Eternal domination in split graphs

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Abstract

The eternal domination number of a split graph is shown to equal either its domination number, or its domination number plus one. A characterization of the split graphs which achieve equality in either instance is given. It is shown that the problem of deciding whether the domination number of a Hamiltonian split graph is at most a given integer k is NP-complete, as is the problem of deciding whether the eternal domination number of a Hamiltonian split graph is at most a given integer k. Finally, the problem of computing the eternal domination number is shown to be polynomial for any subclass of split graphs for which the domination number can be computed in polynomial time, in particular for strongly chordal split graphs.

1 Introduction

In *eternal domination*, the goal is to dynamically maintain a fixed-size dominating set of a graph G subject to the condition that at each time $t = 0, 1, 2, \ldots$ the set be reconfigured so that it contains a specified vertex. This problem was first studied by Goddard, Hedetniemi and Hedetniemi in [9]. Several different models were proposed, depending on how the reconfiguration may take place. We consider the "all guards move" model, in which we say that *a dominating set D_1 can be reconfigured to a dominating set D_2* if and only if there is a bijection $r : D_1 \to D_2$ such that $r(x) \in N[x]$ for all $x \in D_1$. The guarding terminology comes from the analogy of having mobile guards stationed at vertices of D_1 and the reconfiguration involving each guard either staying in place, or moving along an edge of G to a vertex of D_2. A survey of results on eternal domination and related problems can be found in [14]. Reconfiguration problems for dominating sets have been considered by several authors, for example see [5, 8, 10].

An *eternal dominating set* is a dominating set D, starting from which a dominating set of size $|D|$ can be dynamically maintained as above. Note that each set which arises in the process is also an eternal dominating set. The smallest integer d for which there exists an eternal dominating set of cardinality d is called the *eternal domination number* of G, and denoted by $\gamma_{all}^{\infty}(G)$. The problem of computing the eternal domination number is polynomially solvable for several graph families including trees [13], and proper interval graphs [3]. It follows from the recursive construction of cographs that the eternal domination number of a connected cograph which is not complete is two. Since threshold graphs are cographs, the same statement holds for them.

In general it is difficult to determine the complexity of deciding whether a given graph has eternal domination number at most a given integer k. The main difficulty seems to lie in establishing membership in a particular complexity class, as this requires a succinct certificate that a dominating set of the given size can be maintained "forever". Deciding whether a given graph has an eternal dominating set of size at most a given integer k is hard for co-$\mathrm{NP}^{\mathrm{NP}}$, but not known to belong to this complexity class [11].

Our focus is on eternal domination in split graphs. Preliminary results are discussed in the next section. One of these shows that the problem of deciding whether the domination number of a Hamiltonian split graph is at most a given integer k is NP-complete. The complexity of this problem is listed as "unknown" on the website `http://www.graphclasses.org`. The main results of the paper are presented in Section 3. We show that the eternal domination number of a split graph is either its domination number, or its domination number plus one, and characterize the split graphs such that equality holds in either case. It is then shown that, for split graphs, there exists a succinct certificate that there exists an eternal dominating set of a given size. Hence, for split graphs, the problem belongs to NP. This makes it possible to prove that the problem of deciding whether the eternal domination number of a Hamiltonian split graph is at most a given integer k is NP-complete. The section concludes by showing that the problem of computing the eternal domination number is Polynomial for graphs which are simultaneously a split graph and a strongly chordal graph, and some other subclasses of split graphs.

The eternal domination number of a graph in the all guards move model is typically denoted by $\gamma_{\mathrm{m}}^{\infty}$. We prefer to adopt the notation γ_{all}^{∞} used in [7] because it leaves no doubt that all guards can move, rather than only m of them.

2 Definitions and preliminary results

Recall that a *split graph* is a graph G such that $V(G)$ admits a partition $(\mathcal{C}(G), \mathcal{I}(G))$ into a clique $\mathcal{C} = \mathcal{C}(G)$ and an independent set $\mathcal{I} = \mathcal{I}(G)$. A split graph may have several such partitions. It can be assumed that $\mathcal{I}$ is maximal. Thus, every vertex in $\mathcal{C}$ can be assumed to have a neighbour in $\mathcal{I}$.

Also recall that a *dominating set* in a graph G is a subset $D \subseteq V$ such that $N[v] \cap D \neq \emptyset$ for every $v \in V$, where $N[v] = N(v) \cup \{v\}$ is the *closed neighbourhood of* v. Each vertex in $N[v] \cap D$ is said to *dominate* v. The minimum size of a dominating set of G is the *domination number* of G and denoted by $\gamma = \gamma(G)$.

Observation 2.1 *Every connected split graph G has a minimum dominating set which is a subset of $\mathcal{C}$.*

Theorem 2.2 *The problem of deciding whether a split graph with a given Hamilton cycle has a dominating set of size at most a given integer k is NP-complete.*

Proof: Since the Hamilton cycle is given as part of the instance, it is possible to recognize a valid instance of the problem in polynomial time. Since it is easy to verify whether a given set of vertices is a dominating set of size at most k, the problem belongs to NP. The transformation is from the problem of deciding whether a split graph has a dominating set of size at most a given integer k, which was proved to be NP-complete by Bertossi [1].

Suppose a split graph G and an integer $k > 0$ are given. Let $\mathcal{C}(G) = \{c_1, c_2, \ldots, c_p\}$ and $\mathcal{I}(G) = \{i_1, i_2, \ldots, i_q\}$. We may assume G is neither a complete graph nor the complement of a complete graph, so $p > 0$, and $q > 0$. Construct G' from G as follows. For each vertex $i_j \in \mathcal{I}(G)$, add two new vertices a_{i_j} and b_{i_j}. Add edges so that the subgraph of G' induced by $\mathcal{C}(G) \cup \{a_{i_j}, b_{i_j} : i_j \in \mathcal{I}(G)\}$ is complete. Finally, for each $i_j \in \mathcal{I}(G)$, join a_{i_j} and b_{i_j} to i_j. Note that G' is a split graph, and the sequence

$$H = c_1, c_2, \ldots c_p, a_{i_1}, i_1, b_{i_1}, a_{i_2}, i_2, b_{i_2}, \ldots, a_{i_q}, i_q, b_{i_q}, c_1$$

is a Hamilton cycle in G'. The transformed instance of the problem consists of G', the Hamilton cycle, H, and the integer k. The transformation can clearly be accomplished in polynomial time.

It remains to show that G' has a dominating set of size at most k if and only if G has a dominating set of size at most k. Since any dominating set of G is a dominating set of G', if G has a dominating set of size at most k, then so does G'. Since, for every vertex $x \in \{a_{i_j}, b_{i_j} : i_j \in \mathcal{I}(G')\}$ there is a vertex $y \in \mathcal{C}(G)$ such that $N(y) \supseteq N(x)$, any dominating set of G' of size at most k can be transformed into a dominating set of G of size at most k. □

As mentioned in Section 1, a difficulty in determining the complexity of deciding whether a graph G has an eternal dominating set of size at most a *given* integer k is providing a succinct certificate that there is a strategy for dominating the graph "forever". We next define a digraph which will be shown to provide a certificate that such a strategy exists. We will show in the next section that, when G is a split graph, there is a certificate of size polynomial in the number of vertices of G.

Let G be a graph and k be a positive integer. An *eternal dominating k-configuration* for G is an arc-labelled directed graph $\mathcal{D}$ such that:

- the vertices of $\mathcal{D}$ are dominating sets of G of the same cardinality, which is at most k;
- there is an arc from vertex X to vertex Y labelled with the set $Y - X$ if and only if X can be reconfigured to Y; and
- for every vertex X of $\mathcal{D}$, the union of the labels on the arcs leaving X equals $V(G) - X$.

The following proposition can be used to show that, if the integer k is fixed, then the problem of deciding whether a graph G has $\gamma^{\infty}_{all} \leq k$ is Polynomial. The ideas are taken from [12] and carry over to our situation even though the results given there are neither in this form, nor for this variant of eternal domination. One begins with the collection of all dominating sets of $V(G)$ with cardinality k (as k is fixed these can be found in polynomial time) and constructs an arc-labelled digraph with these as vertices using the rule in the second point above. Then, vertices which do not meet the criteria in the third point above are iteratively deleted until an eternal dominating k-configuration is found, or no vertices remain.

Proposition 2.3 ([12]) *A graph G has $\gamma^{\infty}_{all} \leq k$ if and only if it has an eternal dominating k-configuration. A k-subset $D \subseteq V$ is an eternal dominating set of G if and only if it belongs to some eternal dominating k-configuration.*

To compute the eternal domination number, one can use a binary search strategy and employ the above algorithm for $\lceil \log_2(|V|) \rceil$ values of k. Alternatively, since eternal domination can be regarded as a pursuit-evasion game, one can do the same using the relational approach in [2] instead of looking for an eternal dominating k-configuration. Both of these methods are $O(|V|^{f(\gamma_{all}^{\infty})})$. A $O(17.54^{|V|})$ algorithm for computing the eternal domination number is presented in [7].

3 Eternal domination in split graphs

Our main results are presented in this section. There is a sense in which the first lemma is the main result of the paper. It is the essential ingredient that makes all of the other results possible. Using this lemma we give tight bounds on the eternal domination number of split graphs, and a characterization of the situations in which equality holds. We then turn to algorithmic issues and show that it is NP-complete to decide whether the eternal domination number of a Hamiltonian split graph is at most a given integer k. The section and paper concludes by identifying some subclasses of split graphs for which the eternal domination number can be computed in polynomial time.

Lemma 3.1 *Let G be a split graph. Then $\gamma_{all}^{\infty} \leq d$ if and only if for each $x \in V$ there exists a dominating set D_x with $x \in D_x$, and $|D_x| = d$.*

Proof: By definition the elements of an eternal dominating set can be reconfigured to obtain an eternal dominating set containing any specific vertex. Hence the required sets all exists when $d = \gamma_{all}^{\infty}$. Thus they exist for any larger integer.

We shall show that if the sets D_x all exist, then $\gamma_{all}^{\infty} \leq d$. Since G is a split graph, for any $x \in V$, we can assume $D_x - \{x\} \subseteq C$.

Suppose the elements of D_v are required to be reconfigured to obtain the set D_w, where $v \neq w$. We must show that there exists a bijection $r : D_v \to D_w$ such that $r(x) \in N[x]$ for each $x \in D_v$. If $v \in \mathcal{I}$ then, since D_w is a dominating set such that $D_w - \{w\} \subseteq C$, there is a vertex $u \in N[v] \cap D_w \cap C$; set $r(v) = u$. Similarly, if $w \in \mathcal{I}$, since D_v is a dominating set such that $D_v - \{v\} \subseteq C$, there is a vertex $z \in N[w] \cap D_v \cap C$; set $r(z) = w$ (note that if $v \in \mathcal{I}$, then $z \neq v$, and otherwise $r(v)$ was not previously defined). Since $(D_v \cup D_w) - \{v, w\} \subseteq C$, and $|D_v| = |D_w| = d$, the mapping r can be extended to a bijection. It follows that the reconfiguration is possible.

Thus each set D_x, $x \in V$, is an eternal dominating set, and $\gamma_{all}^{\infty} \leq d$. □

It follows that γ_{all}^{∞} is the smallest positive integer d such that, for each

$x \in V$, there exists a dominating set D_x with $x \in D_x$, and $|D_x| = d$.

Corollary 3.2 *Let G be a split graph. Then $\gamma(G) \leq \gamma^{\infty}_{all}(G) \leq \gamma(G) + 1$.*

Proof: The lower inequality is clear. It suffices to prove the upper inequality. The result is clear if G has only one vertex, so assume $|V| \geq 2$. Thus $\gamma < |V|$. Let $D \subseteq C$ be a minimum dominating set, and $w \notin D$.

Let $x \in V$. If $x \in D$, then set $D_x = D \cup \{w\}$. If $x \notin D$, set $D_x = D \cup \{x\}$. Each set D_x is a dominating set of size $\gamma + 1$. The result now follows from Lemma 3.1. □

We next characterize the graphs which achieve equality in Corollary 3.2. A vertex x of a graph G is a *domination critical vertex* if $\gamma(G - \{x\}) < \gamma(G)$.

Corollary 3.3 *Let G be a split graph which is not complete, and in which every vertex in C is adjacent to a vertex in $\mathcal{I}$. Then $\gamma^{\infty}_{all}(G) = \gamma(G)$ if and only if every vertex in $\mathcal{I}$ is a domination critical vertex.*

Proof:

Suppose $\gamma^{\infty}_{all}(G) = \gamma(G) > 1$. Then, by Lemma 3.1, for any $x \in \mathcal{I}$ there exists a dominating set D_x of size $\gamma(G)$ with $x \in D_x$. Since the set $D_x - \{x\}$ is a dominating set of $G - \{x\}$, it follows that x is a domination critical vertex.

Now suppose every vertex in $\mathcal{I}$ is a domination critical vertex. Then, for any $x \in \mathcal{I}$ there exists a minimum dominating set D_x consisting of x and a minimum dominating set of $G - \{x\}$. Hence, by Lemma 3.1, $\gamma^{\infty}_{all}(G) = \gamma(G)$.

□

We now describe examples of split graphs in which $\gamma^{\infty}_{all} = \gamma$. Suppose C is a complete graph on p vertices. For $1 \leq t \leq p$, let $\mathcal{I}$ be the set of t-subsets of the vertices in C. Join a vertex x belonging to C to a vertex T belonging to $\mathcal{I}$ if and only if $x \in T$. Call the resulting split graph $S_{p,t}$. Then the domination number of $S_{p,t}$ is $p - t + 1$, as any set of $p - t + 1$ vertices in C contains a vertex adjacent to each vertex in $\mathcal{I}$, and any set of $p - t$ vertices in C is the complement of the neighbourhood of some vertex in $\mathcal{I}$. Every vertex in C is adjacent to a vertex in $\mathcal{I}$, and for any $T \in \mathcal{I}$ the complement of the neighbourhood of T is a dominating set of $S_{p,t}$ with size $p - t = \gamma - 1$. Thus, by Corollary 3.3, we have $\gamma^{\infty}_{all} = \gamma$.

Let G be a split graph in which every vertex in $\mathcal{I}$ has degree 2. Let $Aux(G)$ be the graph with $V(Aux(G)) = C(G)$ and $xy \in E(Aux(G))$ if and only if there exists $i \in \mathcal{I}(G)$ which is adjacent to both x and y. It is clear that for any graph H there exists a split graph G such that $H = Aux(G)$. Let $D \subseteq C$ be a dominating set in G. (For any dominating set of G, a vertex not in C can be replaced by one of its two neighbours.) Then D corresponds to a vertex cover of H. It follows that $\gamma(G)$ equals the vertex covering number of H, $\tau(H)$. We have therefore proved that the problem of deciding whether a split graph has a dominating set of size at most a given integer k is NP-complete when restricted to the situation where every vertex in $\mathcal{I}$ has degree 2. (This result is almost certainly not new.)

The graph $Aux(G)$ helps provide connections to the problem of characterizing the split graphs with $\gamma_{all}^{\infty} = \gamma$. Suppose $H = Aux(G)$ is connected. Then every vertex in C is adjacent to a vertex in $\mathcal{I}$. By Corollary 3.3, $\gamma_{all}^{\infty}(G) = \gamma(G)$ if and only if every vertex in $\mathcal{I}$ is a domination critical vertex. Equivalently, $\gamma_{all}^{\infty}(G) = \gamma(G)$ if and only if every edge $e \in E(H)$ is a *vertex cover critical edge* (i.e. $\tau(H - e) < \tau(H)$). Since the complement of a vertex cover is an independent set, this is also equivalent to H having the property that the deletion of any edge increases the independence number. Graphs with this property are studied in [4]. A characterization is given, but not one which is easy to check. Our Corollary 3.3 can be regarded as giving another characterization of these graphs which is also not easy to check.

On the other hand, if $H = Aux(G)$ is bipartite, then by König's Theorem we have $\gamma_{all}^{\infty}(G) = \gamma(G)$ if and only if the deletion of any edge of H reduces the size of a maximum matching. This is true if and only if every edge of H belongs to a maximum matching, that is, if and only if H is a disjoint union of copies of K_2.

One can also define $Aux(G)$ when the vertices in $\mathcal{I}$ may have any degree, in which case it is a hypergraph. Again, a dominating set in G corresponds to a vertex cover in $Aux(G)$.

We now show that, in the case of split graphs, an eternal dominating k-configuration provides a succinct certificate that there exists an eternal dominating set of size at most k.

Lemma 3.4 *Let G be a split graph with $\gamma_{all}^{\infty} \leq k$. Then G has an eternal dominating k-configuration with at most $|V(G)|$ vertices.*

Proof: We may assume $k \leq |C|$. For each vertex x, let D_x be a dominating set of size k which contains x and $k - 1$ vertices of $C - \{x\}$. The sets D_x exist because G is a split graph. For any vertices $x, y \in V(G)$, the set D_x can be reconfigured to D_y as in the proof of Lemma 3.1. It follows that the sets D_x, $x \in V(G)$, form the vertices of a dominating configuration. □

Theorem 3.5 *The problem of deciding whether a split graph G with a given Hamilton cycle has an eternal dominating set of size at most a given integer k is NP-complete.*

Proof: We show first that the problem is in NP. Suppose $\gamma_{all}^{\infty}(G) \leq k$. The certificate consists of an eternal k-dominating configuration with $n = |V(G)|$ vertices, which exists by Lemma 3.4. Since each vertex consists of a subset of $V(G)$, there are most $n(n-1)$ arcs, and each arc label is a subset of $V(G)$, the certificate is of size polynomial in n. Further, the certificate can be verified by checking that the definition of a dominating k-configuration is satisfied, and this can be accomplished in time polynomial in n.

The transformation is from the NP-complete problem of deciding whether a given split graph with a given Hamilton cycle has a dominating set of size at most a given positive integer, which is NP-complete by Theorem 2.2. Suppose an instance of this problem, a split graph G with a Hamilton cycle M and a positive integer t, is given. We may assume that G is not a complete graph. Thus, $\mathcal{I} \neq \emptyset$. Let G' be the split graph obtained from G in two steps. First, choose any vertex $c \in \mathcal{C}$ with a neighbour in $\mathcal{I}$, add a new vertex c' and join it to every element of $N[c]$. Second, add a vertex g adjacent to all vertices in $\mathcal{C} \cup \{c'\}$. The transformed instance consists of G', the Hamilton cycle obtained by inserting g, c' into M immediately after c, and the integer $t+1$. The transformation can be accomplished in polynomial time.

Observe that $\gamma(G') = \gamma(G)$. Further, since the vertex g is not a domination critical vertex of G', we have $\gamma_{all}^{\infty}(G') = \gamma(G') + 1$ by Corollary 3.3. Thus, if $\gamma(G) \leq t$, then $\gamma_{all}^{\infty}(G') = \gamma(G) + 1 \leq t + 1$. Similarly, if $\gamma_{all}^{\infty}(G') \leq t + 1$, then $t + 1 \geq \gamma(G) + 1$. □

We conclude by identifying some subclasses of split graphs, different from threshold graphs, for which the problem of determining the eternal domination number is Polynomial.

Lemma 3.6 *Let $\mathcal{G}$ be a subclass of split graphs which is closed with respect to vertex deletion and on which the problem of finding the domination number is Polynomial. Then the problem of finding the eternal domination number is also Polynomial on $\mathcal{G}$.*

Proof: Let $G \in \mathcal{G}$. By Corollary 3.2 the quantity $\gamma_{all}^{\infty}(G)$ is either $\gamma(G)$ or $\gamma(G) + 1$. By Corollary 3.3, it equals

$$\gamma_{all}^{\infty}(G) = 1 + \max\{\gamma_{all}^{\infty}(G - \{x\}) : x \in \mathcal{I}(G)\}.$$

The maximum can be found by solving at most $|\mathcal{I}|$ domination problems. By hypothesis, each vertex-deleted subgraph of G is also in $\mathcal{G}$. Since the domination problem is Polynomial on $\mathcal{G}$, the maximum can be done in time polynomial in $|V(G)|$. □

Corollary 3.7 *The problem of computing the eternal domination number is Polynomial on the class of graphs which are simultaneously a split graph and a strongly chordal graph.*

Proof: Strongly chordal graphs are closed with respect to vertex deletion, as are split graphs. The domination number of a strongly chordal graph can be computed in polynomial time [6]. The result now follows from Lemma 3.6. □

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