strictly semilattice with the rank function $\ell(w, f) = \dim w$ and the parameters*

$$|P_r| = q^{r(r-1)/2} \begin{bmatrix} N \\ r \end{bmatrix}_q, \quad \theta(r, s, t) = \begin{bmatrix} t - r \\ s - r \end{bmatrix}_q.$$

For $1 \le \alpha < \beta \le N$, let P_β be the point set and P_α be the block set. Then we obtain uniform (n, cn, k, m)-CBC with $n = q^{\beta(\beta-1)/2} \begin{bmatrix} N \\ \beta \end{bmatrix}_q$, $c = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}_q$, $m = q^{\alpha(\alpha-1)/2} \begin{bmatrix} N \\ \alpha \end{bmatrix}_q$ and

$$k = \max \left\{ 2\begin{bmatrix} \beta \\ \alpha \end{bmatrix}_q - \begin{bmatrix} \beta - 1 \\ \alpha \end{bmatrix}_q, 3\begin{bmatrix} \beta \\ \alpha \end{bmatrix}_q - 3\begin{bmatrix} \beta - 1 \\ \alpha \end{bmatrix}_q \right\}.$$

Example 4.12 *(The Hermitian Form) Let P be the collection of all pair (w, f), where w is a subspace of $\mathbb{F}_q^N$ and $f : w \to w$ is a Hermitian form on w, where $q = r^2$ is square. Ordered by inclusion, P is a strictly semilattice with the rank function $\ell(w, f) = \dim w$ and the parameters*

$$|P_r| = q^{r^2/2} \begin{bmatrix} N \\ r \end{bmatrix}_q, \quad \theta(r, s, t) = \begin{bmatrix} t - r \\ s - r \end{bmatrix}_q.$$

For $1 \le \alpha < \beta \le N$, let P_β be the point set and P_α be the block set. Then we obtain uniform (n, cn, k, m)-CBC with $n = q^{\beta^2/2} \begin{bmatrix} N \\ \beta \end{bmatrix}_q$, $c = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}_q$, $m = q^{\alpha^2/2} \begin{bmatrix} N \\ \alpha \end{bmatrix}_q$ and

$$k = \max \left\{ 2\begin{bmatrix} \beta \\ \alpha \end{bmatrix}_q - \begin{bmatrix} \beta - 1 \\ \alpha \end{bmatrix}_q, 3\begin{bmatrix} \beta \\ \alpha \end{bmatrix}_q - 3\begin{bmatrix} \beta - 1 \\ \alpha \end{bmatrix}_q \right\}.$$

Example 4.13 *(The Symmetric Bilinear Form) Let P be the collection of all pair (w, f), where w is a subspace of $\mathbb{F}_q^N$ and $f : w \to w$ is a symmetric* *bilinear form on w. Ordered by inclusion, P is a strictly semilattice with the rank function $\ell(w,f) = \dim w$ and the parameters*

$$|P_r| = q^{r(r+1)/2}\begin{bmatrix} N \\ r \end{bmatrix}_q, \quad \theta(r,s,t) = \begin{bmatrix} t-r \\ s-r \end{bmatrix}_q.$$

For $1 \le \alpha < \beta \le N$, let P_β be the point set and P_α be the block set. Then we obtain uniform (n, cn, k, m)-CBC with $n = q^{\beta(\beta+1)/2}\left[\begin{smallmatrix} N \\ \beta \end{smallmatrix}\right]_q, c = \left[\begin{smallmatrix} \beta \\ \alpha \end{smallmatrix}\right]_q, m = q^{\alpha(\alpha+1)/2}\left[\begin{smallmatrix} N \\ \alpha \end{smallmatrix}\right]_q$ and

$$k = \max\left\{2\begin{bmatrix} \beta \\ \alpha \end{bmatrix}_q - \begin{bmatrix} \beta-1 \\ \alpha \end{bmatrix}_q, 3\begin{bmatrix} \beta \\ \alpha \end{bmatrix}_q - 3\begin{bmatrix} \beta-1 \\ \alpha \end{bmatrix}_q\right\}.$$

5 Semilattices from affine spaces

In this section we give four families of examples of strictly semilattices with rank $N+1$. These examples are from an affine space.

Example 5.1 *([19] The Affine Geometry) Let $\mathbb{F}_q^N$ and P be as in Example 4.2. Let P' be the collection of all cosets of subspaces in P together with the empty set $\emptyset$. We define $\ell(\emptyset) = 0$. Ordered by inclusion, P' is a strictly semilattice with the rank function $\ell(x) = \dim x + 1$ and the parameters*

$$|P'_{r+1}| = q^{N-r}\begin{bmatrix} N \\ r \end{bmatrix}_q, \quad \theta(r+1, s+1, t+1) = \begin{bmatrix} t-r \\ s-r \end{bmatrix}_q.$$

Pick $M = (N-2)/2$. For $1 \le \alpha < \beta \le M$, let P_β be the point set and P_α be the block set. Then we obtain uniform (n, cn, k, m)-CBC with $n = |P_\beta|, c = \theta(0,\alpha,\beta), m = |P_\alpha|$ and $k = \max\{2\theta(0,\alpha,\beta) - \theta(0,\alpha,\beta-1), 3\theta(0,\alpha,\beta) - 3\theta(0,\alpha,\beta-1)\}$. For $M < \alpha < \beta \le N$, let P_α be the point set and P_β be the block set. Then we obtain uniform (n, cn, k, m)-CBC with $n = |P_\alpha|, m = |P_\beta|, c = m\theta(0,\alpha,\beta)/n$ and $k = \max\{2c - \Omega, 3c - 3\Omega\}$.

Example 5.2 *(The Affine Attenuated Space) Let $\mathbb{F}_q^{n+N}$ and X be as in Example 4.3. Let P' be the collection of all cosets of subspaces in P together with the empty set $\emptyset$. Ordered by inclusion, P' is a strictly semilattice with the rank function $\ell(x) = \dim x + 1$ and the parameters*

$$|P'_{r+1}| = q^{n+N+rn-r}\begin{bmatrix} N \\ r \end{bmatrix}_q, \quad \theta(r+1, s+1, t+1) = \begin{bmatrix} t-r \\ s-r \end{bmatrix}_q.$$

For $1 \leq \alpha < \beta \leq N \leq n$, *let* P_β *be the point set and* P_α *be the block set. Then we obtain uniform* (n, cn, k, m)*-CBC with* $n = |P_\beta|, c = \theta(0, \alpha, \beta), m = |P_\alpha|$ *and* $k = \max\{2\theta(0, \alpha, \beta) - \theta(0, \alpha, \beta - 1), 3\theta(0, \alpha, \beta) - 3\theta(0, \alpha, \beta - 1)\}$. *For* $n < N$, *pick* $M = (N + n + 1)/2 - 1$. *If* $M \leq \alpha < \beta \leq N$, *let* P_α *be the point set and* P_β *be the block set. Then we obtain uniform* (n, cn, k, m)*-CBC with* $n = |P_\alpha|, m = |P_\beta|, c = m\theta(0, \alpha, \beta)/n$ *and* $k = \max\{2c - \Omega, 3c - 3\Omega\}$.

Example 5.3 *(The Affine Classical Polar Space) Let* $\mathbb{F}_q^n$ *and* X *be as in Example 4.4. Let* P' *be the collection of all cosets of subspaces in* P *together with the empty set* $\emptyset$. *Ordered by inclusion,* P' *is a strictly semilattice with the rank function* $\ell(x) = \dim x + 1$ *and the parameters*

$$|P'_{r+1}| = q^{2N+\delta-r} \begin{bmatrix} N \\ r \end{bmatrix}_q \prod_{i=0}^{r-1} (q^{N+e-i-1} + 1), \quad \theta(r+1, s+1, t+1) = \begin{bmatrix} t-r \\ s-r \end{bmatrix}_q .$$

Pick $M = (2N + e - 2)/3$. *If* $M \leq \alpha < \beta \leq N$, *let* P_α *be the point set and* P_β *be the block set. Then we obtain uniform* (n, cn, k, m)*-CBC with* $n = |P_\alpha|, m = |P_\beta|, c = m\theta(0, \alpha, \beta)/n$ *and* $k = \max\{2c - \Omega, 3c - 3\Omega\}$.

Example 5.4 *(The Affine Attenuated Classical Polar Space). Let* $\mathbb{F}_q^{n+m}$ *and* P *be as in Example 4.5. Let* P' *be the collection of all cosets of subspaces in* P *together with the empty set* $\emptyset$. *Ordered by inclusion,* P' *is a strictly semilattice with the rank function* $\ell(x) = \dim x + 1$ *and the parameters*

$$|P'_{r+1}| = q^{2N+\delta+m+rm-r} \begin{bmatrix} N \\ r \end{bmatrix}_q \prod_{i=0}^{r-1} (q^{N+e-i-1} + 1),$$

$$\theta(r+1, s+1, t+1) = \begin{bmatrix} t-r \\ s-r \end{bmatrix}_q .$$

If $1 \leq \alpha < \beta \leq N \leq m$, *let* P_β *be the point set and* P_α *be the block set. Then we obtain uniform* (n, cn, k, m)*-CBC with* $n = |P_\beta|, c = \theta(0, \alpha, \beta), m = |P_\alpha|$ *and*

$$k = \max\{2\theta(0, \alpha, \beta) - \theta(0, \alpha, \beta - 1), 3\theta(0, \alpha, \beta) - 3\theta(0, \alpha, \beta - 1)\}.$$

For $m < N$, *pick* $M = (2N + m + e - 2)/3$. *If* $M \leq \alpha < \beta \leq N$, *let* P_α *be the point set and* P_β *be the block set. Then we obtain uniform* (n, cn, k, m)*-CBC with* $n = |P_\alpha|, m = |P_\beta|, c = m\theta(0, \alpha, \beta)/n$ *and* $k = \max\{2c - \Omega, 3c - 3\Omega\}$.

6 Semilattices from distance-regular graphs

In this section, we give five families of examples of strictly semilattices with rank N. These examples are from distance-regular graphs.

Let Γ be a connected regular graph. We identify Γ with the set of vertices. For two vertices u and v, let $\partial(u,v)$ denote the usual distance between u and v. The maximum value of the distance function in Γ is called the *diameter* of Γ, denoted by $D(\Gamma)$. For vertices u and v at distance i, define

$$C(u,v) = C_i(u,v) = \{w \mid \partial(u,w) = i-1, \partial(w,v) = 1\},$$
$$A(u,v) = A_i(u,v) = \{w \mid \partial(u,w) = i, \partial(w,v) = 1\}.$$

For the cardinalities of these sets we use lower case letters $c_i(u,v)$ and $a_i(u,v)$. A connected regular graph Γ with diameter D is called *distance-regular* if $c_i(u,v)$ and $a_i(u,v)$ depend only on i for all $1 \le i \le D$.

Let Γ be a distance-regular graph. A r-subset $\{x_1, x_2, \ldots, x_r\} \subseteq \Gamma$ is said to be a *t-clique* of Γ with size r if any two distinct vertices in $\{x_1, x_2, \ldots, x_r\}$ are at distance t.

Example 6.1 *([1] The Johnson Graph) Let $\binom{[n]}{t}$ be the collection of all t-subsets of $[n]$. The Johnson graph $J(n,t)$ is defined on $\binom{[n]}{t}$ such that two vertices A and B are adjacent if and only if $|A \cap B| = t-1$. Let $N = \lfloor n/t \rfloor$ and P be the collection of all t-cliques of the Johnson graph $J(n,t)$ together with the empty set $\emptyset$. Ordered by inclusion, P is a strictly semilattice with the rank function $\ell(x) = |x|$ and the parameters*

$$|P_r| = \binom{n}{rt}\frac{(rt)!}{(t!)^r r!}, \quad \mu(r,s,t) = \binom{t-r}{s-r}.$$

Moreover, by Theorems 3.1 and 3.2 we can obtain uniform CBCs.

Example 6.2 *([1] The Grassmann Graph) Let $\begin{bmatrix}\mathbb{F}_q^n\\ t\end{bmatrix}_q$ be the collection of all t-dimensional subspaces of $\mathbb{F}_q^n$. The Grassmann graph $J_q(n,t)$ is defined on $\begin{bmatrix}\mathbb{F}_q^n\\ t\end{bmatrix}_q$ such that two vertices A and B are adjacent if and only if* $\dim(A \cap B) = t-1$. *A strongly t-clique of $J_q(n,t)$ with size ℓ is a subfamily $\{A_1, A_2, \ldots, A_\ell\} \subseteq \begin{bmatrix}\mathbb{F}_q^n\\ t\end{bmatrix}_q$ such that* $\dim(A_1 + A_2 + \cdots + A_\ell) = t\ell$. *Let $N = \lfloor n/t \rfloor$ and P be the collection of all strongly t-cliques of the Grassmann graph $J_q(n,t)$ together with the empty set $\emptyset$. Ordered by inclusion, P is a strictly semilattice with the rank function $\ell(x) = |x|$ and the parameters*

$$|P_r| = \frac{q^{t^2 r(r-1)/2}}{r!}\prod_{i=0}^{r-1}\begin{bmatrix}n-it\\ t\end{bmatrix}_q, \quad \mu(r,s,t) = \binom{t-r}{s-r}.$$

For $1 \le \alpha < \beta \le N$, *let* P_β *be the point set and* P_α *be the block set. Then we obtain uniform* (n, cn, k, m)*-CBC with* $n = |P_\beta|, c = \theta(0,\alpha,\beta), m = |P_\alpha|$ *and* $k = \max\{2\theta(0,\alpha,\beta) - \theta(0,\alpha,\beta-1), 3\theta(0,\alpha,\beta) - 3\theta(0,\alpha,\beta-1)\}$.

A distance-regular graph Γ with diameter $D \ge 2$ is said to be *antipodal*, if $\partial(x,y) = \partial(x,z) = D$ and $y \ne z$ implies $\partial(y,z) = D$. For $u \in \Gamma$, the size of the set $\{v \in \Gamma \mid \partial(u,v) = D\}$ depends only on D, denoted by k_D.

Example 6.3 *([1, 7] The Antipodal Distance-Regular Graph) Suppose that* Γ *is an antipodal distance-regular graph with diameter* D. *Let* $N = k_D + 1$ *and* P *be the collection of all* D*-cliques of* Γ *together with the empty set* $\emptyset$. *Ordered by inclusion,* P *is a strictly semilattice with the rank function* $\ell(x) = |x|$ *and the parameters*

$$|P_r| = \binom{k_D+1}{r}|\Gamma|/(k_D+1), \quad \mu(r,s,t) = \binom{t-r}{s-r}.$$

Pick $M = \lfloor (k_D+1)/2 \rfloor$. *If* $1 \le \alpha < \beta \le M$, *let* P_β *be the point set and* P_α *be the block set. Then we obtain uniform* (n, cn, k, m)*-CBC with* $n = |P_\beta|, c = \theta(0,\alpha,\beta), m = |P_\alpha|$ *and* $k = \max\{2\theta(0,\alpha,\beta) - \theta(0,\alpha,\beta - 1), 3\theta(0,\alpha,\beta) - 3\theta(0,\alpha,\beta-1)\}$. *If* $M \le \alpha < \beta \le N$, *let* P_α *be the point set and* P_β *be the block set. Then we obtain uniform* (n, cn, k, m)*-CBC with* $n = |P_\alpha|, m = |P_\beta|, c = m\theta(0,\alpha,\beta)/n$ *and* $k = \max\{2c - \Omega, 3c - 3\Omega\}$.

A distance-regular graph Γ is said to be of *order* (l, u) if, for each vertex $x \in \Gamma$, the induced subgraph on $\Gamma(x)$ is a disjoint union of $u+1$ cliques with size l. Then each maximal clique is of size $l+1$, and each vertex is contained in $u+1$ maximal cliques.

Example 6.4 *([1, 7] The Distance-Regular Graph of Order* (l,k)*) Suppose that* Γ *is a distance-regular graph of order* (l,u). *Let* $N = l+1$ *and* P *be the collection of all cliques of* Γ *together with the empty set* $\emptyset$. *Ordered by inclusion,* P *is a strictly semilattice with the rank function* $\ell(x) = |x|$ *and the parameters*

$$|P_r| = \binom{l+1}{r}n(u+1)/(l+1), \quad \mu(r,s,t) = \binom{t-r}{s-r}.$$

Pick $M = \lfloor (l+1)/2 \rfloor$. *If* $1 \le \alpha < \beta \le M$, *let* P_β *be the point set and* P_α *be the block set. Then we obtain uniform* (n, cn, k, m)*-CBC with* $n = |P_\beta|, c = \theta(0,\alpha,\beta), m = |P_\alpha|$ *and* $k = \max\{2\theta(0,\alpha,\beta) - \theta(0,\alpha,\beta -$

$1), 3\theta(0,\alpha,\beta) - 3\theta(0,\alpha,\beta-1)\}$. *If* $M \le \alpha < \beta \le N$, *let* P_α *be the point set and* P_β *be the block set. Then we obtain uniform* (n, cn, k, m)*-CBC with* $n = |P_\alpha|, m = |P_\beta|, c = m\theta(0,\alpha,\beta)/n$ *and* $k = \max\{2c - \Omega, 3c - 3\Omega\}$.

Recall that a subgraph induced on a subset Δ of Γ is called *strongly closed* if $C(u,v) \cup A(u,v) \subseteq \Delta$ for every pair of vertices $u, v \in \Delta$. A distance-regular graph Γ with diameter D is called *D-bounded*, if every strongly closed subgraph of Γ is regular, and any two vertices x and y are contained in a common strongly closed subgraph with diameter $\partial(x,y)$. A regular strongly closed subgraph of Γ is called a *subspace* of Γ. For any two subspaces Δ_1 and Δ_2 of Γ, $\Delta_1 + \Delta_2$ denotes the minimum subspace containing Δ_1 and Δ_2.

Proposition 6.1 *([10, Lemma 2.1]) Let* Γ *be a D-bounded distance-regular graph with diameter* $D \ge 2$. *For* $1 \le i+1 \le i+s \le i+s+t \le D$, *suppose that* Δ *and* Δ' *are two subspaces satisfying* $\Delta \subseteq \Delta'$, $D(\Delta) = i$ *and* $D(\Delta') = i+s+t$. *Then the number of the subspaces with diameter* $i+s$ *containing* Δ *and contained in* Δ', *denoted by* $N(i, i+s, i+s+t)$, *is*

$$\frac{(b_i - b_{i+s+t})(b_{i+1} - b_{i+s+t}) \cdots (b_{i+s-1} - b_{i+s+t})}{(b_i - b_{i+s})(b_{i+1} - b_{i+s}) \cdots (b_{i+s-1} - b_{i+s})}.$$

Example 6.5 *(The D-Bounded Distance-Regular Graph) Let* Γ *be a D-bounded distance-regular graph with* $D = N$. *For* $x \in \Gamma$, *let* P *be the collection of all subspaces* Δ *containing* x *in* Γ. *Ordered by inclusion,* P *is a strictly semilattice with the rank function* $\ell(\Delta) = D(\Delta)$ *and the parameters*

$$|P_r| = N(0, r, D), \quad \mu(r,s,t) = N(r,s,t).$$

Moreover, by Theorems 3.1 and 3.2 we can obtain uniform CBCs.

7 Conclusion

In this paper, we explore a novel construction of CBCs. By investigating a series of semilattices originated from sets, vector spaces and maps, affine spaces and distance-regular graphs, we construct many uniform CBCs. In this sense, our work builds the connection between the CBCs and semilattices. We hope this new point of view can stimulate further research and provide new constructions of CBCs. We have to say that our constructions will provide more choices in solving practical problems.

Huang and Weng [12] introduced pooling spaces, the first author, Ma and Wang [11] introduced partition semilattices. Note that strictly semilattices are not only pooling spaces but also partition semilattices. It seems interesting to construct CBCs using pooling spaces and partition semilattices.

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