Proper Ramsey Numbers of Graphs

Sean English, Daniel Johnston,
Drake Olejniczak and Ping Zhang

Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008-5248, USA
ping.zhang@wmich.edu

Abstract

For graphs F and H, where H has chromatic index t, the proper Ramsey number $PR(F, H)$ is the smallest positive integer n such that every t-edge coloring of K_n results in a monochromatic F or a properly colored H. The proper Ramsey number $PR(F, H)$ is investigated for certain pairs F, H of connected graphs when $t = 2$, namely when F is a complete graph, star or path and when H is a path or even cycle of small order. In particular, $PR(F, H)$ is determined when (1) F is a complete graph and H is a path of order 6 or less, (2) F is a complete graph and H is a 4-cycle, (3) F is a star and H is a 4-cycle or a 6-cycle and (4) F is a star and H is a path of order 8 or less.

Key Words: Ramsey number, proper Ramsey number.

AMS Subject Classification: 05C15, 05C35, 05C55

1 Introduction

One of the major areas in Extremal Graph Theory is Ramsey Theory, which is primarily the study of Ramsey numbers. For two graphs F and H, the *Ramsey number* $R(F, H)$ of F and H is the smallest positive integer n such that every red-blue coloring of the complete graph K_n of order n results in a *red* F (a subgraph isomorphic to F, all of whose edges are red) or a *blue* H. When F and H are both complete, the Ramsey numbers $R(F, H)$ are often referred to as classical Ramsey numbers. When $s, t \geq 3$, only a handful of classical Ramsey numbers $R(K_s, K_t)$ are known. In particular, $R(K_3, K_3) = 6$, $R(K_3, K_4) = 9$, $R(K_4, K_4) = 18$ and $R(K_4, K_5) = 25$; while the exact value of $R(K_5, K_5)$ is unknown. It is a consequence of a theorem of Ramsey that $R(F, H)$ exists for every pair F, H of graphs.

Furthermore, it is a result of Erdös and Szekeres [6] that if F is a graph of order s and H is a graph of order t, then

$$R(F,H) \leq R(K_s, K_t) \leq \binom{s+t-2}{s-1}.$$

Indeed, for every $k \geq 2$ graphs $G_1, G_2, \ldots, G_k$, there exists a least positive integer n such that for every edge coloring of K_n with the colors $1, 2, \ldots, k$, there exists a subgraph of K_n isomorphic to G_i for some i with $1 \leq i \leq k$ such that every edge of this subgraph is colored i. This integer n is the Ramsey number $R(G_1, G_2, \ldots, G_k)$.

Over the years, a number of variations of Ramsey numbers have been introduced. For example, for every two bipartite graphs F and H, the *bipartite Ramsey number* $BR(F,H)$ is the smallest positive integer r such that every red-blue coloring of the r-regular complete bipartite graph $K_{r,r}$ results in a red F or a blue H. It is known that $BR(F,H)$ exists for every two bipartite graphs F and H (see [2]). Furthermore, it is a result of Hattingh and Henning [7] that if $F \subseteq K_{s,s}$ and $H \subseteq K_{t,t}$, then

$$BR(F,H) \leq BR(K_{s,s}, K_{t,t}) \leq \binom{s+t}{s} - 1.$$

Related to the bipartite Ramsey number is the 2-Ramsey number. For every two bipartite graphs F and H, the *2-Ramsey number* $R_2(F,H)$, defined in [1], is the smallest positive integer n such that every red-blue coloring of the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ of order n results in a red F or a blue H. In particular, $R_2(F,H)$ is either $2BR(F,H)$ or $2BR(F,H)-1$. More generally, for every two bipartite graphs F and H and each integer k with $2 \leq k \leq R(F,H)$, the *k-Ramsey number* $R_k(F,H)$, also defined in [1], is the smallest positive integer n such that every red-blue coloring of the balanced complete k-partite graph G of order n (where the numbers of vertices in every two partite sets of G differ by at most 1) results in a red F or a blue H. Certain k-Ramsey numbers have also been shown to exist when F and H are not both bipartite for some values of k.

Another Ramsey number of interest is the rainbow Ramsey number. For graphs F and H, the *rainbow Ramsey number* $RR(F,H)$ is the smallest positive integer n such that every edge coloring of K_n, using an arbitrary number of colors, results in a monochromatic F (all of whose edges are colored the same) or a rainbow H (all of whose edges are colored differently). The conditions under which $RR(F,H)$ exists is a consequence of a result of Erdös and Rado [5].

Theorem 1.1 *Let F and H be two graphs without isolated vertices. The rainbow Ramsey number $RR(F,H)$ exists if and only if F is a star or H is a forest.*

While edge colorings of a graph that result in certain monochromatic or rainbow subgraphs have been the subject of much research, the edge colorings receiving the most attention are proper edge colorings, in which every two adjacent edges are assigned different colors. The minimum number of colors required of a proper edge coloring of a graph G is the *chromatic index* of G, denoted by $\chi'(G)$. It is an immediate observation that for every nonempty graph G, the chromatic index of G is at least as large as the maximum degree $\Delta(G)$ of G, that is, $\chi'(G) \geq \Delta(G)$. The best known and most useful result on edge colorings was obtained by Vizing [8].

Theorem 1.2 (Vizing's Theorem) *For every nonempty graph G,*

$$\chi'(G) \leq \Delta(G) + 1.$$

Thus, by Vizing's theorem, for every nonempty graph G with maximum degree Δ, either $\chi'(G) = \Delta$ or $\chi'(G) = \Delta + 1$.

Let F and H be two nonempty graphs such that $\chi'(H) = t$. The *proper Ramsey number* $PR(F, H)$ of F and H is the smallest positive integer n such that every t-edge coloring of K_n results in either a monochromatic F or a properly colored H. Since the Ramsey number $R(F_1, F_2, \ldots, F_t)$, where $F_t \cong F$ for all $1 \leq i \leq t$, exists and $PR(F, H) \leq R(F_1, F_2, \ldots, F_t)$, it follows that the proper Ramsey number $PR(F, H)$ exists for every two graphs F and H. Here, we investigate the proper Ramsey number $PR(F, H)$ for several pairs F, H of connected graphs of order at least 3 where $\chi'(H) = 2$. For each such pair then,

$$|V(F)| \leq PR(F, H) \leq R(F, F). \tag{1}$$

We refer to the book [4] for graph theory notation and terminology not described in this paper.

2 Complete Graphs Versus Paths

We first determine $PR(K_n, P_k)$ for $n \geq 3$ and $k \in \{3, 4, 5\}$.

Proposition 2.1 *For each integer $n \geq 3$, $PR(K_n, P_3) = n$.*

Proof. First, $PR(K_n, P_3) \geq n$ by (1). Let there be given a red-blue coloring of K_n. If all edges of K_n are colored the same, then a monochromatic K_n results. If not, then there are two adjacent edges of K_n whose colors are different, that is, K_n has a properly colored P_3. Therefore, $PR(K_n, P_3) \leq n$ and so $PR(K_n, P_3) = n$. ∎

Theorem 2.2 *For each integer $n \geq 3$, $PR(K_n, P_4) = n + 1$.*

Proof. Let v be a vertex of the graph K_n. The red-blue coloring of K_n in which each edge incident with v is colored red and all other edges of K_n are colored blue has neither a monochromatic K_n nor a properly colored P_4. Hence, $PR(K_n, P_4) \geq n + 1$.

It remains to show that $PR(K_n, P_4) \leq n + 1$. Assume, to the contrary, that there is a red-blue coloring of $G = K_{n+1}$ that avoids both a monochromatic K_n and a properly colored P_4. By Proposition 2.1, there is a properly colored P_3, say (u, v, w), where uv is colored red and vw is colored blue. Let X be the set consisting of the remaining $n - 2$ vertices of G. Since there is no properly colored P_4 in G, the edge xu is red for each $x \in X$ and xw is blue for each $x \in X$. Assume, without loss of generality, that uw is red. Hence, xv must be blue for each $x \in X$ since there is no properly colored P_4 in G. This is illustrated in Figure 1, where a red edge is indicated by a solid line and a blue edge is indicated by a dashed line.

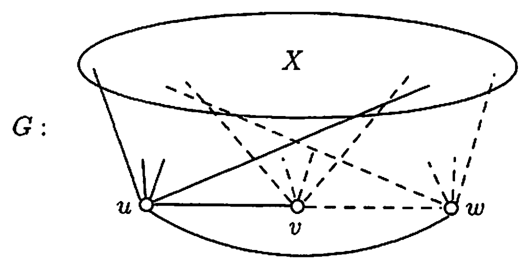

Figure 1: A red-blue coloring of $G = K_{n+1}$

If $n = 3$, then there is a monochromatic K_3, namely a blue K_3. So, we may assume that $n \geq 4$. If any edge of $G[X]$ is red, then there is a properly colored P_4. Thus, all such edges are blue and the subgraph $G[X \cup \{v, w\}]$ is a blue K_n, a contradiction. Therefore, $PR(K_n, P_4) \leq n + 1$ and so $PR(K_n, P_4) = n + 1$. ∎

In order to evaluate $PR(K_n, P_5)$ for $n \geq 3$, we first consider the special case when $n = 3$.

Proposition 2.3 $PR(K_3, P_5) = 5$.

Proof. The red-blue coloring of K_4 in which the red subgraph is C_4 and the blue subgraph is $2K_2$ contains neither a monochromatic K_3 nor a properly colored P_5. Thus, $PR(K_3, P_5) \geq 5$.

Let there be given a red-blue coloring of $G = K_5$ that avoids a monochromatic K_3. Let G_R and G_B be the red and blue subgraphs, respectively, of G. Suppose that the size of G_R is at least that of G_B. Thus, G_R contains a cycle that is not C_3. If $G_R = C_5$, then $G_B = C_5$ and there is a properly colored P_5; while if G_R contains a 4-cycle C, then both of its diagonals are blue and so the vertex of G not on C is adjacent to at least one vertex on C by a red or blue edge, producing a properly colored P_5 in either case and so $PR(K_3, P_5) = 5$. ■

Theorem 2.4 *For every integer $n \geq 4$, $PR(K_n, P_5) = 2n - 2$.*

Proof. Since the red-blue coloring of K_{2n-3}, in which every edge of some $(n-1)$-clique is colored red and all other edges are colored blue, contains neither a monochromatic K_n nor a properly colored P_5, it follows that $PR(K_n, P_5) \geq 2n - 2$.

Next, we show that $PR(K_n, P_5) \leq 2n-2$. Assume, to the contrary, that there is a red-blue coloring of $G = K_{2n-2}$ avoiding a monochromatic K_n and a properly colored P_5. Let G_R and G_B be the red and blue subgraphs, respectively, of G. We consider two cases.

Case 1. $\Delta(G_R) = 2n - 3$ *or* $\Delta(G_B) = 2n - 3$, *say the former.* Let v be a vertex of degree $2n - 3$ in G_R. For each $(n-1)$-subset S of $V(G) - \{v\}$, the subgraph $G[S]$ contains a blue edge; for otherwise, $G[S \cup \{v\}]$ is a red K_n. Hence, G_B contains $\ell \geq \lfloor \frac{n}{2} \rfloor$ independent edges. Suppose that $x_i y_i$ $(1 \leq i \leq \ell)$ are independent edges in G_B. Since there is no properly colored P_5 in G, it follows $x_i y_j$ is blue for all pairs i, j with $1 \leq i \neq j \leq \ell$. Thus, the subgraph induced by $W = \{x_i, y_i : 1 \leq i \leq \ell\}$ is a blue clique of order 2ℓ. If $2\ell \geq n$, then $G[W]$ contains a blue K_n, a contradiction. Hence, we may assume that $\ell = \lfloor \frac{n}{2} \rfloor$ and n is odd. Thus, $\ell = (n-1)/2$ and $G[W]$ is a blue K_{n-1}. Let $G_1 = G[W]$ and $G_2 = G[V(G) - (\{v\} \cup W)]$. Thus, G_2 is a red K_{n-2} and $G[V(G) - W]$ is a red K_{n-1}. Since G contains no monochromatic K_n, there are two vertices p and q in G_1 and a vertex s in G_2 such that ps is red and qs is blue. Let $t \in V(G_1) - \{p, q\}$. However then, (t, p, s, q, v) is a properly colored P_5 in G, a contradiction.

Case 2. $\Delta(G_R) \leq 2n - 4$ *and* $\Delta(G_B) \leq 2n - 4$. We may assume that $\Delta(G_R) \geq \Delta(G_B)$ and so $\Delta(G_R) \geq n - 1$. Let v be a vertex of maximum degree in G_R. Suppose that vx_i is a red edge of G for $1 \leq i \leq \Delta(G_R)$ and vx is a blue edge of G. Let $S = \{x_i : 1 \leq i \leq \Delta(G_R)\}$. Since G contains no red K_n, the subgraph $G[S]$ contains a blue edge, say x_1x_2 is blue. First, suppose that x is joined to a vertex $x_i \in S$ by a red edge. We may assume that $i \neq 1$. If $i = 2$, then (x_1, x_2, x, v, x_3) is a properly colored P_5; while if $i \neq 2$, then (x_1, x_2, v, x, x_i) is a properly colored P_5. In either case, a contradiction is produced. Thus, x is joined to every vertex in $S \cup \{v\}$ by a blue edge. However then, x has degree at least $\Delta(G_R) + 1$ in G_B, contradicting the assumption that $\Delta(G_R) \geq \Delta(G_B)$. ■

In order to determine $PR(K_n, P_6)$ for $n \geq 3$, we first consider the cases when $n = 3, 4, 5$.

Proposition 2.5 $PR(K_3, P_6) = PR(K_4, P_6) = 6$.

Proof. Since the red-blue coloring of K_5 resulting in a red C_5 and a blue C_5 produces neither a monochromatic K_3 nor a properly colored P_6, it follows that $PR(K_4, P_6) \geq PR(K_3, P_6) \geq 6$.

Next, we show that $PR(K_4, P_6) \leq 6$. Assume, to the contrary that, there exists a red-blue coloring of $G = K_6$ that avoids a monochromatic K_4 and a properly colored P_6. Let $V(K_6) = \{u, v, w, x, y, z\}$. Since $PR(K_4, P_5) = 6$ by Theorem 2.4 and G contains no monochromatic K_4, the graph G contains a properly colored P_5, say $P_5 = (u, v, w, x, y)$. We may assume that uv and wx are red and vw and xy are blue and, furthermore, that uy is blue.

* If zu is blue, then (z, u, v, w, x, y) is a properly colored P_6; so zu is red.
* If yz is red, then (u, v, w, x, y, z) is a properly colored P_6; so yz is blue.
* If xz is blue, then (y, u, v, w, x, z) is a properly colored P_6; so xz is red.
* If wy is red, then (x, z, y, w, v, u) is a properly colored P_6; so wy is blue.
* Similarly, if vy is red, then (u, z, y, v, w, x) is a properly colored P_6; so vy is blue.
* If ux is blue, then (v, w, x, u, z, y) is a properly colored P_6; so ux is red.
* If both wz and vz are blue, then $G[\{v, w, y, z\}]$ is a blue K_4; so at least one is red.

  By symmetry, we may assume that wz is red.
* If uw is red, then $G[\{u, w, x, z\}]$ is a red K_4; so uw is blue.
* If vx is blue, then (v, x, w, u, z, y) is a properly colored P_6; so vx is red.
* Now, if vz is red, then $G[\{u, v, x, z\}]$ is a red K_4; while if vz is blue, then (z, v, u, w, x, y) is a properly colored P_6. Hence, a contradiction is produced in either case.

Therefore, $PR(K_3, P_6) = PR(K_4, P_6) = 6$.

Proposition 2.6 $PR(K_5, P_6) = 8$.

Proof. Since the red-blue coloring of K_7, in which every edge of some 4-clique is colored red and all other edges are blue, contains neither a monochromatic K_5 nor a properly colored P_6, it follows that $PR(K_5, P_6) \geq 8$. It remains to show that $PR(K_5, P_6) \leq 8$.

Assume, to the contrary, that there exists a red-blue coloring of $G = K_8$ that avoids a monochromatic K_5 and a properly colored P_6. Let $V(K_8) = \{s, t, u, v, w, x, y, z\}$. Since $PR(K_5, P_5) = 8$ by Theorem 2.4 and G contains no monochromatic K_5, there is a properly colored P_5, say $P_5 = (s, t, u, v, w)$, where st and uv are red and tu and vw are blue. Furthermore, we may assume that sw is blue.

- ⋆ If sx is blue, then (x, s, t, u, v, w) is a properly colored P_6; so sx is red. Similarly, vx is red. Likewise, the edges sy, vy, sz and vz are red.
- ⋆ If wx is red, then (s, t, u, v, w, x) is a properly colored P_6; so wx is blue. Similarly, wy and wz are blue.
- ⋆ If uw is red, then (v, z, w, u, t, s) is a properly colored P_6; so uw is blue. Similarly, tw is blue.
- ⋆ If sv is blue, then (u, t, s, v, z, w) is a properly colored P_6; so sv is red.
- ⋆ If all of xy, yz, and xz are red, then $G[s, v, x, y, z]$ is a red K_5; so at least one of these three edges is colored blue, say xy is blue.
- ⋆ If all of tx, ty, ux, and uy are blue, then $G[t, u, x, y, w]$ is a blue K_5; so at least one of these four edges is colored red, say tx is red. However then, (u, t, x, y, s, w) is a properly colored P_6, a contradiction.

Therefore, $PR(K_5, P_6) = 8$. ■

Theorem 2.7 *For every integer* $n \geq 4$, $PR(K_n, P_6) = 2n - 2$.

Proof. By Propositions 2.5 and 2.6, we may assume that $n \geq 6$. Since $PR(K_n, P_5) = 2n-2$ by Theorem 2.4, it follows that $PR(K_n, P_6) \geq 2n-2$. It remains to show that $PR(K_n, P_6) \leq 2n - 2$.

Assume, to the contrary, that there is a red-blue coloring of $G = K_{2n-2}$ avoiding both a monochromatic K_n and a properly colored P_6. By Theorem 2.4, there is a properly colored P_5 in G, say $P = (v_1, v_2, v_3, v_4, v_5)$, where v_1v_2 and v_3v_4 are red and v_2v_3 and v_4v_5 are blue. Furthermore, we may assume that v_1v_5 is red. Let $X = V(G) - V(P)$ where then $|X| = 2n - 7$. Necessarily, v_1x is red and v_5x is blue for each $x \in X$;

for otherwise, either $(x, v_1, v_2, v_3, v_4, v_5)$ or $(x, v_5, v_4, v_3, v_2, v_1)$ is a properly colored P_6, which is impossible. Likewise, v_2x is blue for each $x \in X$. This is illustrated in Figure 2, where a red edge is indicated by a solid line and a blue edge is indicated by a dashed line.

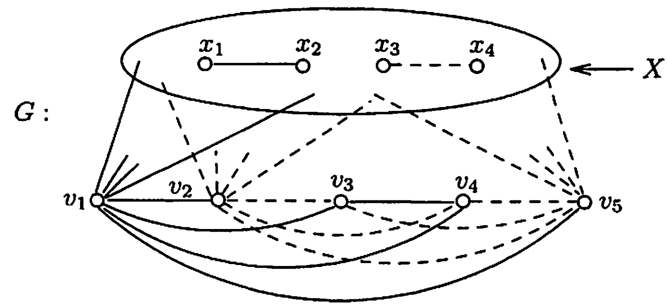

Figure 2: Illustrating a step in a red-blue coloring of $G = K_{2n-2}$

Since $n \geq 6$, it follows that $2n - 7 \geq n - 1$. This implies that $G[X]$ contains a red edge and a blue edge, for otherwise, either $G[X \cup \{v_1\}]$ or $G[X \cup \{v_2\}]$ is a monochromatic K_n. Then $G[X]$ contains nonadjacent edges x_1x_2 and x_3x_4, where x_1x_2 is red and x_3x_4 is blue.

- ⋆ If v_1v_4 is blue, then $(x_3, x_4, v_1, v_4, v_3, v_2)$ is a properly colored P_6; so v_1v_4 is red.
- ⋆ If v_2v_5 is red, then $(x_1, x_2, v_5, v_2, v_3, v_4)$ is a properly colored P_6; so v_2v_5 is blue.
- ⋆ If v_1v_3 is blue, then $(v_5, v_4, v_3, v_1, v_2, x_1)$ is a properly colored P_6; so v_1v_3 is red.
- ⋆ If v_3v_5 is red, then $(v_1, v_2, v_3, v_5, x_1, x_2)$ is a properly colored P_6; so v_3v_5 is blue.
- ⋆ If v_2v_4 is red, then $(v_1, v_5, v_4, v_2, x_1, x_2)$ is a properly colored P_6; so v_2v_4 is blue.

Consequently, every edge incident with v_1 is red and, with the exception of the edges v_1v_2 and v_1v_5, every edge incident with v_2 or v_5 is blue. (See Figure 2).

We now consider the set $S_2 = V(G) - \{v_1, v_2, v_5\}$ where $|S_2| = 2n - 5 \geq n + 1$. Certainly, if $G[S_2]$ is monochromatic, then G contains a monochromatic K_n, a contradiction. Thus, $G[S_2]$ contains a properly colored P_3, say $P_3 = (y_1, y_2, y_3)$, where y_1y_2 is red and y_2y_3 is blue. Then $(v_1, v_5, y_1, y_2, y_3)$ is a properly colored P_5, so, except for v_1y_3, every edge incident with y_3 is blue (see Figure 3). Next, let $S_3 = S_2 - \{y_3\}$, where $|S_3| = 2n - 6 \geq n$. Again, if $G[S_3]$ is monochromatic, then G contains a monochromatic K_n, a contradiction. Hence, $G[S_3]$ contains a properly colored P_3. Applying the argument above, there is a vertex in S_3 that is joined to every vertex in $V(G) - \{v_1\}$ by a blue edge. Deleting this vertex from S_3, we obtain the set S_4.

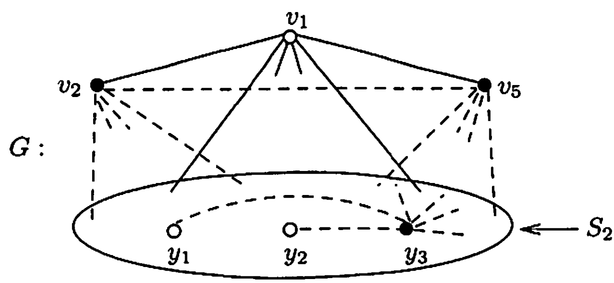

Figure 3: Selecting the vertex y_3 in $G = K_{2n-2}$

In general, for each integer k with $2 \leq k \leq n-2$, let

$$S_k = (V(G) - \{v_1\}) - \{w_1, w_2, \ldots, w_k\}$$

(where $\{w_1, w_2, w_3\} = \{v_2, v_5, y_3\}$). Since $|S_k| = (2n-3) - k \geq n-1$ and G contains no monochromatic K_n, it follows that $G[S_k]$ contains a properly colored P_3 by Proposition 2.1. Thus, there is a vertex $w_{k+1} \in S_k$ such that w_k is joined to every vertex in $V(G) - \{v_1\}$ by a blue edge. Let

$$S_{k+1} = S_k - \{w_k\}.$$

In particular, $|S_{n-2}| = n - 1$. Since G contains no monochromatic K_n, it again follows by Proposition 2.1 that $G[S_{n-2}]$ contains a properly colored P_3. Hence, there is $w_{n-1} \in S_{n-2}$ such that w_{n-1} is joined to every vertex in $V(G) - \{v_1\}$ by a blue edge. Let $S_{n-1} = S_{n-2} - \{w_{n-1}\}$ and let $w_n \in S_{n-1}$. However then, the subgraph $G[\{w_1, w_2, \ldots, w_n\}]$ is a blue K_n in G, a contradiction. Therefore, $PR(K_n, P_6) = 2n - 2$.

3 Complete Graphs Versus a 4-Cycle

In Section 2, we have seen that $PR(K_n, H) = 2n - 2$ for $n \geq 4$, where $H = P_5$ and $H = P_6$. We now show that this is also true $H = C_4$. In fact, $PR(K_n, C_4) = 2n - 2$ when $n = 3$.

Proposition 3.1 $PR(K_3, C_4) = 4$.

Proof. Since a red-blue coloring of K_3 in which not all edges are colored the same avoids both a monochromatic K_3 and a properly colored C_4, it follows that $PR(K_3, C_4) \geq 4$. Next, let there be given a red-blue coloring of $G = K_4$ that contains no monochromatic K_3. We may assume that the size of the red subgraph G_R is at least 3. Thus, G_R either contains $K_{1,3}$ or P_4. If G_R contains $K_{1,3}$, then G has a monochromatic K_3, a contradiction; while if G_R contains $P_4 = (v_1, v_2, v_3, v_4)$, then $(v_1, v_2, v_4, v_3, v_1)$ is a properly colored C_4. Therefore, $PR(K_3, C_4) = 4$. ∎

Theorem 3.2 *For each integer* $n \geq 3$, $PR(K_n, C_4) = 2n - 2$.

Proof. We proceed by induction on $n \geq 3$. By Proposition 3.1, the statement holds for $n = 3$. Assume that $PR(K_{n-1}, C_4) = 2n - 4$ for some integer $n \geq 4$. We show that $PR(K_n, C_4) = 2n - 2$.

Since the red-blue coloring of K_{2n-3} in which every edge of some $(n-1)$-clique is colored red and all other edges are blue, contains neither a monochromatic K_n nor a properly colored C_4, it follows that $PR(K_n, C_4) \geq 2n - 2$. It remains to show that $PR(K_n, C_4) \leq 2n - 2$. Assume to the contrary, that there is a red-blue coloring of $G = K_{2n-2}$ that avoids a monochromatic K_n and a properly colored C_4. By the induction hypothesis, G contains a monochromatic K_{n-1}. We may assume that G contains a red K_{n-1} with vertex set $X = \{x_1, x_2, \ldots, x_{n-1}\}$. Let

$$Y = V(G) - X = \{y_1, y_2, \ldots, y_{n-1}\}.$$

We claim that $G[Y]$ is a blue K_{n-1}. If this were not the case, then $G[Y]$ contains a red edge, say y_1y_2 is red. Since there is no red K_n, it follows that each vertex in Y is joined to at least one vertex in X by a blue edge. We may assume that x_1y_1 is blue where $x_1 \in X$. If x_iy_2 is blue for some $i \in \{2, 3, \ldots, n-1\}$, then $(x_1, y_1, y_2, x_i, x_1)$ is a properly colored C_4. Thus, x_iy_2 is red for each $i \in \{2, 3, \ldots, n-1\}$. Since there is no red K_n, it follows that x_1y_2 is blue. Furthermore, y_1x_i is red for $2 \leq i \leq n-1$; for otherwise, $(y_1, x_i, x_1, y_2, y_1)$ is a properly colored C_4. So, each edge in $[\{y_1, y_2\}, \{x_2, x_3, \ldots, x_{n-1}\}]$ is red. However then, $G[\{x_2, x_3, \ldots, x_{n-1}, y_1, y_2\}]$ is a red K_n, a contradiction. Thus, as claimed, $G[Y]$ is a blue K_{n-1}.

Next, we claim that the vertices of X can be labeled as $u_1, u_2, \ldots, u_{n-1}$ and the vertices of Y can be labeled as $v_1, v_2, \ldots, v_{n-1}$ in such a way that for each integer k with $1 \leq k \leq n-1$, the edge u_iv_j $(1 \leq i, j \leq k)$ is red if and only if $1 \leq i \leq j$. We verify this statement by induction on k.

Since $G[Y]$ is a blue K_{n-1}, every vertex in X must be joined to some vertex in Y by a red edge. Let u_1v_1 is a red edge where $u_1 \in X$ and

$v_1 \in Y$. Hence the statement holds for $k = 1$. Assume for some integer k with $1 \leq k < n-1$ that X contains k vertices $u_1, u_2, \ldots, u_k$ and Y contains k vertices $v_1, v_2, \ldots, v_k$ such that u_iv_j is red if $1 \leq i \leq j \leq k$ and u_iv_j is blue if $1 \leq j < i \leq k$.

We now show that the statement is true for $k+1$. By assumption, v_k is joined to $u_1, u_2, \ldots, u_k$ by red edges. Since v_k cannot be joined to each vertex of X by a red edge, there must be a vertex $u_{k+1} \in X$ such that $u_{k+1}v_k$ is blue. If $u_{k+1}v_i$ were red for some i with $1 \leq i < k$, then $(v_i, u_{k+1}, v_k, u_k, v_i)$ would be a properly colored C_4, which is impossible. Thus, $u_{k+1}v_i$ is blue for all i with $1 \leq i < k$. However, u_{k+1} must be joined to some vertex of Y by a red edge, say $u_{k+1}v_{k+1}$ is red, where $v_{k+1} \in Y$. If u_iv_{k+1} were blue for some i with $1 \leq i \leq k$, then $(v_{k+1}, u_i, v_i, u_{k+1}, v_{k+1})$ would be a properly colored C_4, again impossible. Thus, u_iv_{k+1} is red for all i with $1 \leq i \leq k$. This verifies the claim. In particular then, v_{n-1} is joined to every vertex of X by a red edge. However then, $G[X \cup \{v_{n-1}\}]$ is a red K_n, a contradiction. Therefore, $PR(K_n, C_4) = 2n-2$. ■

4 Stars Versus Cycles

We first determine the value of $PR(K_{1,n}, C_4)$ for each integer $n \geq 3$.

Theorem 4.1 *For every integer $n \geq 3$, $PR(K_{1,n}, C_4) = n+1$.*

Proof. Since the order of $K_{1,n}$ is $n+1$, it follows by (1) that

$$PR(K_{1,n}, C_4) \geq n+1.$$

It remains to show that $PR(K_{1,n}, C_4) \leq n+1$. We proceed by induction on n. For $n = 3$, let there be given a red-blue coloring of K_4 that avoids a monochromatic $K_{1,3}$. Thus, each vertex of K_4 is incident with at least one red edge and at least one blue edge. So, there is a $2K_2, P_4$ or C_4 in each color, which implies that there is a properly colored C_4. Therefore, $PR(K_{1,3}, C_4) \leq 4$, establishing the base step.

Next, suppose that $PR(K_{1,n-1}, C_4) \leq n$ for some integer $n \geq 4$. We show that $PR(K_{1,n}, C_4) \leq n+1$. Assume, to the contrary, there is a red-blue coloring of $G = K_{n+1}$ avoiding both a monochromatic $K_{1,n}$ and a properly colored C_4. Let $u \in V(G)$. By the induction hypothesis, $G[V(G) - \{u\}] = K_n$ contains either a monochromatic $K_{1,n-1}$ or a properly colored C_4. Since G has no properly colored C_4, there is a monochromatic $F = K_{1,n-1}$. We may assume that F is a red $K_{1,n-1}$ whose central vertex is v. Because G has no monochromatic $K_{1,n}$, it follows that uv is blue and u is incident with at least one red edge, say ux. Necessarily, x is incident with at least one blue edge, say xy is blue. However then, (u, v, y, x, u) is a properly colored C_4, which is impossible. Thus, $PR(K_{1,n}, C_4) \leq n+1$.

Therefore, $PR(K_{1,n}, C_4) = n + 1$ for each $n \geq 3$. ∎

Theorem 4.2 [3] *For integers $s, t \geq 2$,*

$$R(K_{1,s}, K_{1,t}) = \begin{cases} s+t-1 & \text{if } s \text{ and } t \text{ are both even} \\ s+t & \text{otherwise.} \end{cases}$$

Since $PR(K_{1,n}, C_6) \leq R(K_{1,n}, K_{1,n}) = 2n - 1$ when $n \geq 4$ is even by (1), it follows that $PR(K_{1,n}, C_6) \leq 2n - 1$ for all even integers $n \geq 4$. In fact, $PR(K_{1,n}, C_6) = 2n - 1$ for each integer $n \geq 4$, as we show next. First, we introduce some useful definitions. Let G be a graph each of whose edges is colored red or blue. For a vertex v of G, the *red neighborhood* $N_R(v)$ is the set of vertices each of which is joined to v by a red edge and the *blue neighborhood* $N_B(v)$ of v is the set of vertices joined to v by blue edges. Because the next result can be readily verified, its proof is omitted. Nevertheless, it is useful so that a more complete result can be presented.

Proposition 4.3 $PR(K_{1,3}, C_6) = 6$, $PR(K_{1,4}, C_6) = 7$, $PR(K_{1,5}, C_6) = 9$.

Theorem 4.4 *For every integer $n \geq 4$, $PR(K_{1,n}, C_6) = 2n - 1$.*

Proof. By Proposition 4.3, we may assume that $n \geq 6$. Since the red-blue coloring of K_{2n-2}, in which the red subgraph is $2K_{n-1}$ and the blue subgraph is $K_{n-1,n-1}$, avoids both a monochromatic $K_{1,n}$ and a properly colored C_6, it follows that $PR(K_{1,n}, C_6) \geq 2n - 1$.

It remains to show that every red-blue coloring of K_{2n-1} produces either a monochromatic $K_{1,n}$ or a properly colored C_6. Assume, to the contrary, that there is a red-blue coloring of $G = K_{2n-1}$ that avoids both a monochromatic $K_{1,n}$ and a properly colored C_6. Necessarily, each vertex is incident with exactly $n - 1$ red edges and exactly $n - 1$ blue edges. Thus, both the red subgraph G_R and the blue subgraph G_B are $(n - 1)$-regular graphs of order $2n - 1$. We first verify three claims.

Claim 1. There is no monochromatic K_n.

Proof of Claim 1. Assume, to the contrary, that G contains a monochromatic $F = K_n$. We may assume that F is a red K_n. Let $x \in V(G) - V(F)$. Since $|V(G) - V(F)| = n - 1$ and x is incident with exactly $n - 1$ red edges, it follows that x is joined to at least one vertex y in F by a red edge. However then, y is incident with at least n red edges, producing a red $K_{1,n}$. This is impossible; so Claim 1 holds.

Claim 2. There is no monochromatic K_{n-1}.

Proof of Claim 2. Assume, to the contrary, that G contains a monochromatic $F = K_{n-1}$. We may assume that F is a red K_{n-1}. Let $X = V(F)$ and let $Y = V(G) - X$; so $|X| = n - 1$ and $|Y| = n$. Since each $x \in X$ is incident with exactly $n - 1$ red edges, it follows that each x is joined to exactly one vertex in Y by a red edge; so $[X, Y]$ contains exactly $n - 1$ red edges. This implies that at least one of the n vertices in Y, say y, is incident with exactly $n - 1$ blue edges in $[X, Y]$. Thus, y is joined to each vertex in Y by a red edge (see Figure 4). Consider the subgraph $H = G[Y - \{y\}]$ of order $n - 1$ in G. Either H is a monochromatic K_{n-1} or H contains a properly colored P_3.

- ⋆ If H is a red K_{n-1}, then $G[Y]$ is a red K_n, which is impossible by Claim 1.
- ⋆ If H is a blue K_{n-1}, then each vertex in H is adjacent to exactly $n - 2$ vertices in X by red edges. This implies that $[X, Y]$ contains $(n - 1)(n - 2)$ red edges. However then, $(n - 1)(n - 2) = n - 1$; so $n = 3$, which is impossible since $n \geq 6$.
- ⋆ If H contains a properly colored $P_3 = (u, v, w)$, where say uv is red and vw is blue, then (u, v, w, y) is a properly colored P_4 (see Figure 4). First, suppose that u is joined to a vertex $x \in X$ by a blue edge. Let $x' \in X - \{x\}$. Then (x', x, u, v, w, y, x') is a properly colored C_6, which is impossible. Hence, u is joined to all vertices in X by red edges. However then, $G[X \cup \{u\}]$ is a red K_n, which is impossible by Claim 1.

Therefore, Claim 2 holds.

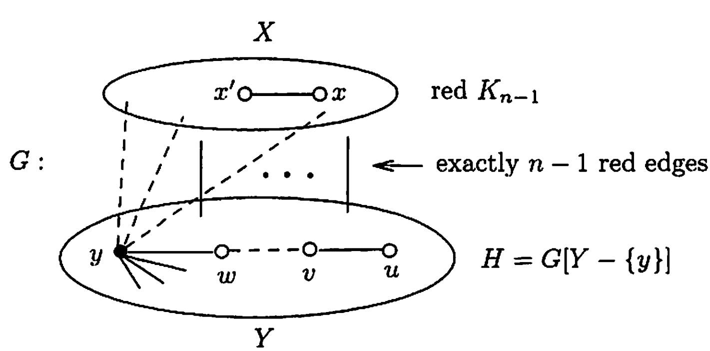

Figure 4: A step in the proof of Claim 2

Claim 3. There is a monochromatic K_{n-2}.

Proof of Claim 3. Since $PR(K_n, P_5) = 2n - 2$ by Theorem 2.4, it follows that G contains either a monochromatic K_n or a properly colored P_5. By Claim 1, the graph G contains a properly colored $P_5 = (u_1, u_2, u_3, u_4, u_5)$. We may assume that u_1u_2 and u_3u_4 are red and u_2u_3 and u_4u_5 are blue and, furthermore, u_1u_5 is red (see Figure 5).

Let $S = \{v_1, v_2, \ldots, v_{2n-6}\} = V(G) - V(P_5)$. Since (i) u_1 is incident with exactly $n-1$ blue edges and (ii) u_1u_2 and u_1u_5 are red, it follows that u_1 is adjacent to at least $n-3$ vertices in S by blue edges. Hence, $|N_B(u_1) \cap S| \geq n-3$. If u_5 is joined to some vertex $v \in N_B(u_1) \cap S$ by a red edge, then $(u_5, v, u_1, u_2, u_3, u_4, u_5)$ is a properly colored C_6, which is impossible. Hence, u_5 is joined to all vertices in $N_B(u_1) \cap S$ by a blue edge. Hence, $N_B(u_1) \cap S \subseteq N_B(u_5) \cap S$ and so $|N_B(u_5) \cap S| \geq n-3$ (see Figure 5). Likewise, since (i) u_5 is incident with exactly $n-1$ red edges and (ii) u_1u_5 is red, it follows that u_5 is joined to at least $n-4$ vertices in S by red edges. That is, $|N_R(u_5) \cap S| \geq n-4 \geq 2$. Furthermore, since $N_B(u_1) \cap S \subseteq N_B(u_5) \cap S$, it follows that $N_B(u_1) \cap S$ and $N_R(u_5) \cap S$ are disjoint. If u_1 is joined to some vertex $w \in N_R(u_5) \cap S$ by a blue edge, then $(u_1, w, u_5, u_4, u_3, u_2, u_1)$ is a properly colored C_6, which is impossible. Thus, u_1 is joined to all vertices in $N_R(u_5) \cap S$ by red edges (see Figure 5).

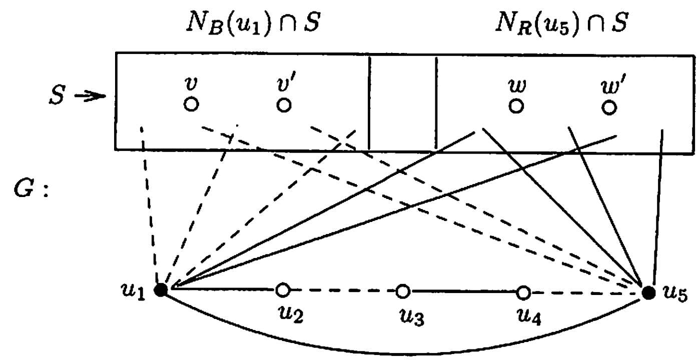

Figure 5: A step in the proof of Claim 3

First, suppose that there is a red edge vv' in $G[N_B(u_1) \cap S]$. If there is also a blue edge in $G[N_R(u_5) \cap S]$, say ww', then $(v, v', u_1, w, w', u_5, v)$ is a properly colored C_6, which is impossible. Hence, $G[N_R(u_5) \cap S]$ is a red clique of order at least $n-4$. Thus, $G_R[N_R(u_5) \cup \{u_5\}]$ contains a red K_{n-2}. Next, suppose that each edge in $G[N_B(u_1) \cap S]$ is blue. Then $G[N_B(u_1) \cap S]$ is a blue clique of order at least $n-3$. Thus, $G[(N_B(u_1) \cap S) \cup \{u_1\}]$ contains a blue K_{n-2}.

Therefore, there is a monochromatic K_{n-2} and so Claim 3 holds.

By Claim 3, the graph $G = K_{2n-1}$ contains a monochromatic K_{n-2}. Assume, without loss of generality, that G contains a red K_{n-2} with vertex set $X = \{u_1, u_2, \ldots, u_{n-2}\}$. Let $Y = V(G) - X$, where then $|Y| = n + 1$. Since $PR(K_{1,n}, C_4) = n + 1$ by Theorem 4.1 and G contains no monochromatic $K_{1,n}$, it follows that $G[Y]$ contains a properly colored $C_4 = (v_1, v_2, v_3, v_4, v_1)$, where say v_1v_2 and v_3v_4 are blue and v_2v_3 and v_1v_4 are red. Consider the vertex u_1. Since u_1 is incident with exactly $n-1$ blue edges, u_1 is joined to $n-1$ vertices in Y by blue edges. Thus, u_1 is joined to at least two vertices of C_4 by blue edges. We may assume, without loss of generality, that u_1v_1 is blue.

- ⋆ If there is $x \in X - \{u_1\}$ such that v_2x is blue, then $(v_1, u_1, x, v_2, v_3, v_4, v_1)$ is a properly colored C_6, which is impossible. Thus, v_2x is red for all $x \in X - \{u_1\}$. Since there is no red K_{n-1} by Claim 2, it follows that v_2u_1 is blue.
- ⋆ If there is $x \in X - \{u_1\}$ such that v_1x is blue, then $(v_1, v_4, v_3, v_2, u_1, x, v_1)$ is a properly colored C_6, which is impossible. Thus, v_1x is red for all $x \in X - \{u_1\}$.

In particular, v_1u_2, v_1u_3, v_2u_2 and v_2u_3 are red (see Figure 6).

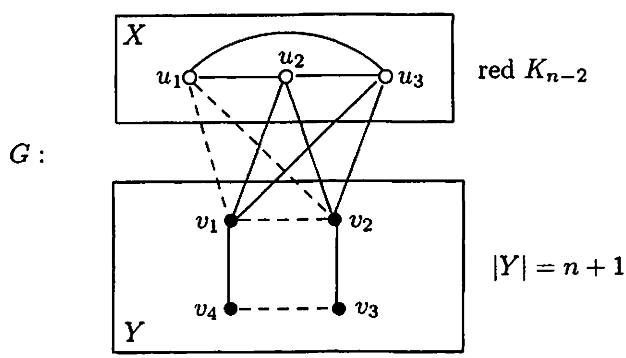

Figure 6: A step in the proof of Theorem 4.4

Since v_1u_2 and v_2u_2 are red, it follows that u_2 is joined to each of the $n-1$ vertices in $Y - \{v_1, v_2\}$ by a blue edge. In particular, u_2v_3 and u_2v_4 are blue. Likewise, u_3v_3 and u_3v_4 are blue. However then, $(u_2, u_3, v_3, v_2, v_1, v_4, u_2)$ is a properly colored C_6, which is impossible. Therefore, $PR(K_{1,n}, C_6) \leq 2n-1$ and so $PR(K_{1,n}, C_6) = 2n-1$. ∎

5 Stars Versus Paths

We begin with a preliminary result concerning stars and the path P_3.

Proposition 5.1 *For each integer $n \geq 3$, $PR(K_{1,n}, P_3) = n + 1$.*

Proof. Since the coloring of K_n in which each edge is colored red avoids both a monochromatic $K_{1,n}$ and a properly colored P_3, it follows that $PR(K_{1,n}, P_3) \geq n + 1$. For any red-blue coloring of K_{n+1}, if all edges are colored same, then there is a monochromatic $K_{1,n}$; otherwise, there are adjacent edges that are colored differently, producing a properly colored P_3. Therefore, $PR(K_{1,n}, P_3) = n + 1$. ∎

Next, we show that $PR(K_{1,n}, P_k) = n+1$ when $n \geq k-1 \geq 3$ for $k \leq 6$.

Proposition 5.2 *For each integer $n \geq 3$, $PR(K_{1,n}, P_4) = n + 1$.*

Proof. Since the coloring of K_n in which each edge is colored red avoids both a monochromatic $K_{1,n}$ and a properly colored P_3 (and so a properly colored P_4), it follows that $PR(K_{1,n}, P_4) \geq n + 1$. To show that $PR(K_{1,n}, P_4) \leq n + 1$, let there be given a red-blue coloring of $G = K_{n+1}$ that avoids a monochromatic $K_{1,n}$. Then every vertex of G is incident with at least one edge of each color and there is a properly colored P_3 in G. Suppose that $P_3 = (u_1, u_2, u_3)$, where u_1u_2 is red and u_2u_3 is blue. We may assume that u_1u_3 is red. Since u_1 is incident with at least one blue edge, there is $x \in V(G) - \{u_1, u_2, u_3\}$ such that u_1x is blue. Then (x, u_1, u_2, u_3) a properly colored P_4. Therefore, $PR(K_{1,n}, P_4) = n + 1$. ∎

Proposition 5.3 *For each integer $n \geq 4$, $PR(K_{1,n}, P_5) = n + 1$.*

Proof. By Proposition 5.2, $PR(K_{1,n}, P_5) \geq n + 1$. It remains to show that $PR(K_{1,n}, P_5) \leq n + 1$. Let there be a red-blue coloring of $G = K_{n+1}$ that avoids a monochromatic $K_{1,n}$. Then every vertex of G is incident with at least one edge of each color. Furthermore, by Proposition 5.2, there is a properly colored $P_4 = (u_1, u_2, u_3, u_4)$. We may assume that u_1u_2 and u_3u_4 are red and u_2u_3 is blue. Let $X = V(K_{n+1}) - V(P_4)$, where then $|X| = n+1-4 = n-3 \geq 1$. If u_1 or u_4 is joined to a vertex in X by a blue edge, then there is a properly colored P_5. Thus, we may assume that each edge in $[\{u_1, u_4\}, X]$ is red. Since each of u_1 and u_4 is incident with at least one blue edge, it follows that either u_1u_4 is blue or both u_1u_3 and u_2u_4 are blue. If u_1u_4 is blue, then for each $x \in X$, the path (x, u_1, u_4, u_3, u_2) is a properly colored P_5; while if u_1u_3 and u_2u_4 are blue, then, for each $x \in X$, the path (x, u_1, u_3, u_4, u_2) is a properly colored P_5. Therefore, $PR(K_{1,n}, P_5) = n + 1$. ∎

In fact, for $k \in \{6, 7, 8\}$, $PR(K_{1,n}, P_k) = n + k - 5$ when $n \geq k - 1$. We verify this next.

Proposition 5.4 *For each integer $n \geq 5$, $PR(K_{1,n}, P_6) = n + 1$.*

Proof. By Proposition 5.3, $PR(K_{1,n}, P_6) \geq n+1$. It remains to show that $PR(K_{1,n}, P_6) \leq n + 1$. Let there be given a red-blue coloring of $G = K_{n+1}$ that avoids a monochromatic $K_{1,n}$. Then every vertex of G is incident with at least one edge of each color. Furthermore, by Proposition 5.3, there is a properly colored $P_5 = (u_1, u_2, u_3, u_4, u_5)$. We may assume that u_1u_2 and u_3u_4 are red, u_2u_3 and u_4u_5 blue and furthermore u_1u_5 is red. Let $X = V(G) - V(P_5)$, where then $|X| = n + 1 - 5 = n - 4 \geq 1$. If

(i) u_1 is joined to a vertex in X by a blue edge or

(ii) one of u_2 and u_5 is joined to a vertex in X by a red edge, then there is a properly colored P_6.

Thus, we may assume that each edge in $[\{u_1\}, X]$ is red and each edge in $[\{u_2, u_5\}, X]$ is blue. Since u_1 is incident with at least one blue edge, it follows that either u_1u_3 or u_1u_4 is blue, say u_1u_3. Now let $x \in X$. Then $(u_2, x, u_1, u_3, u_4, u_5)$ is a properly colored P_6. Therefore, $PR(K_{1,n}, P_6) = n + 1$. ■

Proposition 5.5 *For each integer $n \geq 6$, $PR(K_{1,n}, P_7) = n + 2$.*

Proof. Since the red-blue coloring of K_{n+1}, in which the red subgraph is $K_{n-1} + K_2$ and the blue subgraph $K_{2,n-1}$, avoids both a monochromatic $K_{1,n}$ and a properly colored P_7, it follows that $PR(K_{1,n}, P_7) \geq n + 2$.

Next, we show that $PR(K_{1,n}, P_7) \leq n+2$. Assume, to the contrary, that there exists a red-blue coloring of $G = K_{n+2}$ that avoids both a monochromatic $K_{1,n}$ and a properly colored P_7. Thus,

each vertex of G is incident with at least two red and two blue edges. (2)

By Proposition 5.4 , there is a properly colored $P_6 = (u_1, u_2, u_3, u_4, u_5, u_6)$. We may assume that u_1u_2, u_3u_4 and u_5u_6 are red and u_2u_3 and u_4u_5 are blue. Let $X = V(G) - V(P_6)$, where then $|X| = n + 2 - 6 = n - 4 \geq 2$. Since there is no properly colored P_7, each edge in $[\{u_1, u_6\}, X]$ is red. Furthermore, if u_1u_6 is blue, then for $x \in X$, the path $(x, u_1, u_6, u_5, u_4, u_3, u_2)$ is a properly colored P_7, a contradiction. Thus u_1u_6 is red. By (2), u_1 is joined to at least two vertices in $\{u_3, u_4, u_5\}$ by blue edges and u_6 is joined to at least two vertices in $\{u_2, u_3, u_4\}$ by blue edges. Hence, at least one of u_1u_3 and u_1u_4 is blue. If $G[X]$ contains a blue edge, say x_1x_2 is blue, then either $(u_6, x_2, x_1, u_1, u_3, u_4, u_5)$ or $(u_6, x_2, x_1, u_1, u_4, u_3, u_2)$ is a properly colored P_7. Hence, $G[X]$ is a red K_{n-4}.

First, suppose that at least one of u_1u_3 and u_4u_6 is blue, say u_1u_3.

- ⋆ If u_6u_2 is blue, then, for $x \in X$, $(x, u_1, u_3, u_4, u_5, u_6, u_2)$ is a properly colored P_7; so u_6u_2 is red. By (2), both u_6u_3 and u_6u_4 are blue.
- ⋆ If u_1u_5 is blue, then, for $x \in X$, $(x, u_6, u_4, u_3, u_2, u_1, u_5)$ is a properly colored P_7; so u_1u_5 is red. By (2), u_1u_4 is blue.
- ⋆ If there exists $x \in X$ such that xu_2 or xu_5 is blue, say xu_2, then $(x, u_2, u_6, u_4, u_3, u_1, u_5)$ is a properly colored P_7; so each edge in $[\{u_2, u_5\}, X]$ is red. By (2) then, each edge in $[\{u_3, u_4\}, X]$ is blue. Again, by (2), both u_3u_5 and u_4u_2 are red and so u_2u_5 is blue. However then, $(x, u_2, u_5, u_1, u_3, u_4, u_6)$ is a properly colored P_7.

Next, both u_1u_3 and u_4u_6 are red. It follows by (2) that each of $u_1u_4, u_1u_5, u_6u_3, u_6u_2$ is blue. If there exists $x \in X$ such that xu_2 or x_1u_5 is blue, say xu_2, then $(x, u_2, u_1, u_4, u_3, u_6, u_5)$ is a properly colored P_7. Hence, each edge in $[\{u_2, u_5\}, X]$ is red. By (2) then, each edge in $[\{u_3, u_4\}, X]$ is blue. Now let $x_1, x_2 \in X$ and $x_1 \neq x_2$. Then $(x_2, x_1, u_3, u_1, u_4, u_6, u_2)$ is a properly colored P_7, a contradiction. Therefore, $PR(K_{1,n}, P_7) = n + 2$. ∎

Proposition 5.6 *For each integer $n \geq 7$, $PR(K_{1,n}, P_8) = n + 3$.*

Proof. Since the red-blue coloring of K_{n+2}, in which the red subgraph is $K_{n-1} + K_3$ and the blue subgraph $K_{3,n-1}$, avoids both a monochromatic $K_{1,n}$ and a properly colored P_8, it follows that $PR(K_{1,n}, P_8) \geq n + 3$.

Next, we show that $PR(K_{1,n}, P_8) \leq n+3$. Assume, to the contrary, that there exists a red-blue coloring of $G = K_{n+3}$ that avoids both a monochromatic $K_{1,n}$ and a properly colored P_8. Thus, each vertex of G is incident with at least three red edges and three blue edges. Furthermore, by Proposition 5.5 , there is a properly colored $P_7 = (u_1, u_2, u_3, u_4, u_5, u_6, u_7)$. We may assume that u_iu_{i+1} is red for $i = 1, 3, 5$ and u_iu_{i+1} is blue for $i = 2, 4, 6$; furthermore, u_1u_7 is red. Let $X = V(G) - V(P_7)$, where then $|X| = n+3-7 = n-4 \geq 3$. Since there is no properly colored P_8, each edge in $[\{u_1\}, X]$ is red and each edge in $[\{u_2, u_7\}, X]$ is blue. Since u_1 is incident with at least three blue edges, it follows that u_1 is joined to at least three vertices in $\{u_3, u_4, u_5, u_6\}$ by blue edges. Hence, u_1 is joined to u_3 or u_6 by a blue edge. Let $x \in X$. If u_1u_3 is blue, then $(u_2, x, u_1, u_3, u_4, u_5, u_6, u_7)$ is a properly colored P_8; while if u_1u_6 is blue, then $(u_7, x, u_1, u_6, u_5, u_4, u_3, u_2)$ is a properly colored P_8. In each case, a contradiction is produced.

The results obtained in this section suggest the following conjecture.

Conjecture 5.7 *For integers m and n with $m \geq 4$ and $n \geq \lceil \frac{m}{2} \rceil + 1$,*

$$PR(K_{1,n}, P_m) = n + \left\lfloor \frac{m-3}{4} \right\rfloor + \left\lceil \frac{m-3}{4} \right\rceil .$$

Acknowledgment We are grateful to Professor Gary Chartrand for suggesting the concept of proper Ramsey number to us and kindly providing useful information on this topic.

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