Domination Polynomials of Graph Products

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Abstract

The domination polynomials of binary graph operations, aside from union, join and corona, have not been widely studied. We compute and prove recurrence formulae and properties of the domination polynomials of families of graphs obtained by various products, including both explicit formulae and recurrences for specific families.

1 Introduction and Definitions

This paper discusses simple undirected graphs $G = (V,E)$. A vertex subset $W \subseteq V$ of G is a *dominating set* in G, if for each vertex $v \in V$ of G either v itself or an adjacent vertex is in W.

Definition 1.1. Let $G = (V,E)$ be a graph. The *domination polynomial* $D(G,x)$ is given by

$$D(G,x) = \sum_{i=0}^{|V|} d_i(G)x^i,$$

where $d_i(G)$ is the number of dominating sets of size i in G. The *domination number* of a graph G, denoted $\gamma(G)$, is the smallest i such that $d_i(G) > 0$.

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In [20] we showed that there exist recurrence relations for the domination polynomial which allow for efficient schemes to compute the polynomial for some types of graphs. A recurrence for the domination polynomial of the *path graph* with n vertices (P_n) was shown in [3] to be

$$D(P_{n+1},x) = x(D(P_n,x) + D(P_{n-1},x) + D(P_{n-2},x)) \tag{1}$$

where $D(P_0,x) = 1$, $D(P_1,x) = x$ and $D(P_2,x) = x^2 + 2x$. Note that the complete graphs $K_j \cong P_j$ for $0 \leq j \leq 2$ and that $D(K_r,x) = (x+1)^r - 1$.

Given any two graphs G and H we define the *Cartesian product*, denoted $G \square H$, to be the graph with vertex set $V(G) \times V(H)$ and edges between two vertices (u_1, v_1) and (u_2, v_2) if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$, or $u_1 u_2 \in E(G)$ and $v_1 = v_2$. As in [14], if $u \in G$ then the subgraph of $G \square H$ induced by the vertices (u, v) such that $v \in H$ will be the H-layer through u and this will be denoted by H^u. We define G^v analagously.

The Cartesian product is well known to be commutative and, if G is a disconnected graph with components G_1 and G_2, then $G \square H = (G_1 \square H) \cup (G_2 \square H)$, so that

$$D(G \square H,x) = D(G_1 \square H,x) D(G_2 \square H,x).$$

Despite these properties, it is difficult to determine much about this product, even in such simple cases as the grid graphs $P_n \square P_m$, especially in the case of dominating sets. The strong product ($G \boxtimes H$) is the graph which is formed by taking the graph $G \square H$ and then additionally adding edges between vertices (u_1, v_1) and (u_2, v_2) if both $u_1 u_2 \in E(G)$ and $v_1 v_2 \in E(H)$.

The domination numbers of graph products have been extensively studied in the literature, see e.g [1, 5, 9, 10, 12, 13, 16, 19, 22, 23, 25, 26]. In particular, a large number of papers have addressed the domination number of Cartesian products, inspired by the conjecture by V. G. Vizing [27] that $\gamma(G \square H) \geq \gamma(G) \times \gamma(H)$ (see [6] for a recent survey). In contrast, although the domination polynomial has been actively studied in recent years, almost no attention has been given to the domination polynomials of graph products.

The *closed neighbourhood* $N_G[W]$ of a vertex set W in G contains W and all vertices adjacent to vertices in W. When $W = \{v\}$ we will write $N_G[v]$ or just $N[v]$ if the graph we are working in is obvious. We define $N_G(W)$ as the *open neighbourhood* which includes all neighbours of W that are not in W, so that $N_G(W) := N_G[W] \setminus W$. If S is a set of vertices from G we use $G - S$ to mean the graph resulting from the deletion of all vertices in S from G, and let $G - v$ be $G - \{v\}$. The *vertex contraction* G/v denotes the graph obtained from G by the removal of v and the addition of edges between any pair of non-adjacent neighbours of v.

The general reduction formula for any $u \in V(G)$ given in [20] is the following:

$$D(G,x) = xD(G/u,x) + D(G-u,x) + xD(G-N[u],x) - (1+x)p_u(G,x), \tag{2}$$

where $p_u(G,x)$ is the polynomial which counts the dominating sets of $G-u$ which do not include any vertex from $N_G(u)$. Note that if the vertices of $N(u)$ induce a complete graph then $G/u \cong G-u$ and so

$$D(G,x)=(x+1)D(G-u,x)+xD(G-N[u],x)-(1+x)p_u(G,x). \quad (3)$$

An outline of the paper is as follows. In section 2 we give decomposition formulae for the domination polynomials of the Cartesian product of an arbitrary graph G with K_2 and of the strong product of G with K_r, then generalise these results. Section 3 gives a recurrence relation for the domination polynomial of any graph which contains $P_n \square K_2$ that uses only six smaller graphs. A generalisation of the result in section 3 is given in section 4, where we give a recurrence for $P_n \square K_r$. In section 5 we give the polynomial for a family of graphs which generalise path graphs.

2 Domination Polynomials of Products with Complete Graphs

Let us suppose that $V(K_2) := \{u,v\}$ in the product $G \square K_2$ and let G be any non-null graph. We will concentrate first on the vertices in G^u: every vertex subset W of G^u can be a subset of some dominating set S in $G \square K_2$ if some vertices in G^v are also in S. Let $W \subseteq V(G)$, so, by definition, all vertices (y,u) are dominated for $y \in N_G[W]$ as well as the vertices (w,v) for $w \in W$. If S is a dominating set for $G \square K_2$ such that $S \cap V(G^u) = W$, all vertices (y,u) such that $y \in V(G) \backslash N[W]$ must then be dominated by (y,v), their only neighbour outside of G^u.

Theorem 2.1. *Let J_W be formed from the subgraph of G induced by $N_G[W]$ by adding a new vertex z joined to the union of W and $N(V(G)-N_G[W])$. The domination polynomial for $D(G \square K_2,x)$ is then equal to*

$$\frac{x^{|V(G)|}}{x+1} \times \sum_{W \subseteq V(G)} \frac{(D(J_W/z,x)+D(J_W-N_{J_W}[z],x)+D(J_W,x)-D(J_W-z,x))}{x^{|N_G(W)|}}.$$

Proof. Suppose that $W \subseteq V(G)$, so that we know that, in any dominating set for $G \square K_2$, if the only vertices from G^u are W then we must also include all vertices $(y,v) \in G^v$ where $y \notin N_G[W]$. In this way all vertices in G^u are dominated by

$$|W|+|V(G) \setminus N_G[W]| = |W|+|V(G)|-|N_G[W]| = |V(G)|-|N_G(W)|$$

vertices, giving the powers of x as in the theorem.

It now remains to ensure that all of the vertices in G^v are dominated. Using the vertices forced to dominate G^u we see that, in G^v, every vertex in either W or in $N[V(G)-N[W]]$ is dominated. The only vertices not dominated are therefore those which are in $N(W)$ but have no neighbours outside of $N[W]$. Let us call this set T_W.

We now introduce the graph J_W which is formed by taking the subgraph of G induced by $N[W]$ and adding a new vertex z which is adjacent to every vertex either in W or $N(V(G)-N[W])$. The vertices which z is joined to are exactly those *not* in T_W. Thus we want to count all sets of vertices in $J_W \setminus \{z\}$ such that T_W is dominated.

As defined in Equation (2), $p_z(J_W,x)$ generates the dominating sets for $J_W - N[z]$ which additionally dominate the vertices of $N(z)$. Each of these sets when combined with z is a dominating set for J_W in which T_W is dominated and z is only dominated by itself. All other sets which dominate T_W must then include a vertex from $N(z)$ and hence they will be a dominating set for both J_W and $J_W - z$. The difference of domination polynomials $D(J_W,x) - D(J_W - z,x)$ generates all such sets which include z and so $p_z(J_W,x) + D(J_W,x) - D(J_W - z,x)$ generates all sets of vertices in J_W that dominate T_W and include z.

Since z is not adjacent to any vertex of T_W the generating function counting all sets of vertices in $J_W \setminus \{z\}$ such that T_W is dominated satisfies the following relation, using Equation (2) for the expansion of $p_z(J_W,x)$:

$$\begin{aligned}
&\frac{p_z(J_W,x) + D(J_W,x) - D(J_W - z,x)}{x} \\
= \; &\frac{xD(J_W/z,x) + xD(J_W - N_{J_W}[z],x) + D(J_W - z,x) - D(J_W,x)}{x(x+1)} \\
&+ \frac{D(J_W,x) - D(J_W - z,x)}{x} \\
= \; &\frac{D(J_W/z,x) + D(J_W - N_{J_W}[z],x)}{x+1} + \frac{D(J_W,x) - D(J_W - z,x)}{x+1}.
\end{aligned}$$

Putting this together with our first observation finishes the proof. □

Since the graphs involved in the summation have at most around half the number of vertices of the product it is significantly faster to use Theorem 2.1 to calculate the domination polynomial even with the summation over all subsets. Additionally, it can be used to get a closed form solution for some highly symmetric graphs as we show in Corollary 2.2.

Corollary 2.2. *For* $r \geq 1$, $D(K_r \square K_2,x) = ((x+1)^r - 1)^2 + 2x^r$.

Proof. When $G = K_r$ we have $J_W/z = J_W - z$ for all W since all vertices in G are joined to all others. Unless $W = \varnothing$ or $W = V(G)$ the sum is therefore $(x+1)((x+1)^r - 1)$ since J_W is then K_r with z joined to the vertices in W; if we combine any non-empty subset of W with or without z we get a dominating set for exactly one of J_W or $J_W - N_{J_W}[z]$. By Theorem 2.1 we then have

$$\begin{aligned}
D(K_r \square K_2,x) &= \frac{x^r}{x+1}\left(x+1+(x+1)^{r+1} + \sum_{j=1}^{r-1}\binom{r}{j}\frac{(x+1)((x+1)^r-1)}{x^{r-j}}\right) \\
&= x^r\left(1+(x+1)^r + ((x+1)^r-1)\sum_{j=1}^{r-1}\binom{r}{j}\frac{1}{x^{r-j}}\right) \\
&= x^r + x^r(x+1)^r + ((x+1)^r-1)\left(\left(\sum_{j=0}^{r}\binom{r}{j}x^j\right) - (1+x^r)\right) \\
&= x^r + ((x+1)^r-1)^2 + x^r.
\end{aligned}$$

□

The following result was also proven independently in [7] as their Lemma 3:

Theorem 2.3. *For any graph G*

$$D(G \boxtimes K_r, x) = D(G, (x+1)^r - 1).$$

Proof. Let u be a vertex of G and $v \in V(K_r)$; the closed neighbourhood of the vertex (u,v) is $(N_G[u], K_r)$. For any $X \subseteq V(G)$, let $\{A_x \mid x \in X\}$ be a family of arbitrary non-empty subsets of $V(K_r)$. We then have that such a set X is a dominating set of G if and only if

$$\bigcup_{x \in X} \{(x,v) \mid v \in A_x\}$$

is a dominating set of $G \boxtimes K_r$. Consequently, each vertex u of a dominating set of G corresponds to all non-empty subsets of the K_r through u in $G \boxtimes K_r$, which are counted by the generating function $(x+1)^r - 1$. □

Theorem 2.3 can be used to generalise recurrence relations for the domination polynomial of any families of graphs, such as for $H_{n,r} := P_n \boxtimes K_r$ as follows:

Corollary 2.4. *For any integers $n \geq 3$ and $r \geq 1$,*

$$D(H_{n+1,r}, x) = ((x+1)^r - 1)\,(D(H_{n,r}, x) + D(H_{n-1,r}, x) + D(H_{n-2,r}, x)).$$

Proof. From Equation (1) and using Theorem 2.3 we have

$$\begin{aligned} D(H_{n+1,r}, x) &= D(P_{n+1} \boxtimes K_r, x) \\ &= D(P_{n+1}, (x+1)^r - 1) \\ &= ((x+1)^r - 1)(D(P_n, (x+1)^r - 1) + D(P_{n-1}, (x+1)^r - 1) \\ &\quad + D(P_{n-2}, (x+1)^r - 1)) \\ &= ((x+1)^r - 1)(D(H_{n,r}, x) + D(H_{n-1,r}, x) + D(H_{n-2,r}, x)) \end{aligned}$$

as required. □

Note that, as shown in [3], the same recurrence as Equation (1) holds for the cycle graphs C_n hence there is an identical generalisation for the domination polynomial of $C_n \boxtimes K_r$.

Corollary 2.5. *For any integers $n > 3$ and $r \geq 1$, $D(C_{n+1} \boxtimes K_r, x) =$*

$$((x+1)^r - 1)\,(D(C_n \boxtimes K_r, x) + D(C_{n-1} \boxtimes K_r, x) + D(C_{n-2} \boxtimes K_r, x)).$$

Corollary 2.2 can be generalised in the following way:

Theorem 2.6. *The domination polynomial for $K_r \square K_s$ is, for $r \geq 2$ and $s \geq 2$,*

$$D(K_r \square K_s, x) = ((x+1)^r - 1)^s - \sum_{k=1}^{s-1} \binom{s}{k} (-1)^k \left((x+1)^{s-k} - 1\right)^r.$$

Proof. We can imagine the vertices of $K_r \square K_s$ as elements of an $r \times s$ matrix; for a dominating set in this graph we need to have at least one element in every row and column. The simplest way this can be achieved is to have at least one vertex in every column and the ordinary generating function that generates such sets is $((x+1)^r - 1)^s$. However, it is also possible to have empty sets in some columns, so long as each row contains at least one element:

There are s choices for the case of one empty column and, given that choice, the generating function counting non-empty rows of $s-1$ elements is $\left((x+1)^{s-1} - 1\right)^r$. However, some of the sets counted in this way will have more than one empty column; by the principle of inclusion-exclusion, we now need to subtract the $\binom{s}{2}$ ways to choose a pair of columns to be empty.

The polynomial counting dominating sets with at least two columns empty is

$$\left((x+1)^{s-2} - 1\right)^r$$

but this then includes sets with more than two empty columns and so the inclusion-exclusion process will continue. The final case will be when we have all but one column empty, in which case the only possible dominating set contains all r vertices from one column. The term counting all such sets will be $sx^r = \binom{s}{s-1}((x+1)-1)^r$, which matches the term in the sum in the theorem when $k = s-1$. Combining all of these cases together completes the proof. □

Corollary 2.7. *The domination polynomial for $K_r \square K_3$ is, for $r \geq 1$,*

$$((x+1)^r - 1)^3 + 3x^r((x+2)^r - 1).$$

Proof. Substituting $s = 3$ into Theorem 2.6 we get

$$\begin{aligned} D(K_r \square K_3, x) &= ((x+1)^r - 1)^3 - \sum_{k=1}^{2} \binom{3}{k} (-1)^k \left((x+1)^{3-k} - 1\right)^r \\ &= ((x+1)^r - 1)^3 + 3\left(\left((x+1)^2 - 1\right)^r - ((x+1) - 1)^r\right) \\ &= ((x+1)^r - 1)^3 + 3(((x(x+2))^r - x^r) \\ &= ((x+1)^r - 1)^3 + 3x^r((x+2)^r - 1) \end{aligned}$$ □

3 The Domination Polynomial for $P_n \square K_2$

Let L_n be the graph $P_n \square K_2$ and label the vertices of the two copies of P_n as $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ where u_i and v_i are adjacent, $i = 1, \ldots, n$. Note that the graph L_{n-1} is formed from L_n by deletion of u_n and v_n. The domination polynomials of the first six graphs in the family are given in Table 1.

We first prove a small result which will be used in the main theorem of this section.

Table 1: The domination polynomials for the graphs $P_n \square K_2$

n	$D(P_n \square K_2, x)$
1	x^2+2x
2	$x^4+4x^3+6x^2$
3	$x^6+6x^5+15x^4+16x^3+3x^2$
4	$x^8+8x^7+28x^6+52x^5+48x^4+12x^3$
5	$x^{10}+10x^9+45x^8+116x^7+178x^6+148x^5+47x^4+2x^3$
6	$x^{12}+12x^{11}+66x^{10}+216x^9+453x^8+604x^7+470x^6+168x^5+17x^4$

Lemma 3.1. *The polynomial $A_n(x)$ counting the dominating sets of L_n such that both u_n and v_n are included is*

$$A_n(x) := x^2\left(D(L_{n-1},x)+D(L_{n-2},x)-A_{n-2}(x)\right).$$

Proof. Every dominating set for either L_{n-1} or L_{n-2} will be a dominating set for L_n when combined with u_n or v_n since these two vertices dominate themselves and their neighbours. Any set S which is a dominating set in both L_{n-1} and L_{n-2} cannot contain either u_{n-1} or v_{n-1} since they are not in L_{n-2} and hence S must contain both u_{n-2} and v_{n-2} in order for the former pair of vertices to be dominated. Thus exactly $x^2A_{n-2}(x)$ sets are counted twice and this is subtracted to give our result. □

Theorem 3.2. *The domination polynomial for L_n satisfies this recurrence for $n \geq 6$:*

$$\begin{aligned} D(L_n,x) &= x(x+2)D(L_{n-1},x)+x(x+1)D(L_{n-2},x) \\ &\quad +x^2(x+1)D(L_{n-3},x)-x^3D(L_{n-4},x)-x^3D(L_{n-5},x) \end{aligned}$$

Proof. Let T be a dominating set for L_n and set $T_1 := T\backslash\{u_n,v_n\}$. If $T_1 = T$ then (in order to have u_n and v_n dominated) we can conclude that $|T\cap\{u_{n-1},v_{n-1}\}| = 2$ and the polynomial counting such sets will be $A_{n-1}(x)$ as in Lemma 3.1. This gives us the contribution $x^2\left(D(L_{n-2},x)+D(L_{n-3},x)-A_{n-3}(x)\right)$ for our summation.

Now suppose $|T\cap\{u_n,v_n\}| \geq 1$; if T_1 is a dominating set for L_{n-1} then T will be a dominating set for L_n. Thus we get the term $x(x+2)D(L_{n-1},x)$, the $x(x+2)$ coming from that we can use u_n and/or v_n with T_1 to form a dominating set.

However, there are circumstances under which T_1 does not have to be a dominating set for L_{n-1}, since u_{n-1} and v_{n-1} in L_{n-1} might be only dominated by u_n or v_n in T. Let us now consider the ways that exist such that u_{n-1} and v_{n-1} are not dominated in T_1 but dominated in T.

If both u_{n-1} and v_{n-1} are undominated by T_1 then we must have $|T\cap\{u_n,v_n\}| = 2$ to dominate those vertices and also $|T\cap\{u_{n-3},v_{n-3}\}| = 2$ to dominate u_{n-2} and v_{n-2}, (and neither u_{n-1} nor v_{n-1}) giving the term $x^2A_{n-3}(x)$ which will cancel that term introduced at the start of the proof.

We are now left to count just the dominating sets for L_{n-2} which include only one of u_{n-2} and v_{n-2}. These sets will make a previously uncounted dominating set for L_n when combined with v_n and/or u_n respectively. These are the four different possibilities, defining $S := T \cap \{u_n, v_n, u_{n-1}, v_{n-1}, u_{n-2}, v_{n-2}\}$:

(i) $S = \{u_n, v_n, v_{n-2}\}$

(ii) $S = \{u_n, v_{n-2}\}$

(iii) $S = \{v_n, u_{n-2}\}$

(iv) $S = \{u_n, v_n, u_{n-2}\}$

To count these possibilities we can now consider the different ways that exactly one of u_n or v_n can be combined with a dominating set for L_{n-2} which will lead to the contribution of the term $xD(L_{n-2},x)$ to our sum. Suppose Q is a dominating set for L_{n-2}; we will split into subcases depending on $r := |Q \cap \{u_{n-2}, v_{n-2}\}|$ as follows:

Every set Q satisfying $r = 2$ can be converted into a set of the type of possibility (i) (by adding u_n and switching v_n for u_{n-2}), but this new set will not be a dominating set for L_n when u_{n-3} is solely dominated by u_{n-2} in Q; that is when $Q \cap \{u_{n-3}, v_{n-3}, u_{n-4}\} = \varnothing$. Let the sets of this form which have $v_{n-4} \in Q$ be counted by the polynomial $J(x)$ and such sets which also do not include v_{n-4} are necessarily $x^2 A_{n-5}(x)$ as in Lemma 3.1.

When $r = 1$ we can add u_n or v_n as appropriate and have possibilities (ii) and (iii) for S. In the case when $r = 0$, Q must include both u_{n-3} and v_{n-3} to be dominating. No such set can be combined with just one more vertex to make a dominating set for L_n, and we can count the sets with $r = 0$ (and one additional unspecified vertex) using the polynomial $xA_{n-3}(x)$. Putting these terms together, we see that possibilities (i),(ii) and (iii) are counted by

$$x(D(L_{n-2},x) - J(x) - x^2 A_{n-5} - A_{n-3}(x)).$$

Finally, we can count the dominating sets for L_n with S as in possibility (iv) by using $x^3 D(L_{n-3},x) + xJ(x)$. We make a slight adjustment in the same way as in the subcase when $r = 0$ since a set in which only u_{n-3} is not dominated in L_{n-3} will still be a dominating set in L_n when combined with this S, and the polynomial counting such sets exactly matches the definition of $xJ(x)$.

Using Lemma 3.1 again, we get that

$$x^3 A_{n-5}(x) + xA_{n-3}(x) = x^3 \left(D(L_{n-4},x) + D(L_{n-5},x)\right)$$

and so, summing all of our terms together, we can count all possible dominating sets T for L_n by using the polynomial in the statement in the theorem. □

Note that at no point did we either concern ourselves with the structure beyond u_{n-5} and v_{n-5} or utilise the symmetry of $P_n \square K_2$, and hence this same recurrence also holds for any family of graphs with $P_6 \square K_2$ as a pendant subgraph.

We can again use Theorem 2.3 as in Corollary 2.4 to find the domination polynomial for the strong product $Z_{n,r} := L_n \boxtimes K_r$:

Corollary 3.3. *For any integers $n \geq 6$ and $r \geq 1$,*

$$\begin{aligned} D(Z_{n,r},x) &= ((x+1)^{2r}-1)\,D(Z_{n-1,r},x) \\ &\quad +((x+1)^r-1)(x+1)^r D(Z_{n-2,r},x) \\ &\quad +((x+1)^r-1)^2(x+1)^r D(Z_{n-3,r},x) \\ &\quad -((x+1)^r-1)^3(D(Z_{n-4,r},x)+D(Z_{n-5,r},x)). \end{aligned}$$

Proof. Let us substitute $y := (x+1)^r - 1$ to simplify calculations.

$$\begin{aligned} D(Z_{n,r},x) &= D(L_n,(x+1)^r-1) \\ &= D(L_n,y) \\ &= y(y+2)D(L_{n-1},y)+y(y+1)D(L_{n-2},y) \\ &\quad +y^2(y+1)D(L_{n-3},y)-y^3D(L_{n-4},y)-y^3D(L_{n-5},y) \\ &= y(y+2)D(Z_{n-1,r},x)+y(y+1)D(Z_{n-2,r},x) \\ &\quad +y^2(y+1)D(Z_{n-3,r},x)-y^3D(Z_{n-4,r},x)-y^3D(Z_{n-5,r},x). \end{aligned}$$

Utilising now that $y+1 := (x+1)^r$ we get the desired result. □

4 The Domination Polynomial for $P_n \square K_r$

We denote by $M_{n,r} := P_n \square K_r$ the Cartesian product of the path P_n and the complete graph K_r, where n and r are non-negative integers. We will utilise the linear structure of P_n and refer to the copy of K_r corresponding to one of the vertices of degree one in P_n as at the *first K_r layer* and the copy of K_r adjacent to it as the *second K_r layer.* Let $m^t_{n,r}(x)$ be the polynomial counting the vertex subsets of $M_{n,r}$ such that all vertices outside of the first K_r layer are dominated and a particular subset of t of the r vertices of the first K_r layer is not necessarily dominated.

Let $\delta_{t,r} := [t = r]$ denote the Kronecker delta function. The graph $M_{0,r}$ is the null graph and $M_{1,r} = K_r$ and so only the case of the empty dominating set needs to be considered carefully. For $n = 2$ the case $t = 0$ and $r > 0$ corresponds to Corollary 2.2 and the proof of Theorem 2.6 can be generalised to give the result here.

$$\begin{aligned} m^t_{0,r}(x) &= 1 \\ m^t_{1,r}(x) &= (x+1)^r - 1 + \delta_{t,r} \qquad (4) \\ m^t_{2,r}(x) &= (x+1)^{2r} - 2(x+1)^r + x^r + 1 + x^{(r-t)}(x+1)^t - \delta_{t,r} \end{aligned}$$

From these equations we can establish the following recurrence relations for $m^t_{n,r}$ in general and $D(M_{n,r},x) = m^0_{n,r}(x)$ in particular.

Theorem 4.1. *The domination polynomial for $P_n \square K_r$ (with $n \geq 3$ and $r \geq 3$) satisfies*

$$D(M_{n,r},x) = \sum_{p=1}^{r}\binom{r}{p}x^p m^p_{n-1,r}(x) + \sum_{i=0}^{t}\binom{t}{i}x^{r-i}m^{r-i}_{n-2,r}(x) + \delta_{r,t}(m^0_{n-1,r} - m^r_{n-1,r})$$

where the $m^t_{j,r}(x)$ terms can be evaluated recursively.

Proof. We consider the graph $M_{n,r}$ for $n \geq 3$ and note that, on deletion of its first K_r layer, we get a copy of the graph $M_{n-1,r}$. For $m^r_{n,r}(x)$ we are looking for sets of vertices in which all of the vertices of the first K_r layer of $M_{n,r}$ are not necessarily dominated but all other vertices are. We can combine a set of q vertices in the first K_r layer with a set in $M_{n-1,r}$ counted by $m^q_{n-1,r}(x)$ and form a subset of the vertices in $M_{n,r}$ such that none of the vertices in the first K_r layer of $M_{n,r}$ are necessarily dominated but all other vertices are and so

$$m^r_{n,r}(x) = \sum_{q=0}^{r} \binom{r}{q} x^q m^q_{n-1,r}(x). \tag{5}$$

Now suppose that $0 \leq t < r$ and note that this implies that, in every subset S of $M_{n,r}$ which is counted by $m^t_{n,r}(x)$, there must be at least one vertex in the second K_r layer and therefore all vertices outside of the first K_r layer are automatically dominated. We need to consider two cases as to whether or not there is a vertex in S from the first K_r layer.

1. If not then it will be possible to have a set in $M_{n,r}$ in which the t particular vertices are not necessarily dominated by adding a set including all of the other $r-t$ vertices (and perhaps some others) from the second K_r layer to a subset of $M_{n-2,r}$ in which the corresponding vertices in the first K_r layer of $M_{n-2,r}$ are not necessarily dominated; this will contribute a term equal to

$$x^{r-t} \sum_{j=0}^{t} \binom{t}{j} x^j m^{r-t+j}_{n-2,r}(x) = \sum_{i=0}^{t} \binom{t}{i} x^{r-i} m^{r-i}_{n-2,r}(x).$$

2. Now we can assume there is at least one vertex in the first K_r layer of S and so all vertices will be dominated and the value of t becomes immaterial. We can take any $p > 0$ vertices in the first K_r layer and combine them with a set counted by $m^p_{n-1,r}(x)$ to count such sets. Putting these cases together we get that, for $0 \leq t < r$,

$$m^t_{n,r}(x) = \sum_{p=1}^{r} \binom{r}{p} x^p m^p_{n-1,r}(x) + \sum_{i=0}^{t} \binom{t}{i} x^{r-i} m^{r-i}_{n-2,r}(x). \tag{6}$$

It is possible to combine these two results as follows, after noting that, when $t = r$ we can apply Equation (5) as follows:

$$\begin{aligned}
\sum_{i=0}^{t} \binom{t}{i} x^{r-i} m^{r-i}_{n-2,r}(x) &= \sum_{i=0}^{r} \binom{r}{i} x^{r-i} m^{r-i}_{n-2,r}(x) \\
&= \sum_{j=0}^{r} \binom{r}{j} x^j m^j_{n-2,r}(x) \\
&= m^r_{n-1,r}.
\end{aligned}$$

Thus, combining Equations (5) and (6) using the Kronecker delta and noting that the first summations in the two equations differ only by one term,

$$m^t_{n,r}(x) = \sum_{p=1}^{r} \binom{r}{p} x^p m^p_{n-1,r}(x) + \sum_{i=0}^{t} \binom{t}{i} x^{r-i} m^{r-i}_{n-2,r}(x) + \delta_{r,t}(m^0_{n-1,r} - m^r_{n-1,r}) \quad (7)$$

Equations (4) and (7) can produce all polynomials necessary for this result. □

5 Domination Polynomials of k-path graphs

A different way of combining complete graphs and paths was introduced by Beineke and Pippert in [4]. The k-path graph of length $n \geq k$ is defined as follows; P^k_n is an n-vertex graph with vertices v_1 to v_k all being joined to each other and for $j > k$ add edges from vertex v_j to all vertices from v_{j-1} to v_{j-k}.

In [17] the domination polynomial was given for P^k_n with $n \leq 2k+6$, but we can simplify and extend the results given as follows:

Theorem 5.1. *For $k \geq 2$ and $k < n \leq 2k+1$ there is the following recursion:*

$$D(P^k_n, x) = D(P^{k-1}_{n-1}, x) + x(1+x)^{n-1}.$$

Proof. Since $k+1 \leq n \leq 2k+1$, v_{k+1} is adjacent to every other vertex and its removal leaves the $(k-1)$-path graph with $n-1$ vertices. Any other vertices when combined with v_{k+1} give a dominating set, leading to the term $x(1+x)^{n-1}$ and v_{k+1} will be dominated by any other vertex in a dominating set without it, giving us $D(P^{k-1}_{n-1}, x)$. □

It is possible to find a recursive formula for the polynomial for large n compared to k:

Theorem 5.2. *For $n \geq 3k+2$ we have*

$$D(P^k_n, x) = (x+1)D(P^k_{n-1}, x) - xD(P^k_{n-2(k+1)}, x).$$

Proof. We use Equation (3) with vertex $u := v_n$, after noting that u is adjacent to all vertices from v_{n-k} to v_{n-1} we thus have $P^k_n - N[v_n] = P^k_{n-k-1}$ and hence

$$D(P^k_n, x) = (x+1)D(P^k_{n-1}, x) + xD(P^k_{n-k-1}, x) - (1+x)p_u(P^k_n, x). \quad (8)$$

By the definition of P^k_n, the polynomial $p_u(P^k_n, x)$ counts the dominating sets for P^k_{n-1} which do not include any of v_{n-k} to v_{n-1}. Aside from these vertices v_{n-1} is only adjacent to v_{n-k-1} and so $p_u(P^k_n, x)$ actually counts the dominating sets for P^k_{n-k-1} which include v_{n-k-1}.

Let S be such a set; either $S - v_{n-k-1}$ is a dominating set for P^k_{n-k-1} or it does not contain any of the vertices v_{n-2k-1} to v_{n-k-2} and thus is a dominating set for P^k_{n-2k-2}. Therefore we can say that, by counting both the sets S and $S - v_{n-k-1}$,

$$\frac{(1+x)p_{v_n}(P^k_n, x)}{x} = D(P^k_{n-k-1}, x) + D(P^k_{n-2k-2}, x),$$

and so we have the relation in the theorem by substituting this into Equation (8). □

For $n = 2k+2$ the expression given is correct:

$$D(P^k_{2k+2},x) = (x+1)D(P^k_{2k+1},x) + xD(P^k_{k+1},x) - x(1+x)^k.$$

Theorem 5.3. *The expression given in [17] for the range $2k+3 \leq n \leq 2k+6$ which simplifies to*

$$D(P^k_n,x) = (x+1)D(P^k_{n-1},x) + xD(P^k_{n-k-1},x) - x(1+x)^{n-2k-2}((1+x)^{k+1} - 1)$$

is actually true for $2k+3 \leq n \leq 3k+3$.

Proof. After using Equation (8) in this case,

$$p_{v_n}(P^k_n,x) = x(1+x)^{n-2k-3}((1+x)^{k+1} - 1).$$

This is because, as in Theorem 5.2, vertex v_{n-k-1} has to be in our dominating set S for P^k_{n-k-1}, and the vertices v_{n-2k-2} to v_{n-k-2} are therefore dominated by v_{n-k-1}. However, since $n \geq 2k+3$, vertex v_1 is not dominated by this vertex.

The vertices v_1 to v_{k+1} form a clique and so will be dominated so long as there is at least one of these vertices in S, giving the factor of $(1+x)^{k+1} - 1$ in our expression. As $n \leq 3k+3$, we have $n-2k-2 \leq k+1$ and hence all of the other $(n-k-1)-1-(k+1) = n-2k-3$ vertices are dominated, and so any combination of them can be in S, giving the term $(1+x)^{n-2k-3}$. □

This completes the calculation of $D(P^k_n,x)$ for all n and k.

6 Future Work

In this paper we investigated the domination polynomials of families of graphs given by products. In a future paper we will be outlining why such recurrence relations can be deduced to exist for many graph products and show implications of their existence to properties of sequences of coefficients of the domination polynomial. Additionally the computional complexity of domination polynomial can be studied and, intriguingly, the evaluation of $D(G,-2)$ can be shown to be potentially significant.

While our results cover some important families of graphs obtained by products, there remain some open problems which we believe deserve attention.

Problem 6.1.

1. How can Theorem 2.1 be extended to deal with basic Cartesian product families such as $G \square K_s$, $G \square P_s$, $G \square C_s$, etc.?

2. Can analogues of Theorem 2.3 be found for $G \boxtimes P_s$, $G \boxtimes C_s$, etc.?

3. What other families of graphs obtained using graph products have simple explicit formulae in the spirit of Theorem 2.6?

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