Preimages of Small Geometric Cycles

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Abstract

A graph G is a homomorphic preimage of another graph H, or equivalently G is *H-colorable*, if there exists a graph homomorphism $f : G \to H$. A classic problem is to characterize the family of homomorphic preimages of a given graph H. A *geometric graph* $\overline{G}$ is a simple graph G together with a straight line drawing of G in the plane with the vertices in general position A geometric homomorphism (resp. isomorphism) $\overline{G} \to \overline{H}$ is a graph homomorphism (resp. isomorphism) that preserves edge crossings (resp. and non-crossings). The homomorphism poset $\mathcal{G}$ of a graph G is the set of isomorphism classes of geometric realizations of G partially ordered by the existence of injective geometric homomorphisms. A geometric graph $\overline{G}$ is $\mathcal{H}$-colorable if $\overline{G} \to \overline{H}$ for some $\overline{H} \in \mathcal{H}$. In this paper, we provide necessary and sufficient conditions for $\overline{G}$ to be C_n-colorable for $3 \leq n \leq 5$.

1 Basic Definitions

A graph homomorphism $f : G \to H$ is a vertex function such that for all $u, v \in V(G)$, $uv \in E(G)$ implies $f(u)f(v) \in E(H)$. If such a function exists, we write $G \to H$ and say that G is homomorphic to H, or equivalently, that G is a homomorphic preimage of H. A proper n-coloring of a graph G is a homomorphism $G \to K_n$; thus, G is n-colorable if and only if G is a homomorphic preimage of K_n. (For an excellent overview of the theory of graph homomorphisms, see [4].)

In 1981, Maurer, Salomaa and Wood [9] generalized this notion by defining G to be H-colorable if and only if $G \to H$. They used the notation $\mathcal{L}(H)$ to denote the family of H-colorable graphs. For example, G is C_5-colorable if and only if $G \to C_5$; this means there exists a proper 5-coloring of G such a vertex of color 1 can only be adjacent to vertices of color 2 or 5, but not to vertices of color 3 or 4, *etc.* Maurer *et al.* noted that for odd m and n, C_m is C_n-colorable (*i.e.* $C_m \to C_n$) if and only if $m \geq n$. Since any composition of graph homomorphisms is also a graph homomorphism, this generates the following hierarchy among color families of cliques and odd cycles.

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$$\ldots \mathcal{L}(C_{2n+1}) \subsetneq \mathcal{L}(C_{2n-1}) \subsetneq \cdots \subsetneq \mathcal{L}(C_5) \subsetneq \mathcal{L}(C_3) =$$
$$= \mathcal{L}(K_3) \subsetneq \mathcal{L}(K_4) \subsetneq \cdots \subsetneq \mathcal{L}(K_n) \subsetneq \mathcal{L}(K_{n+1}) \ldots$$

For a given graph H, the H-coloring problem is the decision problem, "Is a given graph H-colorable?" In 1990, Hell and Nešetřil showed that if $\chi(H) \leq 2$, then this problem is polynomial and if $\chi(H) \geq 3$, then it is NP-complete [3].

The concept of H-colorability can been extended to directed graphs. Work has been done by Hell, Zhu and Zhou in characterizing homomorphic preimages of certain families of directed graphs, including oriented cycles [12], [8], [5], oriented paths [7] and local acyclic tournaments [6].

In [1], Boutin and Cockburn generalized the notion of graph homomorphisms to geometric graphs. A *geometric graph* $\overline{G}$ is a simple graph G together with a straight-line drawing of G in the plane with vertices in general position (no three vertices are collinear and no three edges cross at a single point). A geometric graph $\overline{G}$ with underlying abstract graph G is called a *geometric realization* of G. The definition below formalizes what it means for two geometric realizations of G to be considered the same.

Definition 1.1. A *geometric isomorphism*, denoted $f : \overline{G} \to \overline{H}$, is a function $f : V(\overline{G}) \to V(\overline{H})$ such that for all $u, v, x, y \in V(\overline{G})$,

1. $uv \in E(\overline{G})$ if and only if $f(u)f(v) \in E(\overline{H})$, and
2. xy crosses uv in $\overline{G}$ if and only if $f(x)f(y)$ crosses $f(u)f(v)$ in $\overline{H}$.

Relaxing the biconditionals to implications yields the following.

Definition 1.2. A *geometric homomorphism*, denoted $f : \overline{G} \to \overline{H}$, is a function $f : V(\overline{G}) \to V(\overline{H})$ such that for all $u, v, x, y \in V(\overline{G})$,

1. if $uv \in E(\overline{G})$, then $f(u)f(v) \in E(\overline{H})$, and
2. if xy crosses uv in $\overline{G}$, then $f(x)f(y)$ crosses $f(u)f(v)$ in $\overline{H}$.

If such a function exists, we write $\overline{G} \to \overline{H}$ and say that $\overline{G}$ is homomorphic to $\overline{H}$, or equivalently that $\overline{G}$ is a homomorphic preimage of $\overline{H}$.

An easy consequence of this definition is that no two vertices that are adjacent or co-crossing (*i.e.* incident to distinct edges that cross each other) can have the same image (equivalently, can be identified) under a geometric homomorphism.

Boutin and Cockburn define $\overline{G}$ to be *n-geocolorable* if $\overline{G} \to \overline{K}_n$, where $\overline{K}_n$ is some geometric realization of the n-clique. The *geochromatic number* of $\overline{G}$, denoted $X(\overline{G})$, is the smallest n such that $\overline{G}$ is n-geocolorable.

Observe that if a geometric graph of order n has the property that no two of its vertices can be identified under any geometric homomorphism, then $X(\overline{G}) = n$. The existence of multiple geometric realizations of the n-clique for $n > 3$ necessarily complicates the definition of geocolorability, but there is additional structure we can take advantage of.

Definition 1.3. Let $\overline{G}$ and $\widehat{G}$ be geometric realizations of G. Then set $\overline{G} \preceq \widehat{G}$ if there exists a (vertex) injective geometric homomorphism $f : \overline{G} \to \widehat{G}$. The set of isomorphism classes of geometric realizations of G under this partial order, denoted $\mathcal{G}$, is called the *homomorphism poset* of G.

Hence, $\overline{G}$ is n-geocolorable if $\overline{G}$ is homomorphic to some element of the homomorphism poset $\mathcal{K}_n$. In [2], it is shown that $\mathcal{K}_3, \mathcal{K}_4$ and $\mathcal{K}_5$ are all chains. Hence, for $3 \leq n \leq 5$, $\overline{G}$ is n-geocolorable if and only if $\overline{G} \to \overline{K}_n$, where $\overline{K}_n$ is the last element of the chain. By contrast, $\mathcal{K}_6$ has three maximal elements, so $\overline{G}$ is 6-geocolorable if and only if it is homomorphic to one of these three realizations.

Definition 1.4. Let $\mathcal{H}$ denote the homomorphism poset of geometric realizations of a simple graph H. Then $\overline{G}$ is $\mathcal{H}$-geocolorable if and only if $\overline{G} \to \overline{H}$ for some maximal $\overline{H} \in \mathcal{H}$.

In this paper, we provide necessary and sufficient conditions for $\overline{G}$ to be $\mathcal{C}_n$-geocolorable, where $3 \leq n \leq 5$. The structure of the homomorphism posets $\mathcal{C}_n$ for $3 \leq n \leq 5$ is given in [2]. It is worth noting that the geometric cycles are richer than than abstract cycles. All even cycles are homomorphically equivalent to K_2, and as noted earlier, $C_{2k+1} \to C_{2\ell+1}$ if and only if $k \geq \ell$. However, since geometric homomorphisms preserve edge crossings, and both K_2 and C_3 have only plane realizations, this is not true even for small non-plane geometric cycles, as shown in Figure 1.

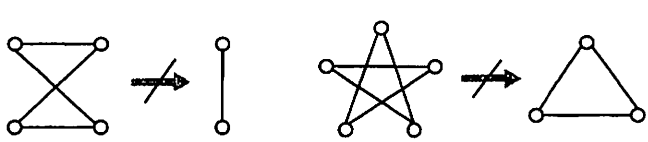

Figure 1: $\widehat{C}_4 \not\to \overline{K_2}$ and $\widehat{C}_5 \not\to \overline{C_3}$

2 Edge-Crossing Graph and Thickness Edge Colorings

Definition 2.1. [2] The *edge-crossing graph* of a geometric graph $\overline{G}$, denoted by $EX(\overline{G})$, is the abstract graph whose vertices correspond to the edges of $\overline{G}$, with adjacency when the corresponding edges of $\overline{G}$ cross.

Clearly, non-crossing edges of $\overline{G}$ correspond to isolated vertices of $EX(\overline{G})$. In particular, $\overline{G}$ is plane if and only if $EX(\overline{G}) \to K_1$. To focus on the crossing structure of $\overline{G}$, we let $\overline{G}_\times$ denote the geometric subgraph of $\overline{G}$ induced by its crossing edges. Note that $EX(\overline{G}_\times)$ is simply $EX(\overline{G})$ with any isolated vertices removed. From [2], a geometric homomorphism $\overline{G} \to \overline{H}$ induces a geometric homomorphism $\overline{G}_\times \to \overline{H}_\times$ as well as graph homomorphisms $G \to H$ and $EX(\overline{G}) \to EX(\overline{H})$.

Definition 2.2. [1] A *thickness edge m-coloring* ϵ of a geometric graph $\overline{G}$ is a coloring of the edges of $\overline{G}$ with m colors such that no two edges of the same color cross. The *thickness* of $\overline{G}$ is the minimum number of colors required for a thickness edge coloring of $\overline{G}$.

From these two definitions, a thickness edge m-coloring ϵ of $\overline{G}$ is a graph homomorphism $\epsilon : EX(\overline{G}) \to K_m$. This can be generalized as follows.

Definition 2.3. A *thickness edge C_m-coloring* ϵ on $\overline{G}$ is a graph homomorphism $\epsilon : EX(\overline{G}) \to C_m$.

Observe that under a thickness edge C_m-coloring, edges are colored with colors numbered $1, 2, \ldots, m$ such that colors assigned to edges that cross each other must be consecutive mod m. Equivalently, edges of color i may only be crossed by edges of colors $i-1$ and $i+1$ mod m. Note also that if $\overline{G}$ has a thickness edge C_m-coloring for $m > 3$, then $\overline{G}$ cannot have three mutually crossing edges.

Definition 2.4. Let ϵ be a thickness edge coloring $\overline{G}$. The plane subgraph of $\overline{G}$ induced by all edges of a given color is called a *monochromatic* subgraph of $\overline{G}$ under ϵ. The monochromatic subgraph corresponding to edge color i is called the i-subgraph of $\overline{G}$ under ϵ.

We assume from now on that $\overline{G}$ has no isolated vertices, which implies that every vertex belongs to at least one monochromatic subgraph of $\overline{G}$ under any thickness edge coloring.

3 Easy Cases: $n = 3$ and $n = 4$

The smallest (simple) cycle is $C_3 = K_3$. As noted in [1], $\overline{G} \to \overline{K}_3$ if and only if $\overline{G}$ is a 3-colorable plane geometric graph. Thus $\overline{G}$ is C_3-geocolorable if and only if G is 3-colorable and $EX(\overline{G})$ is 1-colorable, or more concisely,

$$\overline{G} \to \overline{C}_3 \iff G \to K_3 \text{ and } EX(\overline{G}) \to K_1.$$

Next, C_4 has two geometric realizations, one plane and the other with a single crossing, which we denote $\overline{C}_4$ and $\widehat{C}_4$ respectively. Since $\overline{C}_4 \to \widehat{C}_4$, the homomorphism poset C_4 consists of a two element chain, as shown in Figure 2. Hence $\overline{G}$ is C_4-geocolorable if and only if $\overline{G} \to \widehat{C}_4$.

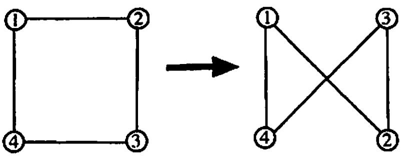

Figure 2: $\overline{C}_4 \to \widehat{C}_4$

If $\overline{G} \to \widehat{C}_4$, then $G \to C_4$ and $EX(\overline{G}) \to EX(\widehat{C}_4) = K_2 \cup 2K_1$. Since any bipartite graph is a preimage of K_2,

$$\overline{G} \to \widehat{C}_4 \implies G \to K_2 \text{ and } EX(\overline{G}) \to K_2,$$

which merely says that any C_4-geocolorable geometric graph is bipartite and thickness-2. In [1], Boutin and Cockburn show that the converse is false, by describing a family of bipartite, thickness-2 geometric graphs of arbitrarily large order with the property that no two vertices can be identified under any geometric homomorphism. The authors do, however, provide necessary and sufficient conditions for $\overline{G} \to \widehat{C}_4$; to describe them requires a definition.

Definition 3.1. The *crossing component graph* $C_\times$ of a geometric graph $\overline{G}$ is the abstract graph whose vertices correspond to the connected components $\overline{C}_1, \overline{C}_2, \ldots, \overline{C}_m$ of $\overline{G}_\times$, with an edge between vertices $\overline{C}_i$ and $\overline{C}_j$ if an edge of $\overline{C}_i$ crosses an edge of $\overline{C}_j$ in $\overline{G}$.

Theorem 3.1. *[1] A geometric graph $\overline{G}$ is homomorphic to $\widehat{C}_4$ if and only if*

1. *$\overline{G}$ is bipartite;*
2. *each component $\overline{C}_i$ of $\overline{G}_\times$ is a plane subgraph;*
3. *$C_\times$ is bipartite.*

If each component of $\overline{G}_\times$ is a plane subgraph, then we can thickness edge color $\overline{G}_\times$ by coloring all the edges in a given component the same color, provided components corresponding to adjacent vertices in $C_\times$ are assigned different colors. Moreover, in this thickness edge coloring, every vertex of $\overline{G}_\times$ appears in only one monochromatic subgraph. Conversely, if there exists a thickness edge m-coloring of $\overline{G}_\times$ in which the monochromatic subgraphs are vertex disjoint, then each component of $\overline{G}_\times$ must be contained in a monochromatic subgraph, and hence be plane. Thus Theorem 3.1 can be rephrased more simply as follows.

Theorem 3.2. *A geometric graph $\overline{G}$ is C_4-geocolorable if and only if*

1. $G \to K_2$, *and*
2. *there exists a thickness edge 2-coloring of $\overline{G}_\times$ in which the two monochromatic subgraphs are vertex disjoint.*

4 Harder Case: $n = 5$

From [2], the homomorphism poset C_5 consists of a chain of five elements, the last of which is the convex realization $\widehat{C}_5$, as shown in Figure 3. Thus if $\overline{G}$ is C_5-geocolorable if and only if $\overline{G} \to \widehat{C}_5$.

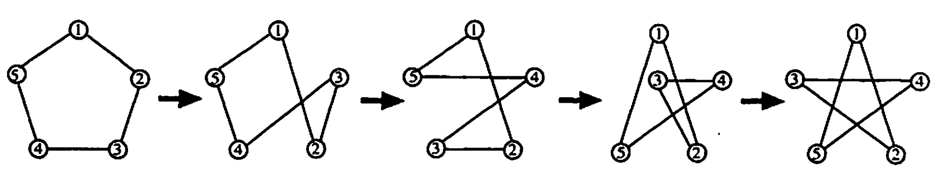

Figure 3: Homomorphism poset C_5

Note that every edge of $\widehat{C}_5$ is a crossing edge, so $(\widehat{C}_5)_\times = \widehat{C}_5$. Moreover, $EX(\widehat{C}_5) = C_5$; see Figure 4, where vertex labels are in bold and edge labels are in italics. (For example, edge *1* is $\{4,5\}$.) Note that with the labeling shown, and with the understanding that all labels are modulo 5, edge i is incident with vertices $2i+2, 2i+3$ and vertex k is incident with edges $3k-1, 3k+1$. Moreover, every vertex label is the sum of the edge labels on the vertex's two incident edges.

Hence if $\overline{G}$ is C_5-geocolorable, then both G and $EX(\overline{G})$ are C_5-colorable. Verifying that this necessary condition is satisfied is no easy matter, however. Maurer *et al.* showed in 1981 that determining whether an abstract graph is C_5-colorable is NP-complete [10]. In 1979, Vesztergombi

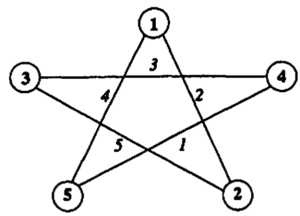

Figure 4: $\widehat{C}_5$ with vertex and edge labels

related C_5-colorability and 5-colorability by proving that for G nonbipartite, $G \to C_5$ if and only if $\chi(G \boxtimes C_5) = 5$, where $\boxtimes$ denotes the strong product [11]. Combined with Vesztergombi's result, we obtain that if $\overline{G}$ is C_5-geocolorable and both G and $EX(\overline{G})$ are nonbipartite, then $\chi(G \boxtimes C_5) = \chi(EX(\overline{G}) \boxtimes C_5) = 5$.

However, as was the case with $n = 4$, $G \to C_5$ and $EX(\overline{G}) \to C_5$ together are not sufficient for $\overline{G}$ to be C_5-geocolorable. For example, $\overline{G}$ in Figure 5 has a C_5-coloring (as indicated by the vertex labels, in bold) as well as a thickness edge C_5-coloring (as indicated by edge labels, in italics). However, since any two vertices of $\overline{G}$ are either adjacent or co-crossing, no two vertices can have the same homomorphic image. In particular, $X(\overline{G}) = 7$.

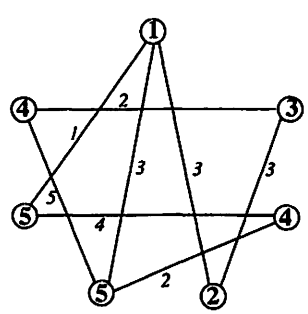

Figure 5: $G \to C_5$ and $EX(\overline{G}) \to C_5$, but $\overline{G} \not\to \widehat{C}_5$

The following theorem provides necessary and sufficient conditions for $\overline{G}$ to be C_5-geocolorable. Unlike Theorem 3.2, the conditions involve only thickness edge colorings, not vertex colorings.

Theorem 4.1. *A geometric graph $\overline{G}$ is C_5-geocolorable if and only if there exists a thickness edge C_5-coloring ϵ of $\overline{G}$ such that*

1. *any vertex of $\overline{G}$ belongs to at most two monochromatic subgraphs under ϵ;*

2. *two monochromatic subgraphs can intersect (i.e. have common vertices) only if the corresponding colors are not consecutive mod 5 (equivalently, the i-subgraph can intersect only with the $(i+2)$-subgraph and $(i+3)$-subgraph);*

3. *each monochromatic subgraph is bipartite and moreover, there exists a partition in the i-subgraph such that all vertices also in the $(i+2)$-subgraph (if any) belong to one partite set, and all those also in the $(i+3)$-subgraph (if any) belong to the other.*

Proof. Assume $f : \overline{G} \to \widehat{C}_5$. This induces an abstract graph homomorphism $EX(\overline{G}) \to C_5$. We can pull back the edge colors shown in Figure 4 to obtain a thickness edge C_5-coloring on $\overline{G}$. Note that every vertex of $\widehat{C}_5$ is incident to edges of exactly two colors that are not consecutive mod 5, so under ϵ, $\overline{G}$ must satisfy conditions (1) and (2).

Since the i-subgraph of $\overline{G}$ maps onto the single i-colored edge $\{2i+2, 2i+3\}$ of $\widehat{C}_5$, by transitivity it is homomorphic to K_2 and is thus bipartite. Moreover, all vertices also in the $(i+2)$-subgraph get mapped to $2i+2$ and all vertices also in the $(i+3)$-subgraph get mapped to $2i+3$. Hence, $\overline{G}$ satisfies (3).

For the converse, assume $\overline{G}$ has a thickness edge C_5-coloring ϵ satisfying conditions (1) - (3). First label all vertices that are in two monochromatic subgraphs with the sum of the two corresponding colors mod 5. To label the vertices that are in only one monochromatic subgraph, say the i-subgraph, first break this bipartite subgraph into connected components. By condition (3), if a component has vertices that have already been labeled, then we can label the remaining vertices either $2i+2$ or $2i+3$ according to the partite set they are in. If a component of the i-subgraph has no vertices that are already labeled, then we can arbitrarily assign the the label $2i+2$ to vertices in one partite set and $2i+3$ to those in the other.

To show that f is a graph homomorphism, let $u, v \in V(\overline{G})$ be adjacent vertices. WLOG edge uv is colored i, so u and v both belong to the i-subgraph. WLOG again, $f(u) = 2i+2$ and $f(v) = 2i+3 \mod 5$. Since these are consecutive mod 5, $f(u)$ and $f(v)$ are adjacent in $\widehat{C}_5$.

Next we show that f is a geometric homomorphism. Suppose that in $\overline{G}$, edge ux crosses edge vy. Since ϵ is a thickness edge C_5-coloring, crossing edges must be assigned consecutive colors mod 5. Assume ux is colored i and vy is colored $i+1$. Then WLOG, $f(u) = 2i+2$, $f(x) = 2i+3$, $f(v) = 2(i+1)+2 = 2i+4$ and $f(y) = 2(i+1)+3 = 2i$. Set $j = 2i+2$ and notice that all pairs of edges of the form $\{j, j+1\}$ and $\{j+2, j+3\}$ cross in $\widehat{C}_5$. □

We show how this theorem can be applied to $\overline{G}$ in Figure 5. The thickness edge C_5-coloring shown violates condition (1) of the theorem because both vertices of degree 3 are incident to edges of 3 different colors. In fact, no thickness edge C_5-coloring on this geometric graph will satisfy all 3 conditions of Theorem 4.1. We being by noting that in any thickness edge C_5-coloring, any 5-cycle of crossings will have to involve all 5 colors. WLOG, we can start with the edge colors shown in Figure 6.

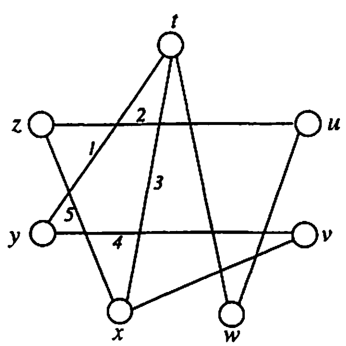

Figure 6: Recoloring $\overline{G}$

Since edge tw crosses edges of colors 2 and 4, it must be colored 3. Next, vertex u is incident to an edge colored 2, so to satisfy condition (2), edge uw must be colored either 2, 4 or 5. Since uw crosses vy which is colored 4, uw must be colored 5. Edge xv crosses edges of color 3 and 5, so it must be colored 4. However, now vertex x appears in 3 monochromatic subgraphs, violating condition (1).

Consider the graph $\overline{H}$ obtained from $\overline{G}$ by deleting xv, shown in Figure 7, with edges colored as required in the previous paragraph. We still have a problem; u and z are vertices in the 2-subgraph belonging also to the 5-subgraph, yet they are an odd distance apart. Hence $\overline{H}$ is also not C_5-colorable.

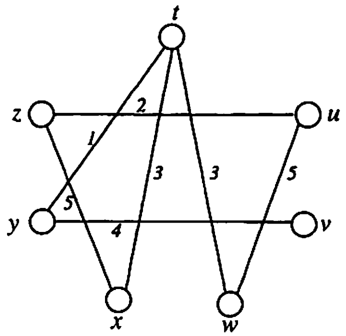

Figure 7: $\overline{H}$

5 5-geocolorability

Recall that $\overline{G}$ is n-geocolorable if and only if $\overline{G}$ is homomorphic to some realization of K_n. In [1], Boutin and Cockburn give a set of necessary but not sufficient conditions (Theorem 4), as well as a set of sufficient but not necessary conditions (Corollary 5.1) for $\overline{G}$ to be 4-geocolorable. Finding necessary and sufficient conditions for a geometric graph to be 5-geocolorable is likely to be even more difficult. However, the work in the previous section allows us to make some progress.

From [2], the homomorphism poset $\mathcal{K}_5$ is chain of length 3, with last element $\widehat{K_5}$, shown in Figure 8. Hence $\overline{G}$ is 5-geocolorable if and only if $\overline{G} \to \widehat{K_5}$.

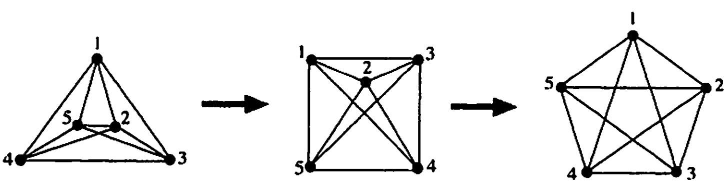

Figure 8: The homomorphism poset $\mathcal{K}_5$.

From [1], $\overline{G} \to \overline{H}$ implies $\overline{G}_\times \to \overline{H}_\times$, although the converse is false. Hence if $\overline{G}$ is 5-geocolorable, then $\overline{G}_\times \to \widehat{C}_5$; equivalently, if $\overline{G}$ is 5-geocolorable then $\overline{G}_\times$ is C_5-geocolorable. The contrapositive is, of course, if $\overline{G}_\times$ is not C_5-geocolorable, then $\overline{G}$ is not 5-geocolorable.

6 Future Work

Finding necessary and sufficient conditions for a geometric graph to be C_6-colorable is complicated by the fact that the homomorphism poset C_6 has two maximal elements, shown in Figure 9 (see [2]). The one on the left is bipartite and thickness-2, while the one on the right is bipartite and thickness-3. We investigate these in a future paper.

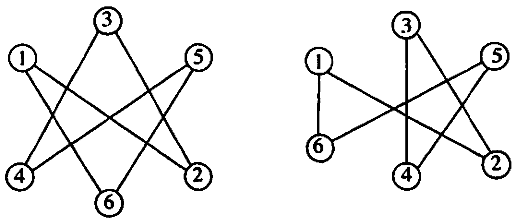

Figure 9: Two maximal elements of C_6.

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