On the λ-fold Spectra for Bipartite Subgraphs of $^2K_4$

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Abstract

For a graph H and a positive integer λ, let $^{\lambda}H$ denote the multigraph obtained by replacing each edge of H with λ parallel edges. Let G be a multigraph with edge multiplicity 2 and with C_4 as its underlying simple graph. We find necessary and sufficient conditions for the existence of a G-decomposition of $^{\lambda}K_n$ for all positive integers λ and n.

1 Introduction

If a and b are integers with $a \leq b$, we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$. Let $\mathbb{Z}_n$ be the group of integers modulo n. For a finite set S and a positive integer λ, we let $^{\lambda}S$ denote the multiset that contains every element of S exactly λ times. For example, $^3[a, b]$ is the multiset $\{a, a, a, a+1, a+1, a+1, ..., b-1, b-1, b-1, b, b, b\}$. Similarly for a graph H, we let $^{\lambda}H$ denote the multigraph obtained by replacing each edge in H with λ parallel edges. Thus $^{\lambda}K_n$ denotes the λ-fold complete multigraph of order n. We note that a multigraph is not required to contain multiple edges. However, our graphs contain no loops. If we wish to emphasize that a given graph does not contain parallel edges, then we refer to it as a simple graph. For positive integers r and s, let $K_{r\times s}$ denote the complete multipartite graph

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with r parts of cardinality s each. The *order* and *size* of a multigraph G refer to $|V(G)|$ and $|E(G)|$, respectively.

Let $V({}^{\lambda}K_n) = [0, n-1]$. The *label* of an edge $\{i, j\}$ in ${}^{\lambda}K_n$ is defined to be $|i-j|$. The *length* of an edge $\{i, j\}$ in ${}^{\lambda}K_n$ is defined to be $\min\{|i-j|, n-|i-j|\}$. Thus if the elements of $V({}^{\lambda}K_n)$ are placed in order as vertices of an equisided n-gon, then the length of edge $\{i, j\}$ is the shortest distance around the polygon between i and j. Note that if n is odd, then ${}^{\lambda}K_n$ consists of λn edges of length i for $i \in [1, \frac{n-1}{2}]$. Furthermore if n is even, then ${}^{\lambda}K_n$ consists of λn edges of length i for $i \in [1, \frac{n}{2}-1]$ and $\frac{\lambda n}{2}$ edges of length $\frac{n}{2}$.

Let $V({}^{\lambda}K_n) = \mathbb{Z}_n$ and let G be a subgraph of ${}^{\lambda}K_n$. By *rotating* G, we mean applying the permutation $i \mapsto i+1$ to $V(G)$. Note that rotating an edge does not change its length.

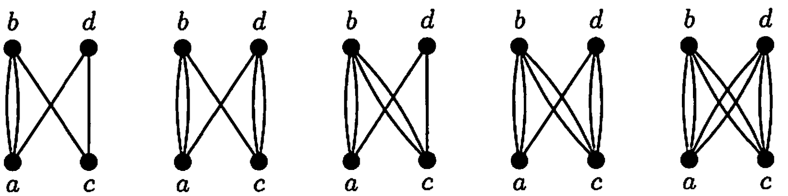
Alternatively, we may let $V({}^{\lambda}K_n) = \mathbb{Z}_{n-1} \cup \{\infty\}$. As expected, rotating a subgraph G of ${}^{\lambda}K_n$ in this case continues to mean applying the permutation $i \mapsto i+1$ to $V(G)$, with the convention that $\infty + 1 = \infty$. If neither i nor j is the ∞-vertex, then the label and length of the edge $e = \{i, j\}$ are defined as if e is in ${}^{\lambda}K_{n-1}$. The label and length of an edge $\{i, \infty\}$ are both defined to be ∞. Again, rotating an edge does not change its length. In this case, if n is odd, then ${}^{\lambda}K_n$ consists of $\lambda(n-1)$ edges of length ∞ along with $\lambda(n-1)$ edges of length i for $i \in [1, \frac{n-3}{2}]$ and $\frac{\lambda(n-1)}{2}$ edges of length $\frac{n-1}{2}$. Furthermore if n is even, then then ${}^{\lambda}K_n$ consists of $\lambda(n-1)$ edges of length ∞ along with $\lambda(n-1)$ edges of length i for $i \in [1, \frac{n-2}{2}]$.

Let K and G be graphs with G a subgraph of K. A *G-decomposition* of K is a set (or multiset) $\Delta = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of K each of which is isomorphic to G (and is called a *G-block*) and such that each edge of K appears in exactly one G-block. If there exists a G-decomposition of K, then we say G *divides* K and write $G|K$. A G-decomposition of K is also known as a *(K, G)-design*. A $({}^{\lambda}K_n, G)$-design is called a *G-design of order n and index λ*. A $({}^{\lambda}K_n, G)$-design Δ is said to be *cyclic* if rotating a G-block in Δ yields another G-block in Δ. If $V({}^{\lambda}K_n) = \mathbb{Z}_{n-1} \cup \{\infty\}$, then a cyclic $({}^{\lambda}K_n, G)$-design is also called a *1-rotational $({}^{\lambda}K_n, G)$-design*. The study of graph decompositions is generally known as the study of graph designs, or G-designs. For recent surveys on G-designs of index 1, see [1] and [3].

Let G be a graph. A classical problem in the study of graph designs is to find necessary and sufficient conditions for the existence of a G-decomposition of ${}^{\lambda}K_n$. This is known as the *spectrum problem* for G. The set of all such n is called the *spectrum for G-designs of index λ*, or alternatively the *index λ spectrum for G*. The spectra for G-designs of index 1 has been determined for several classes of graphs including cycles, paths, stars and all graphs of order at most 5 (see [1]).

In recent years, there have been some investigations of G-designs of index λ where G is a multigraph with edge multiplicity at least 2. For example, in [6] Carter determined the spectrum for G-designs of index λ for all connected cubic multigraphs G of order at most 6. Sarvate and various co-authors have investigated G-designs of index λ for various multigraphs G of small order (see for example [7], [10], [12], and [13]). See also [5] and [8] for the spectrum for G-designs where G is a multigraph of small order.

In this article, we investigate G-decompositions of ${}^{\lambda}K_n$, where G is a multigraph with edge multiplicity 2 and with C_4 as the simple graph underlying G. Figure 1 shows the five possibilities for such a G. We find necessary and sufficient conditions for the existence of a G-decomposition of ${}^{\lambda}K_n$ for all integers $\lambda \geq 2$.



$G_1[a,b,c,d]$ $G_2[a,b,c,d]$ $G_3[a,b,c,d]$ $G_4[a,b,c,d]$ $G_5[a,b,c,d]$

Figure 1: The five multigraphs with edge multiplicity 2 and C_4 as the underlying simple graph.

Figure 1 gives a key that denotes a labeled copy for each of the five multigraphs of interest. For example, $G_1[a,b,c,d]$ refers to the multigraph with vertex set $\{a,b,c,d\}$ and edge multiset $\big\{\{a,b\},\{a,b\},\{b,c\},\{c,d\},\{d,a\}\big\}$.

2 Main Results

The index λ spectra for G_1 and G_2 are settled in [12] and in [6], respectively. Thus we will focus on the three remaining multigraphs. The case $\lambda = 2$ for all bipartite subgraphs of ${}^{2}K_4$ is settled in [2].

2.1 $({}^{\lambda}K_n, G_3)$-designs

We begin with some obvious necessary conditions.

Lemma 1. *Let $\lambda \geq 2$ and $n \geq 4$ be integers. If there exists a $({}^{\lambda}K_n, G_3)$-design, then the following necessarily hold:*

1. *if* $\gcd(\lambda, 6) = 1$, *then* $n \equiv 0,\ 1,\ 4,$ *or* $9 \pmod{12}$*;*
2. *if* $\gcd(\lambda, 6) = 2$, *then* $n \equiv 0$ *or* $1 \pmod 3$*;*

3. *if* $\gcd(\lambda, 6) = 3$, *then* $n \equiv 0$ *or* $1 \pmod 4$*;*

4. *if* $\gcd(\lambda, 6) = 6$, *then* $n \geq 4$.

Proof. Let λ and n be as stated and suppose there exists a $({}^{\lambda}K_n, G_3)$-design. Since the number of edges in G_3 is 6, we must have that $6|\lambda n(n-1)/2$, and thus $12|\lambda n(n-1)$. If $\gcd(\lambda, 6) = 1$, then $12|n(n-1)$, and thus $n \equiv 0$, 1, 4, or 9 (mod 12). If $\gcd(\lambda, 6) = 2$, then $6|n(n-1)$, and thus $n \equiv 0$ or 1 (mod 3). Similarly, if $\gcd(\lambda, 6) = 3$, then $4|n(n-1)$, and thus $n \equiv 0$ or 1 (mod 4). Finally, if $\gcd(\lambda, 6) = 6$, then $2|n(n-1)$, which is always true. ■

From Allen et al. [2], we have the following for index 2.

Lemma 2. *There exists a* $({}^2K_n, G_3)$*-design for all* $n \equiv 0$ *or* $1 \pmod 3$ *where* $n \neq 3$.

Next, we settle both the index 3 and index 6 spectra for G_3.

Lemma 3. *There exists a* $({}^3K_n, G_3)$*-design for all* $n \equiv 0$ *or* $1 \pmod 4$.

Proof. We consider two cases.

CASE 1: $n \equiv 0 \pmod 4$.
Let $n = 4x$ and let $V({}^3K_{4x}) = \mathbb{Z}_{4x-1} \cup \{\infty\}$. Let

$$\begin{aligned}\Delta = &\{G_3[\infty, j, 2+j, 1+j] : 0 \leq j \leq 4x-2\} \\ &\cup \{G_3[4i+j, j, 4i+2+j, 1+j] : 1 \leq i \leq x-1,\ 0 \leq j \leq 4x-2\}.\end{aligned}$$

It is easily checked that Δ is a 1-rotational $({}^3K_{4x}, G_3)$-design.

CASE 2: $n \equiv 1 \pmod 4$.
Let $n = 4x+1$ and let $V({}^3K_{4x+1}) = \mathbb{Z}_{4x+1}$. Let

$$\Delta = \{G_3[4i+j, j, 4i-2+j, 1+j] : 1 \leq i \leq x,\ 0 \leq j \leq 4x\}.$$

It is easily checked that Δ is a cyclic $({}^3K_{4x}, G_3)$-design. ■

Lemma 4. *There exists a* $({}^6K_n, G_3)$*-design for all* $n \geq 4$.

Proof. We consider four cases.

CASE 1: $n \equiv 0$ or $1 \pmod 3$.
By Lemma 2 there exists a $({}^2K_n, G_3)$-design. Hence, we can obtain a $({}^6K_n, G_3)$-design from three copies of a $({}^2K_n, G_3)$-design.

CASE 2: $n \equiv 5$ or $8 \pmod{12}$.
By Lemma 3 there exists a $({}^3K_n, G_3)$-design. Hence, we can obtain a $({}^6K_n, G_3)$-design from two copies of a $({}^3K_n, G_3)$-design.

CASE 3: $n \equiv 2 \pmod{12}$.
Let $n = 12x + 14$. Then we are looking to show that G_3 divides ${}^6K_{12x+14}$.

We view our ${}^6K_{12x+14}$ as ${}^6K_6 \cup {}^6K_{12x+8} \cup {}^6K_{6,12x+8}$. It is proved in the above cases that $G_3|{}^6K_6$ and $G_3|{}^6K_{12x+8}$. We now must show that $G_3|{}^6K_{6,12x+8}$. Clearly ${}^2K_{3,2}$ divides ${}^6K_{6,12x+8}$, so all that remains to be shown is that $G_3|{}^2K_{3,2}$. Let ${}^2K_{3,2}$ have vertex bipartition $\{\{u_1, u_2, u_3\}, \{v_1, v_2\}\}$. Then $\{G_3[v_1, u_1, v_2, u_3], G_3[v_1, u_2, v_2, u_3]\}$ is a $({}^2K_{3,2}, G_3)$-design.

CASE 4: $n \equiv 11 \pmod{12}$.
Let $n = 12x + 11$. Then we are looking to show that G_3 divides ${}^6K_{12x+11}$. We view our ${}^6K_{12x+11}$ as ${}^6K_5 \cup {}^6K_{12x+6} \cup {}^6K_{5,12x+6}$. It is proved in the above cases that $G_3|{}^6K_5$ and $G_3|{}^6K_{12x+6}$. We now must show that $G_3|{}^6K_{5,12x+6}$. Clearly ${}^3K_{5,2}$ divides ${}^6K_{5,12x+6}$, so all that remains to be shown is that $G_3|{}^3K_{5,2}$. Let ${}^3K_{5,2}$ have vertex bipartition $\{\{u_1, u_2, u_3, u_4, u_5\}, \{v_1, v_2\}\}$. Then $\{G_3[v_1, u_1, v_2, u_5], G_3[v_1, u_2, v_2, u_1], G_3[v_1, u_3, v_2, u_2], G_3[v_1, u_4, v_2, u_3], G_3[v_1, u_5, v_2, u_4]\}$ is a $({}^3K_{5,2}, G_3)$-design. ■

Finally, we have all the necessary building blocks to settle the index λ spectrum for G_3.

Theorem 5. *For any positive integers $\lambda \geq 2$ and $n \geq 4$, there exists a $({}^\lambda K_n, G_3)$-design if and only if $12|\lambda n(n-1)$.*

Proof. The necessary conditions are established by the fact that the number of edges in G_3 must divide the number of edges in ${}^\lambda K_n$. To show sufficiency, we use the following 4-case breakdown prescribed by Lemma 1.

CASE 1: $\lambda \equiv 0 \pmod 6$.
Let $\lambda = 6t$. By Lemma 1, we need to show that G_3 divides ${}^{6t}K_n$ for $n \geq 4$. By Lemma 4 there exists a $({}^6K_n, G_3)$-design. Hence, we can obtain a $({}^{6t}K_n, G_3)$-design from t copies of a $({}^6K_n, G_3)$-design.

CASE 2: $\lambda \equiv 1$ or $5 \pmod 6$.
We note that $\lambda = 5$ is the least possible edge multiplicity that meets the criterion for this case of the proof. Thus $\lambda = 2t + 3$ for some integer $t \geq 1$. By Lemma 1, we need to show that G_3 divides ${}^{2t+3}K_n$ for $n \equiv 0$, 1, 4, or 9 (mod 12). By Lemmas 2 and 3 there exist both a $({}^2K_n, G_3)$-design and a $({}^3K_n, G_3)$-design. Hence, we can obtain a $({}^{2t+3}K_n, G_3)$-design from t copies of a $({}^2K_n, G_3)$-design and a single $({}^3K_n, G_3)$-design.

CASE 3: $\lambda \equiv 2$ or $4 \pmod 6$.
Let $\lambda = 2t$ such that $t \not\equiv 0 \pmod 3$. By Lemma 1, we need to show that G_3 divides ${}^{2t}K_n$ for $n \equiv 0$ or $1 \pmod 3$. By Lemma 2 there exists a $({}^2K_n, G_3)$-design. Hence, we can obtain a $({}^{2t}K_n, G_3)$-design from t copies of a $({}^2K_n, G_3)$-design.

CASE 4: $\lambda \equiv 3 \pmod 6$.
Let $\lambda = 6t + 3$. By Lemma 1, we need to show that G_3 divides ${}^{6t+3}K_n$ for $n \equiv 0$ or $1 \pmod 4$. By Lemma 3 there exists a $({}^3K_n, G_3)$-design. Hence,

we can obtain a $({}^{6t+3}K_n, G_3)$-design from $2t+1$ copies of a $({}^{3}K_n, G_3)$-design. ■

2.2 $({}^{\lambda}K_n, G_4)$-designs

Again, we begin with some necessary conditions.

Lemma 6. *Let $\lambda \geq 2$ and $n \geq 4$ be integers. If there exists a $({}^{\lambda}K_n, G_4)$-design, then the following necessarily hold:*

1. *if $\gcd(\lambda, 7) = 1$, then $n \equiv 0$ or $1 \pmod 7$;*
2. *if $\gcd(\lambda, 7) = 7$, then $n \geq 4$.*

Proof. Let λ and n be as stated and suppose there exists a $({}^{\lambda}K_n, G_4)$-design. Since the number of edges in G_4 is 7, we must have that $7|\lambda n(n-1)/2$, and thus $14|\lambda n(n-1)$. If $\gcd(\lambda, 7) = 1$, then $14|n(n-1)$, and thus $n \equiv 0$, 1, 7, or 8 (mod 14). If $\gcd(\lambda, 7) = 7$, then $2|n(n-1)$, which is always true. ■

From Allen et al. [2], we have the following for index 2.

Lemma 7. *There exists a $({}^{2}K_n, G_4)$-design for all $n \equiv 0$ or $1 \pmod 7$.*

Next, we show the only insufficiencies of the necessary conditions in Lemma 6 (i.e., when λ is 3 or 5) before settling the index 7 spectrum for G_4.

Lemma 8. *There does not exist a $({}^{3}K_n, G_4)$-design for any n.*

Proof. Suppose Δ is a $({}^{3}K_n, G_4)$-design. We note that each G_4-block in Δ contains exactly one edge of multiplicity 1 and three edges with multiplicity 2. Since each edge in ${}^{3}K_n$ has edge multiplicity 3, each pair of vertices must be incident with at least one edge of multiplicity 1 within a G_4-block of Δ. This leads to a contradiction, as the number of vertex pairings in ${}^{3}K_n$ (i.e., the size of K_n) exceeds the number of G_4-blocks in Δ. ■

Lemma 9. *There does not exists a $({}^{5}K_n, G_4)$-design for any n.*

Proof. Suppose Δ is a $({}^{5}K_n, G_4)$-design. Then the proof proceeds similarly to that of Lemma 8. ■

Lemma 10. *There exists a $({}^{7}K_n, G_4)$-design for all $n \geq 4$.*

Proof. We consider four cases.

CASE 1: $n \equiv 0 \pmod 4$.
Let $n = 4x$ and let $V({}^7K_{4x}) = \mathbb{Z}_{4x-1} \cup \{\infty\}$. Let

$$\begin{aligned}\Delta = &\Big\{G_4[\infty, j, 1+j, 2+j], G_4[1+j, j, \infty, 2+j] : 0 \le j \le 4x-2\Big\}\\ &\cup \Big\{G_4[4i-1+j, j, 4i+1+j, 1+j],\\ &\qquad G_4[4i+1+j, j, 4i-1+j, 1+j] :\\ &\qquad\qquad 1 \le i \le x-1,\ 0 \le j \le 4x-2\Big\}.\end{aligned}$$

It is easily checked that Δ is a 1-rotational $({}^7K_{4x}, G_4)$-design.
CASE 2: $n \equiv 1 \pmod 4$.
Let $n = 4x+1$ and let $V({}^7K_{4x+1}) = \mathbb{Z}_{4x+1}$. Let

$$\begin{aligned}\Delta = &\Big\{G_4[4i-2+j, j, 4i+j, 1+j], G_4[4i+j, j, 4i-2+j, 1+j] :\\ &\qquad 1 \le i \le x,\ 0 \le j \le 4x\Big\}.\end{aligned}$$

It is easily checked that Δ is a cyclic $({}^7K_{4x+1}, G_4)$-design.
CASE 3: $n \equiv 2 \pmod 4$.
Let $n = 4x+2$ and let $V({}^7K_{4x+2}) = \mathbb{Z}_{4x+1} \cup \{\infty\}$. Let

$$\begin{aligned}\Delta = &\Big\{G_4[\infty, j, 2+j, 1+j], G_4[j, 1+j, \infty, 2+j],\\ &\qquad G_4[2+j, j, 1+j, 3+j] : 0 \le j \le 4x\Big\}\\ &\cup \Big\{G_4[4i+j, j, 4i+2+j, 1+j],\\ &\qquad G_4[4i+2+j, j, 4i+j, 1+j] : 1 \le i \le x-1,\ 0 \le j \le 4x\Big\}.\end{aligned}$$

It is easily checked that Δ is a 1-rotational $({}^7K_{4x+2}, G_4)$-design.
CASE 4: $n \equiv 3 \pmod 4$.

Let $n = 4x + 3$ and let $V({}^7K_{4x+3}) = \mathbb{Z}_{4x+3}$. Let

$$\begin{aligned}\Delta = \Big\{&G_4[3+j, j, 2+j, 1+j], G_4[3+j, 1+j, 2+j, j],\\ &\qquad G_4[3+j, j, 1+j, 4+j] : 0 \le j \le 4x+2\Big\}\\ \cup \Big\{&G_4[4i+1+j, j, 4i+3+j, 1+j],\\ &\qquad G_4[4i+3+j, j, 4i+1+j, 1+j] :\\ &\qquad\qquad 1 \le i \le x-1,\ 0 \le j \le 4x+2\Big\}.\end{aligned}$$

It is easily checked that Δ is a cyclic $({}^7K_{4x+3}, G_4)$-design. ■

Finally, we have all the necessary building blocks to settle the index λ spectrum for G_4.

Theorem 11. *For positive integers $\lambda \ge 2$ and $n \ge 4$, there exists a $({}^\lambda K_n, G_4)$-design if and only if $14|\lambda n(n-1)$ and $\lambda \notin \{3, 5\}$.*

Proof. The necessary condition that $14|\lambda n(n-1)$ is established by the fact that the number of edges in G_4 must divide the number of edges in ${}^\lambda K_n$. The latter condition is proved in Lemmas 8 and 9. To show sufficiency, we consider three cases.

CASE 1: $\lambda \equiv 0 \pmod 7$.
Let $\lambda = 7t$. By Lemma 6, we need to show that G_4 divides ${}^{7t}K_n$ for $n \ge 4$. By Lemma 10 there exists a $({}^7K_n, G_4)$-design. Hence, we can obtain a $({}^{7t}K_n, G_4)$-design from t copies of a $({}^7K_n, G_4)$-design.

CASE 2: $\lambda \not\equiv 0 \pmod 7$ and λ is even.
Let $\lambda = 2t$. By Lemma 6, we need to show that G_4 divides ${}^{2t}K_n$ for $n \equiv 0$ or $1 \pmod 7$. By Lemma 7 there exists a $({}^2K_n, G_4)$-design. Hence, we can obtain a $({}^{2t}K_n, G_4)$-design from t copies of a $({}^2K_n, G_4)$-design.

CASE 3: $\lambda \not\equiv 0 \pmod 7$ and λ is odd.
We note that $\lambda = 9$ is the least possible edge multiplicity that meets the criteria for this case of the proof. Thus $\lambda = 2t + 7$ for some integer $t \ge 1$. By Lemma 6, we need to show that G_4 divides ${}^{2t+7}K_n$ for $n \equiv 0$ or $1 \pmod 7$. By Lemmas 7 and 10 there exist both a $({}^2K_n, G_4)$-design and a $({}^7K_n, G_4)$-design. Hence, we can obtain a $({}^{2t+7}K_n, G_4)$-design from t copies of a $({}^2K_n, G_4)$-design and a single $({}^7K_n, G_4)$-design. ■

2.3 $({}^\lambda K_n, G_5)$-designs

Since G_5 is isomorphic to ${}^2C_4$, we first give the index λ spectrum for C_4 (see [11] and [9]).

Theorem 12. *For any positive integers λ and n, there exists a $({}^{\lambda}K_n, C_4)$-design if and only if* (a) 2 *divides* $\lambda(n-1)$, (b) 8 *divides* $\lambda n(n-1)$, *and* (c) $n \geq 4$.

It is easy to see that for all graphs G and K we have $G|K$ if and only if ${}^2G|{}^2K$. Thus, we have the following.

Theorem 13. *For any positive integers λ and n, there exists a $({}^{\lambda}K_n, G_5)$-design if and only if* (a) 4 *divides* $\lambda(n-1)$, (b) 16 *divides* $\lambda n(n-1)$, (c) $n \geq 4$, *and* (d) λ *is even.*

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