The covering numbers of $\mathbb{A}_9$ and $\mathbb{A}_{11}$

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Abstract

A collection $\mathbb{S}$ of proper subgroups of a group G is said to be a *cover* (or *covering*) for G if the union of the members of $\mathbb{S}$ is all of G. A cover C of minimal cardinality is called a *minimal cover* for G and $|C|$ is called the *covering number* of G, denoted by $\sigma(G)$. In this paper we determine the covering numbers of the alternating groups $\mathbb{A}_9$ and $\mathbb{A}_{11}$.

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1 Introduction

A collection $\mathbb{S}$ of proper subgroups of a group G is said to be a *cover* for G if the union of the members of $\mathbb{S}$ is all of G. An immediate consequence of the definition is that cyclic groups have no covers. A cover C of minimal cardinality is called a *minimal cover* [15] for G, and $|C|$ the *covering number* of G, denoted by $\sigma(G)$ [5]. It is clear that any finite non-cyclic group has a finite cover, hence a finite covering number.

C.E. Praeger [13] discussed group coverings of the form $\{H^\alpha : \alpha \in A\}$ where $Inn(G) \leq A \leq Aut(G)$. In 1997, M.J. Tomkinson [15] showed that the covering number of a solvable group is of the form p^k+1 where p is a prime, and suggested the investigation of the covering number for families of finite simple groups. D.

Bubboloni in [4] investigated group coverings with members from only two conjugacy classes of subgroups. Covering numbers of several types of linear groups and Suzuki groups are addressed in [2, 3] and [11], respectively.

For sporadic simple groups, the covering numbers for the Mathieu groups M_{11}, M_{22} and M_{23}, as well as for Ly and $O'N$ were determined by P.E. Holmes in [8]. In the same paper, Holmes gave estimates for the Janko group J_1 and the McLaughlin group M^cL. Recently, in [10], L. C. Kappe, D. Nikolova-Popova, and E. Swartz determined the covering number for the Mathieu group M_{12}, and improved the Holmes estimate for J_1.

In [12] Maróti investigates the covering numbers of symmetric and alternating groups. It is shown that $\sigma(\mathbb{S}_n) = 2^{n-1}$ if n is odd, unless $n = 9$, and $\sigma(\mathbb{S}_n) \leq 2^{n-2}$ for n even. Concerning small values of n, it was shown in [1] that $\sigma(\mathbb{S}_6) = 13$, and for $n = 8, 9, 10, 12$, covering numbers for $\mathbb{S}_n$ were established in [10]. In particular, showing that $\sigma(\mathbb{S}_9) = 256$ establishes that Maróti's result holds uniformly for all odd n.

Turning to alternating groups, it was already shown in [5] that $\sigma(\mathbb{A}_5) = 10$, and it follows from [3] that $\sigma(\mathbb{A}_6) = 16$. For $n \neq 7, 9$ it is shown in [12] that $\sigma(\mathbb{A}_n) \geq 2^{n-2}$ with equality holding if and only if n is even but not divisible by 4. Furthermore, it is shown that $\sigma(\mathbb{A}_7) \leq 31$, and $\sigma(\mathbb{A}_9) \geq 80$. In [9] it is established that $\sigma(\mathbb{A}_7) = 31$, $\sigma(\mathbb{A}_8) = 71$, and $127 \leq \sigma(\mathbb{A}_9) \leq 157$.

One would think that the problem of determining $\sigma(G)$ for small groups like $\mathbb{A}_9$ or $\mathbb{A}_{11}$ would be child's play, but in fact, for these and other small simple groups, the corresponding problems have proved to be rather hard, and remained unanswered for a number of years. In this paper we determine the covering numbers of the alternating groups $\mathbb{A}_9$ and $\mathbb{A}_{11}$. In the case of $G = \mathbb{A}_9$, although it was almost trivial to establish a good upper bound for $\sigma(G)$, it was much harder to show that this upper bound was in fact the covering number.

The topic of this paper is to show that $\sigma(\mathbb{A}_{11}) = 2751$ and $\sigma(\mathbb{A}_9) = 157$.

2 Preliminaries

Throughout, we use standard notation and terminology about groups, as for example in J.J. Rotman [14], M. Hall [7] or the ATLAS [6], except that we use $N{\cdot}C$ for a split extension of a group N by a group C, and $N\backslash C$ for a general extension of N by C. If $G|X$ is a group action and $A \subseteq X$, we denote by $G_{[A]}$ the pointwise stabilizer of A in G, and by $G_{\{A\}}$ the setwise stabilizer of A in G.

Let G be a group. If $x \in G$ and $\langle x \rangle$ is maximal cyclic, we will say that $\langle x \rangle$ is a *principal* subgroup of G, and that x is a *principal* element. We denote by $\mathbb{S}$

the collection of all proper subgroups of G, by $\mathcal{M}$ the collection of all maximal subgroups of G and by $\mathcal{P}$ the collection of all principal subgroups of G. Further, we let $s = |\mathbb{S}|$, $m = |\mathcal{M}|$ and $p = |\mathcal{P}|$. If $x \in H \in C \subseteq \mathbb{S}$, we say that C covers x, and also that C covers H. If X and Y are sets, an *incidence* relation between X and Y is a subset $\mathcal{I} \subseteq X \times Y$. The elements $(x, y) \in \mathcal{I}$ are also called the *flags* of $\mathcal{I}$. It is an easy task to establish the following:

Lemma 2.1 *Suppose that G is a finite non-cyclic group, with $\mathbb{S}$, $\mathcal{M}$ and $\mathcal{P}$ as above. Then,*

(i) For $C \subseteq \mathbb{S}$, C is a cover for G if and only if C covers all principal subgroups.

(ii) If C is any cover for G, there exists a cover $C' \subseteq \mathcal{M}$, such that $|C'| \leq |C|$.

(iii) There is a minimal cover C for G consisting solely of maximal subgroups of G.

Proof. Statement (i) is obvious. To prove (ii), suppose that C is a cover for G. If we replace each $H \in C$ by a maximal subgroup subgroup M in $\mathcal{M}$ containing H we obtain a multiset $C'' \subseteq \mathcal{M}$ which covers all the subgroups $H \in C$, and therefore covers G. Further, if we keep all M of multiplicity 1, and a single occurrence of those M which appear with multiplicity higher than 1 in C'', we obtain a subcollection $C' \subseteq C''$ which also covers G. Then $|C'| \leq |C''| = |C|$, and $C' \subseteq \mathcal{M}$. Statement (iii) follows immediately from (ii). □

In view of the above lemma, to determine $\sigma(G)$ for a given group G, it suffices to determine a minimal cover consisting solely of maximal subgroups of G, that is a collection $C \subseteq \mathcal{M}$ of minimal size, covering all principal subgroups. We begin by ordering $\mathcal{P}$ and $\mathcal{M}$ in some arbitrary but fixed way, say $\mathcal{P} = \{P_1, P_2, \ldots, P_p\}$ and $\mathcal{M} = \{M_1, M_2, \ldots, M_m\}$.

Next, we proceed to define an incidence structure $\mathcal{I} \subset \mathcal{P} \times \mathcal{M}$, where $P_i \in \mathcal{P}$ is incident with $M_j \in \mathcal{M}$ if and only if $P_i \leq M_j$. This structure is equivalent to a bipartite graph and a $p \times m$ incidence matrix A, where $A(i, j) = 1$ if $P_i \leq M_j$, 0 otherwise. The problem of determining $\sigma(G)$ can now be phrased in terms of A as follows:

Problem 2.1 *For $X = (x_1, \ldots, x_m) \in \{0,1\}^m$, and $C = (c_1, \ldots, c_p)$ defined by:*

$$C = XA^T \tag{2.1}$$

determine a lowest weight vector X such that all $c_j > 0$, $1 \leq j \leq p$.

Essentially, the above formulation says: " Select a smallest possible number of columns of A whose sum is a vector with all entries positive", that is, select a minimal cover consisting of maximal subgroups of G.

It is now clear that once the matrix A has been constructed for a given group G, a linear programming approach could be used to provide a solution. Abusing standard terminology, we will say that an $n \times m$ real matrix A is *row-stochastic* (*column-stochastic*) if A has constant row-sums k (column-sums ℓ) respectively.

The group action $G|G$ of G on itself by conjugation induces actions $G|\mathcal{P}$ and $G|\mathcal{M}$. We now consider the decompositions of $\mathcal{P}$ and $\mathcal{M}$ into G-orbits under these actions:

$$\mathcal{P} = \mathcal{P}_1 + \cdots + \mathcal{P}_s, \tag{2.2}$$

$$\mathcal{M} = \mathcal{M}_1 + \cdots + \mathcal{M}_t, \tag{2.3}$$

respectively, and let $|\mathcal{P}_i| = p_i$, $|\mathcal{M}_j| = m_j$. The matrix A can be reorganized according to G-orbits into an $s \times t$ matrix of $p_i \times m_j$ block matrices $A_{\mathcal{P}_i,\mathcal{M}_j}$, which describe the induced incidence between the principal subgroups in $\mathcal{P}_i$ and the maximal subgroups in orbit $\mathcal{M}_j$. It is not hard to see that each $A_{\mathcal{P}_i,\mathcal{M}_j}$ is row-stochastic, where the row sums depend only on i and j, and represent the number of maximal subgroups in $\mathcal{M}_j$ containing any $P \in \mathcal{P}_i$. We denote by $\bar{a}_{i,j}$ the row sum of $A_{\mathcal{P}_i,\mathcal{M}_j}$ and form an $s \times t$ *fused* matrix $\bar{A} = (\bar{a}_{i,j})$. Each matrix $A_{\mathcal{P}_i,\mathcal{M}_j}$ is also column-stochastic with column sum $\bar{b}_{i,j}$ which counts the number of principal subgroups P in $\mathcal{P}_i$ contained in any fixed $M \in \mathcal{M}_j$, thus we obtain a second fused $s \times t$ matrix $\bar{B} = (\bar{b}_{i,j})$. By counting the number of flags joining $\mathcal{P}_i$ to $\mathcal{M}_j$ in two different ways we see that the following condition holds:

$$p_i \bar{a}_{i,j} = m_j \bar{b}_{i,j} \quad 1 \leq i \leq s, \quad 1 \leq j \leq t. \tag{2.4}$$

If $\mathcal{C} \subseteq \mathcal{M}$ is a cover for G, let $\mathcal{C}_j = \mathcal{C} \cap \mathcal{M}_j$, and $y_j = |\mathcal{C}_j|$. Since $\mathcal{C}$ covers $\mathcal{P}$, we must have that

$$\sum_{j=1}^{t} \bar{b}_{i,j} y_j \geq p_i \quad \text{for each } i,\ 1 \leq i \leq s,$$

that is,

$$Y\bar{B}^T \geq (p_1, \ldots, p_s) \tag{2.5}$$

where $Y = (y_1, \ldots, y_t)$, $0 \leq y_j \leq m_j$. Since y_j is the number of maximal subgroups in $\mathcal{M}_j$ that are members of the cover $\mathcal{C}$, the vectors $X = (x_1, \ldots, x_m)$ and $Y = (y_1, \ldots, y_t)$ are related in the following way: y_j is the sum of all the x_i over all the indices i corresponding to the members of $\mathcal{M}_j$.

Let $m_0 = 0$ and consider the $m \times t$ matrix D which in the j^{th} column has 1's for the indices of rows in the interval $[1+\sum_{k=0}^{j-1} m_k\ ,\ \sum_{k=0}^{j} m_k]$ and 0's everywhere else. Then

$$Y = XD. \tag{2.6}$$

Putting equations (2.5) and (2.6) together yields $XD\bar{B}^T \geq (p_1, \ldots, p_s)$, that is

$$XE \geq (p_1, \ldots, p_s), \tag{2.7}$$

where $E = D\bar{B}^T$.

It is convenient to introduce some notation as follows: If $R \subseteq \{1, 2, \dots, r\}$ we write $\mathcal{P}_R = \cup_{i \in R} \mathcal{P}_i$, and $\mathcal{M}_R = \cup_{i \in R} \mathcal{M}_i$, moreover we further simplify notation by dropping the brackets, for example we write $\mathcal{M}_{2,4,7}$ for $\mathcal{M}_{\{2,4,7\}} = \mathcal{M}_2 \cup \mathcal{M}_4 \cup \mathcal{M}_7$, and $\mathcal{P}_{4,5}$ for $\mathcal{P}_{\{4,5\}} = \mathcal{P}_4 \cup \mathcal{P}_5$.

3 The $\mathbb{A}_9$ case

3.1 The maximal subgroups

Let $X = \{1, \dots, 9\}$, $G \cong \mathbb{A}_9$, and consider the action of G on X. There are precisely 8 conjugacy classes of maximal subgroups of G (see [6]) which we label as $\{\mathcal{M}_1, \dots, \mathcal{M}_8\}$, listed in ascending order of the $|\mathcal{M}_i|$. The vector of cardinalities of the $\mathcal{M}_i$ is $(m_1, \dots, m_8) = (9, 36, 84, 120, 120, 126, 280, 840)$. G acts primitively on X, $\binom{X}{2}$, $\binom{X}{3}$ and $\binom{X}{4}$, and the members of $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ and $\mathcal{M}_6$ are the stabilizers in the respective actions. If we select a representative $M_i \in \mathcal{M}_i$, then $M_1 = G_1 \cong \mathbb{A}_8$, $M_2 = G_{\{1,2\}} \cong \mathbb{S}_7$, $M_3 = G_{\{1,2,3\}} \cong (\mathbb{A}_6 \times \mathbb{Z}_3) \cdot \mathbb{Z}_2$, and $M_6 = G_{\{1,2,3,4\}} \cong (\mathbb{A}_4 \times \mathbb{A}_5) \cdot \mathbb{Z}_2$. There are two distinct conjugacy classes of groups of order 1512, which are the normalizers of groups isomorphic to $PSL_2(8)$, thus $M_4 \cong M_5 \cong PSL_2(8) \cdot \mathbb{Z}_3$. A representative M_7 is the normalizer in G of an elementary abelian group of order 27, a split extension of $\mathbb{Z}_3^3$ by $\mathbb{S}_4$, i.e. $M_7 \cong \mathbb{Z}_3^3 \cdot \mathbb{S}_4$. Finally a representative $M_8 \in \mathcal{M}_8$ is of order 216, and is the normalizer of an elementary abelian group of order 9, a non-split extension of $\mathbb{Z}_3^2$ by a group of order 24 and type $\mathbb{Z}_2 \backslash \mathbb{A}_4$.

3.2 The principal subgroups

There are also 8 conjugacy classes of principal subgroups $\{\mathcal{P}_1, \dots, \mathcal{P}_8\}$ which we list in ascending order of the $|P_i|$, $P_i \in \mathcal{P}_i$. It is easy to establish that generators of the P_i are of cycle types $1^1 4^2$, $1^1 2^1 6^1$, $1^2 7^1$, 9^1, 9^1, $2^2 5^1$, $2^1 3^1 4^1$, and $1^1 3^1 5^1$ respectively, and that the vector of cardinalities of the $\mathcal{P}_i$ is $(p_1, \dots, p_8) = (5670,\ 15200,\ 4320,\ 3360,\ 3360, 2268,\ 3780,\ 3024)$. There is a single conjugacy class of principal subgroups for each cycle type, except for the case of cycle type 9^1 for which there are precisely two conjugacy classes, $\mathcal{P}_4$ and $\mathcal{P}_5$ of principal subgroups of order 9. Interestingly, $\mathcal{P}_4 \cup \mathcal{P}_5$ is covered by $\mathcal{M}_4 \cup \mathcal{M}_5$, but no members of $\mathcal{P}_4$ are covered by $\mathcal{M}_5$, and similarly, no members of $\mathcal{P}_5$ are covered by $\mathcal{M}_4$, thus, $A_{\mathcal{P}_4, \mathcal{M}_5} = A_{\mathcal{P}_5, \mathcal{M}_4} = (0)_{3360 \times 120}$.

3.3 The computation of incidence matrices

Computation of the incidence matrix A is undertaken by using the software system "KNUTH" developed by S. Magliveras in APL to compute with permutation groups and combinatorial objects. We begin by computing one representative $P_i \in \mathcal{P}_i$, for each i, $1 \leq i \leq 8$, and one representative $M_j \in \mathcal{M}_j$ for each $1 \leq j \leq 8$. Further, for each $(i,j) \in \{1,\dots,8\}^2$ we store a single generator for each of the distinct conjugates of P_i, and a a set of generators for each of the distinct conjugates of the M_j. We then compute the matrix $A_{\mathcal{P}_i,\mathcal{M}_j}$ by generating each conjugate of M_j using a variant of the Schreier-Sims algorithm, and then running through all principal subgroups in $\mathcal{P}_i$, testing for membership of the single generator of each of the conjugates of P_i. We repeat this for each member of the conjugacy class $\mathcal{M}_j$. Once the $A_{\mathcal{P}_i,\mathcal{M}_j}$ are computed, the matrices $A_{\mathcal{P},\mathcal{M}_j}$ consisting of the concatenation of all $A_{\mathcal{P}_i,\mathcal{M}_j} : i \in \{1,\dots,8\}$ as well the complete matrix A can be formed by splicing together the component matrices $A_{\mathcal{P}_i,\mathcal{M}_j}$. We considered trying to exhibit these matrices in this paper, but did not find a reasonable way to concisely encode the 40902×1615 matrix A, or the submatrices $A_{\mathcal{P}_i,\mathcal{M}_j}$. Instead, we exhibit below the fused matrices $\bar{A}$, and $\bar{B}$ in the form of two tables.

	$\|\langle x\rangle\|$	type	$\mathcal{P}_i\backslash\mathcal{M}_j$ $p_i\backslash m_j$	$\mathcal{M}_1$ 9	$\mathcal{M}_2$ 36	$\mathcal{M}_3$ 84	$\mathcal{M}_4$ 120	$\mathcal{M}_5$ 120	$\mathcal{M}_6$ 126	$\mathcal{M}_7$ 280	$\mathcal{M}_8$ 840
$\mathcal{P}_1$	4	$1^1 4^2$	5670	1	0	0	0	0	2	0	4
$\mathcal{P}_2$	6	$1^1 2^1 6^1$	15200	1	1	1	2	2	0	1	2
$\mathcal{P}_2$	7	$1^2 7^1$	4320	2	1	0	1	1	0	0	0
$\mathcal{P}_4$	9a	9^1	3360	0	0	0	3	0	0	1	0
$\mathcal{P}_5$	9b	9^1	3360	0	0	0	0	3	0	1	0
$\mathcal{P}_6$	10	$2^2 5^1$	2268	0	2	0	0	0	1	0	0
$\mathcal{P}_7$	12	$2^1 3^1 4^1$	3780	0	1	1	0	0	1	2	0
$\mathcal{P}_8$	15	$1^1 3^1 5^1$	3024	1	0	1	0	0	1	0	0

Matrix $\bar{A}$ for $\mathbb{A}_9$

	$\|\langle x\rangle\|$	type	$\mathcal{P}_i\backslash\mathcal{M}_j$ $p_i\backslash m_j$	$\mathcal{M}_1$ 9	$\mathcal{M}_2$ 36	$\mathcal{M}_3$ 84	$\mathcal{M}_4$ 120	$\mathcal{M}_5$ 120	$\mathcal{M}_6$ 126	$\mathcal{M}_7$ 280	$\mathcal{M}_8$ 840
$\mathcal{P}_1$	4	$1^1 4^2$	5670	630	0	0	0	0	90	0	27
$\mathcal{P}_2$	6	$1^1 2^1 6^1$	15200	1680	420	180	252	252	0	54	36
$\mathcal{P}_3$	7	$1^2 7^1$	4320	960	120	0	36	36	0	0	0
$\mathcal{P}_4$	9a	9^1	3360	0	0	0	84	0	0	12	0
$\mathcal{P}_5$	9b	9^1	3360	0	0	0	0	84	0	12	0
$\mathcal{P}_6$	10	$2^2 5^1$	2268	0	126	0	0	0	18	0	0
$\mathcal{P}_7$	12	$2^1 3^1 4^1$	3780	0	105	45	0	0	30	27	0
$\mathcal{P}_8$	15	$1^1 3^1 5^1$	3024	336	0	36	0	0	24	0	0

Matrix $\bar{B}$ for $\mathbb{A}_9$

3.4 An upper bound for $\sigma(\mathbb{A}_9)$

This upper bound for $\sigma(\mathbb{A}_9)$ was first established in [9]. With the exception of the elements of order 9, every principle element of $\mathbb{A}_9$ fixes a point or a subset of size 2. Thus, the 9 + 36 members of $\mathcal{M}_1 \cup \mathcal{M}_2$ cover all elements except for the elements of order 9. There are two conjugacy classes of elements of order 9 corresponding to two classes of principal subgroups $\mathcal{P}_4 = 9a$, and $\mathcal{P}_5 = 9b$, each of size 3360. Class $\mathcal{P}_4$ is covered by members of $\mathcal{M}_4$ and class $\mathcal{P}_5$ by members of $\mathcal{M}_5$. Also, $\mathcal{M}_4$ covers none of the members of $\mathcal{P}_5$ and $\mathcal{M}_5$ covers none of the members of $\mathcal{P}_4$ (the classes of 9's split among the $\mathcal{M}_4$ and $\mathcal{M}_5$). Interestingly, the elements of order 9 are also covered by $\mathcal{M}_7$.

Proposition 3.1 *There is a cover C for $\mathbb{A}_9$ consisting of $\mathcal{M}_{1,2} \cup \mathcal{D} \cup \mathcal{E}$, where $\mathcal{D} \subset \mathcal{M}_4$, $\mathcal{E} \subset \mathcal{M}_5$, and $|\mathcal{D}| = |\mathcal{E}| = 56$. Consequently, $\sigma(\mathbb{A}_9) \leq 157$.*

Proof. We construct a cover $C = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{D} \cup \mathcal{E}$, where $\mathcal{D} \subset \mathcal{M}_4$ and $\mathcal{E} \subset \mathcal{M}_5$. We determine a collection $\mathcal{D} \subset \mathcal{M}_4$, which covers optimally $\mathcal{P}_4$ with $|\mathcal{D}| = 56$ by running a small LP, using only the incidence matrix $A_{\mathcal{P}_4,\mathcal{M}_4}$ for minimizing the number of members of $\mathcal{M}_4$ which cover $\mathcal{P}_4$. Similarly, we determine $\mathcal{E} \subset \mathcal{M}_5$ which covers optimally $\mathcal{P}_5$ with $|\mathcal{E}| = 56$. Thus, $|C| = 9 + 36 + 56 + 56 = 157$, and $\sigma(\mathbb{A}_9) \leq 157$. □

It will turn out that the cover constructed in the proposition above is indeed a minimal cover for $\mathbb{A}_9$. To begin with we observe that the above cover could conceivably be non-minimal because the cover size could potentially decrease if optimization is sought over a larger initial collection of maximal subgroups. We note that $\mathcal{M}_4$ and $\mathcal{M}_5$ cover other principle subgroups besides the ones of order 9, hence it is conceivable that a smaller cover could be obtained if we seek an optimal cover of $\mathcal{P}$ over $\mathcal{M}_{1,2,4,5}$.

A new LP using the matrix $A_{\mathcal{P},\mathcal{M}_{1,2,4,5}}$ yields the following result.

Proposition 3.2 *Determining an optimal cover over the collection of maximal subgroups in $\mathcal{M}_{1,2,4,5}$ yields a cover of exactly the same size as the cover C above.*

Up to this point we avoided running a "large" LP using the complete set of possible maximal subgroups, i.e. $\mathcal{M}_{1,2,4,5,6,7,8}$, however, since $\mathcal{M}_7$ also covers the elements of order 9, and since we were not able to rule out members of $\mathcal{M}_6$, $\mathcal{M}_7$ or $\mathcal{M}_8$ in a minimal cover, we run a large LP using the full 40902×1615 incidence matrix A. The resulting LP over all of $\mathcal{M}$ produced an optimal cover of the same size as the cover C above.

Remark 3.1 Perhaps a note concerning the computational effort for the "large" LP is in order here. We had altogether two independent runs, using two different software packages, to determine a minimal cover, using the complete 40902×1615 incidence matrix. The two runs produced the same result for $\sigma(\mathbb{A}_9)$, but the second, using GUROBI, was much faster and took approximately one day to complete.

Proposition 3.3 *A minimal cover for* $\mathbb{A}_9$ *has size 157. That is,* $\sigma(\mathbb{A}_9) = 157$.

4 The $\mathbb{A}_{11}$ case

In what follows we let $X = \{j \in \mathbb{Z} \mid 1 \leq j \leq 11\}$ and $G = \mathbb{A}_{11}$.

4.1 The maximal subgroups

There are seven conjugacy classes of maximal subgroups of $\mathbb{A}_{11}$ which we denote by $\mathcal{M}_1, \ldots, \mathcal{M}_7$, with cardinalities 11, 55, 165, 330, 462, 2520, and 2520 respectively. We note that the natural action of $\mathbb{A}_{11}$ on X as well as the induced actions of $\mathbb{A}_{11}$ on $\binom{X}{k}, k = 2, 3, 4, 5$, are all primitive and that the maximal subgroups contained in classes $\mathcal{M}_1, \ldots, \mathcal{M}_5$ are the stabilizers in the actions of $\mathbb{A}_{11}$ on X, $\binom{X}{2}, \binom{X}{3}, \binom{X}{4}$, and $\binom{X}{5}$ respectively. The isomorphism types of representatives $M_i \in \mathcal{M}_i, i = 1, 2, 3, 4, 5$, are as follows: $M_1 \cong \mathbb{A}_{10}, M_2 \cong \mathbb{S}_9, M_3 \cong (\mathbb{A}_8 \times \mathbb{Z}_3) \cdot \mathbb{Z}_2, M_4 \cong (\mathbb{A}_7 \times \mathbb{A}_4) \cdot \mathbb{Z}_2$, and $M_5 \cong (\mathbb{A}_6 \times \mathbb{A}_5) \cdot \mathbb{Z}_2$. The remaining two classes, $\mathcal{M}_6$, and $\mathcal{M}_7$, consist of subgroups which are isomorphic to the Mathieu group M_{11}. Specifically, these subgroups are self-normalizing in $\mathbb{A}_{11}$.

4.2 The principal subgroups

$\mathbb{A}_{11}$ has 14 conjugacy classes of principal subgroups, $\mathcal{P}_1, \ldots, \mathcal{P}_{14}$, which are generated by elements of cycle types $1^1 5^2$, $2^1 3^1 6^1$, $1^3 2^1 6^1$, $1^1 2^1 8^1$, $1^2 9^1$, 11^1, $1^1 4^1 6^1, 3^1 4^2, 1^2 2^1 3^1 4^1, 2^2 7^1, 3^2 5^1, 1^3 3^1 5^1, 2^1 4^1 5^1$, and $1^1 3^1 7^1$ respectively. We also have $(p_1, \ldots, p_{14}) =$ (199584, 554400, 277200, 623700, 369600, 362880, 415800, 103950, 207900, 118800, 55440, 55440, 124740, 158400).

	$\|\langle x\rangle\|$	type	$\mathcal{P}_i\backslash\mathcal{M}_j$ / $p_i\backslash m_j$	$\mathcal{M}_1$ / 11	$\mathcal{M}_2$ / 55	$\mathcal{M}_3$ / 165	$\mathcal{M}_4$ / 330	$\mathcal{M}_5$ / 462	$\mathcal{M}_6$ / 2520	$\mathcal{M}_7$ / 2520
$\mathcal{P}_1$	5	$1^1 5^2$	199584	18144	0	0	0	864	396	396
$\mathcal{P}_2$	6a	$2^1 3^1 6^1$	554400	0	0	3360	0	1200	660	660
$\mathcal{P}_3$	6b	$1^3 2^1 6^1$	277200	75600	20160	6720	2520	600	0	0
$\mathcal{P}_4$	8	$1^1 2^1 8^1$	623700	56700	11340	3780	0	0	495	495
$\mathcal{P}_5$	9	$1^2 9^1$	369600	67200	6720	0	0	0	0	0
$\mathcal{P}_6$	11	11^1	362880	0	0	0	0	0	144	144
$\mathcal{P}_7$	12a	$1^1 4^1 6^1$	415800	37800	0	0	1260	900	0	0
$\mathcal{P}_8$	12b	$3^1 4^2$	103950	0	0	630	630	0	0	0
$\mathcal{P}_9$	12c	$1^2 2^1 3^1 4^1$	207900	37800	7560	3780	2520	1800	0	0
$\mathcal{P}_{10}$	14	$2^2 7^1$	118800	0	4320	0	360	0	0	0
$\mathcal{P}_{11}$	15a	$3^2 5^1$	55440	0	0	672	0	120	0	0
$\mathcal{P}_{12}$	15b	$1^3 3^1 5^1$	55440	15120	3024	672	504	480	0	0
$\mathcal{P}_{13}$	20	$2^1 4^1 5^1$	124740	0	2268	0	378	270	0	0
$\mathcal{P}_{14}$	21	$1^1 3^1 7^1$	158400	14400	0	960	480	0	0	0

Matrix $\bar{B}$ for $\mathbb{A}_{11}$

Proposition 4.1 *The 2520 subgroups from class $\mathcal{M}_6$ (or $\mathcal{M}_7$) are sufficient to cover the cyclic subgroups of order 11. Moreover, any collection of maximal subgroups of $\mathbb{A}_{11}$ which covers all of the elements of order 11 necessarily contains at least 2520 subgroups from $\mathcal{M}_{6,7}$.*

Proof. Let $H \in \mathcal{M}_6$ and let $C \leq H$ be any cyclic subgroup of H of order 11. Then C is a Sylow 11-subgroup of A_{11} and so is conjugate to all of the other cyclic subgroups of order 11 in A_{11}. For any $\sigma \in A_{11}$, $C^\sigma \leq H^\sigma \in \mathcal{M}_6$.

Note that the elements of order 11 in A_{11} appear only in the maximal subgroups from classes classes $\mathcal{M}_6$ and $\mathcal{M}_7$, so it suffices to show that if $H_1, ..., H_n \in \mathcal{M}_6 \cup \mathcal{M}_7$ is a collection of subgroups covering all of the cyclic subgroups of order 11, then $n \geq 2520$. Now, there are a total of 10! elements of order 11 in A_{11} and each H_i contains exactly 1440 of these. Consequently, $1440n \geq 10!$, that is, $n \geq 10!/1440 = 2520$. □

We will now consider an arbitrary covering C of A_{11} by maximal subgroups. For $i \in \{1, 2, 3, 4, 5\}$ let $y_i = |C \cap \mathcal{M}_i|$.

Proposition 4.2 *The following inequalities hold:*

i) $y_3 + y_4 \geq 165$

ii) $y_3 + y_5 \geq 83$

Proof. The only maximal subgroups containing elements of type $3^1 4^2$ are those from classes $\mathcal{M}_3$ and $\mathcal{M}_4$. In particular, each subgroup $H \in \mathcal{M}_3 \cup \mathcal{M}_4$ contains exactly 2520 elements of this type. Since $\mathcal{C}$ covers A_{11}, each of the 415800 elements of type $3^1 4^2$ is contained in some $H \in \mathcal{C} \cap (\mathcal{M}_3 \cup \mathcal{M}_4)$. Consequently, $2520(y_3 + y_4) \geq 415800$, and thus $y_3 + y_4 \geq 165$.

The elements of type $3^2 5^1$ in A_{11} appear only in the maximal subgroups from classes $\mathcal{M}_3$ and $\mathcal{M}_5$. Each subgroup from class $\mathcal{M}_3$ contains exactly 5376 of these elements, and each subgroup from class $\mathcal{M}_5$ contains 960 of them. Since there are a total of 443520 elements of this type in A_{11}, we must have $5376y_3 + 960y_5 \geq 443520$, and hence $28y_3 + 5y_5 \geq 2310$. Now $28(y_3 + y_5) \geq 28y_3 + 5y_5 \geq 2310$, so $y_3 + y_5 \geq 82.5$. Since $y_3, y_5 \in \mathbb{Z}$, $y_3 + y_5 \geq 83$. □

Proposition 4.3 *If* $y_1 < 11$ *then* $y_3 + y_4 + y_5 \geq 330$

Proof. Since $y_1 < 11$, there is $G \in \mathcal{M}_1 \setminus \mathcal{C}$, which we may assume without loss of generality is the stabilizer of 1 in A_{11}. Since G is not used in the cover, there are 172800 elements of type $1^1 3^1 7^1$ fixing 1 which must be covered by some collection of subgroups from classes $\mathcal{M}_3$ and $\mathcal{M}_4$, and 151200 elements of type $1^1 4^1 6^1$ fixing 1 which must be covered by some collection of subgroups from classes $\mathcal{M}_4$ and $\mathcal{M}_5$.

i) If $A \in \binom{X}{3}$ then G_A contains elements of type $1^1 3^1 7^1$ fixing 1 if and only if $1 \notin A$, in which case G_A contains exactly 1440 elements of type $1^1 3^1 7^1$ fixing 1. There are 120 $A \in \binom{X}{3}$ such that $1 \notin A$.

ii) If $B \in \binom{X}{4}$ then G_B contains elements of type $1^1 3^1 7^1$ fixing 1 if and only if $1 \in B$, in which case it contains exactly 1440 elements of type $1^1 3^1 7^1$ fixing 1. There are 120 $B \in \binom{X}{4}$ such that $1 \in B$.

iii) Also, if $B \in \binom{X}{4}$ then G_B contains elements of type $1^1 4^1 6^1$ fixing 1 if and only if $1 \notin B$, in which case it contains 720 elements of type $1^1 4^1 6^1$ fixing 1. There are 210 sets $B \in \binom{X}{4}$ such that $1 \notin B$.

iv) If $C \in \binom{X}{5}$ then G_C contains elements of type $1^1 4^1 6^1$ fixing 1 if and only if $1 \in C$, in which case it contains 720 elements of type $1^1 4^1 6^1$ fixing 1. There are 210 sets $C \in \binom{X}{5}$ such that $1 \in C$.

Let y_4' be the number of $G_B \in \mathcal{C}$ such that $1 \in B \in \binom{X}{4}$ and y_4'' be the number of $G_B \in \mathcal{C}$ such that $1 \notin B \in \binom{X}{4}$. Then,

$$1440(y_3 + y_4') \geq 172800, \quad \text{and} \quad 720(y_4'' + y_5) \geq 151200$$

Consequently,

$$y_3 + y_4' \geq 120, \text{ and } y_4'' + y_5 \geq 210.$$

Therefore,

$$y_3 + y_4 + y_5 = y_3 + y_4' + y_4'' + y_5 \geq 120 + 210 = 330.$$

□

Proposition 4.4 *If* $34 \leq y_2 < 55$ *then* $y_2 + y_3 + y_4 + y_5 \geq 221$.

Proof. Let $A = \{1, 2\}$. We may suppose without loss of generality that the stabilizer G_A of A in A_{11} is not among the subgroups from class $\mathcal{M}_2$ used in the cover $\mathcal{C}$. Then the 18144 elements of type $2^1 4^1 5^1$ fixing A must be covered by some collection of subgroups from classes $\mathcal{M}_4$ and $\mathcal{M}_5$.

i) For $B \in \binom{X}{4}$, G_B contains 144 elements of type $2^1 4^1 5^1$ fixing A if $B \cap A = \emptyset$, and none otherwise.

ii) Similarly, if $C \in \binom{X}{5}$, then G_C contains 144 elements of type $2^1 4^1 5^1$ fixing A if $C \cap A = \emptyset$, and none otherwise.

Thus, $144(y_4 + y_5) \geq 18144$ which implies that $y_4 + y_5 \geq 126$. From Proposition 4.2 we have that $y_3 + y_4 \geq 165$, and $y_3 + y_5 \geq 83$. Consequently, $2(y_3 + y_4 + y_5) \geq 165 + 83 + 126 = 374$, and so $y_3 + y_4 + y_5 \geq 187$. Since also $y_2 \geq 34$, we have $y_2 + y_3 + y_4 + y_5 \geq 221$. □

Proposition 4.5 *If* $y_2 \leq 33$, *then there are three pairwise disjoint sets in* $\binom{X}{2}$ *whose stabilizers are not in* $\mathcal{C}$.

Proof. Consider the graph $\mathcal{G} = (V, E)$, where $V = \binom{X}{2}$ and $E = \{\{A, B\} \subseteq V : |A \cap B| = 1\}$. This is the well known triangular graph, $\mathcal{T}_{11}$, i.e. the line graph of the complete graph $\mathcal{K}_{11}$, with parameters $(v, k, \lambda, \mu) = (55, 18, 9, 4)$ as a strongly regular graph. We observe that :

i) $\mathcal{G}$ is regular of degree 18, and

ii) If $x, y, z \in X$ are distinct, then $\{\{x, y\}, \{x, z\}, \{y, z\}\}$ is a maximal clique in $\mathcal{G}$. Consequently, if K is any clique in $\mathcal{G}$ with at least 4 vertices, then there is $x \in X$ such that for all $A \in K$, $x \in A$, and a maximum clique in $\mathcal{G}$ has 10 vertices.

Since $y_2 \leq 33$, there is $T \subseteq V$ such that $|T| = 22$ and such that for all $A \in T$, $G_A \notin \mathcal{C}$.

Let $\mathcal{H}$ be the subgraph of $\mathcal{G}$ induced by T. For $A \in T$, let $N_{\mathcal{H}}(A) = \{B \in T \mid A \neq B, A \cap B \neq \emptyset\}$, and $N^*_{\mathcal{H}}(A) = N_{\mathcal{H}}(A) \cup \{A\}$.

By degree considerations, there exist $A, B \in T$ such that $A \cap B = \emptyset$. If $T \neq N^*_{\mathcal{H}}(A) \cup N^*_{\mathcal{H}}(B)$ then the proposition follows, so suppose that $T = N^*_{\mathcal{H}}(A) \cup N^*_{\mathcal{H}}(B)$. Necessarily then both $N_{\mathcal{H}}(A)$ and $N_{\mathcal{H}}(B)$ are nonempty. We claim that $N_{\mathcal{H}}(A) \setminus N_{\mathcal{H}}(B)$ and $N_{\mathcal{H}}(B) \setminus N_{\mathcal{H}}(A)$ cannot both be cliques in $\mathcal{G}$.

Suppose by way of contradiction that they are both cliques. Then so are $N^*_{\mathcal{H}}(A) \setminus N_{\mathcal{H}}(B)$ and $N^*_{\mathcal{H}}(B) \setminus N_{\mathcal{H}}(A)$. Consequently, $|N^*_{\mathcal{H}}(A) \setminus N_{\mathcal{H}}(B)| \leq 10$ and $|N^*_{\mathcal{H}}(B) \setminus N_{\mathcal{H}}(A)| \leq 10$. However, $|N_{\mathcal{H}}(A) \cap N_{\mathcal{H}}(B)| \leq 4$ so we must have $|N^*_{\mathcal{H}}(A) \setminus N_{\mathcal{H}}(B)| + |N^*_{\mathcal{H}}(B) \setminus N_{\mathcal{H}}(A)| \geq 18$.

Then, $|N^*_{\mathcal{H}}(A) \setminus N_{\mathcal{H}}(B)| \geq 8$ and $|N^*_{\mathcal{H}}(B) \setminus N_{\mathcal{H}}(A)| \geq 8$. Since $N^*_{\mathcal{H}}(A) \setminus N_{\mathcal{H}}(B)$ and $N^*_{\mathcal{H}}(B) \setminus N_{\mathcal{H}}(A)$ are cliques of at least 8 elements, there are $x \in A$ and $y \in B$ such that for all $C \in N^*_{\mathcal{H}}(A) \setminus N_{\mathcal{H}}(B)$ and all $D \in N^*_{\mathcal{H}}(B) \setminus N_{\mathcal{H}}(A)$, $x \in C$ and $y \in D$. Then, $8 \leq |N^*_{\mathcal{H}}(A) \setminus N_{\mathcal{H}}(B)| \leq |\{\{x, z\} \in \binom{X}{2} \mid z \in X \setminus B\}| \leq 8$, and $8 \leq |N^*_{\mathcal{H}}(B) \setminus N_{\mathcal{H}}(A)| \leq |\{\{y, z\} \in \binom{X}{2} \mid z \in X \setminus A\}| \leq 8$. Thus, we have $22 = |T| = |N^*_{\mathcal{H}}(A) \setminus N_{\mathcal{H}}(B)| + |N^*_{\mathcal{H}}(B) \setminus N_{\mathcal{H}}(A)| + |N_{\mathcal{H}}(A) \cap N_{\mathcal{H}}(B)| \leq 8 + 8 + 4 = 20$, a contradiction, thereby establishing the claim.

Now one of $N_{\mathcal{H}}(A) \setminus N_{\mathcal{H}}(B)$ and $N_{\mathcal{H}}(B) \setminus N_{\mathcal{H}}(A)$ is not a clique in $\mathcal{G}$. Without loss of generality, suppose $N_{\mathcal{H}}(A) \setminus N_{\mathcal{H}}(B)$ is not a clique. Then there are $C, D \in N_{\mathcal{H}}(A) \setminus N_{\mathcal{H}}(B)$ such that $C \cap D = \emptyset$. Since $C, D \notin N_{\mathcal{H}}(B)$, $B \cap C = B \cap D = C \cap D = \emptyset$. □

Proposition 4.6 *If* $y_2 \leq 33$, *then* $y_3 + y_4 + y_5 \geq 232$.

Proof. By the previous proposition there are pairwise disjoint sets $A, B, C \in \binom{X}{2}$ such that $G_A, G_B, G_C \notin \mathcal{C}$. Then there are $18144 \cdot 3 = 54432$ elements of type $2^1 4^1 5^1$ in $G_A \cup G_B \cup G_C$ that must be covered by subgroups from classes $\mathcal{M}_4$ and $\mathcal{M}_5$. If $D \in \binom{X}{4} \cup \binom{X}{5}$, then G_D contains elements of this type fixing A (respectively B or C) if and only if $D \cap A = \emptyset$ (respectively $D \cap B = \emptyset$ or $D \cap C = \emptyset$), in which case it contains exactly 144 of them. Thus, $G_D \in \mathcal{M}_4 \cup \mathcal{M}_5$ will cover $144 \cdot |\{E \in \{A, B, C\} \mid D \cap E = \emptyset\}|$ of these 54432 elements. Let us define

$$\begin{cases} f : \binom{X}{4} \cup \binom{X}{5} \to \mathbb{Z}, & \text{by} \\ f(D) = |\{E \in \{A, B, C\} : D \cap E = \emptyset\}|. \end{cases}$$

Now for $i = 4, 5$ and $j = 1, 2, 3$, let $y_{i,j} = |\{D \in \binom{X}{i} \mid G_D \in \mathcal{C}, f(D) = j\}|$. Then $y_i \geq y_{i,1} + y_{i,2} + y_{i,3}$ for $i = 4, 5$. Also,

i) There are only 5 $D \in \binom{X}{4}$ such that $D \cap A = D \cap B = D \cap C = \emptyset$, so $y_{4,3} \leq 5$.

ii) There is only one $D \in \binom{X}{5}$ such that $D \cap A = D \cap B = D \cap C = \emptyset$, so $y_{5,3} \leq 1$.

iii) There are a total of 90 $D \in \binom{X}{4}$ with $f(D) = 2$, so $y_{4,2} \leq 90$.

iv) There are 60 $D \in \binom{X}{5}$ with $f(D) = 2$, so $y_{5,2} \leq 60$.

Since all 54432 elements of type $2^1 4^1 5^1$ in $G_A \cup G_B \cup G_C$ are covered by subgroups from classes $\mathcal{M}_4$ or $\mathcal{M}_5$, we have $432(y_{4,3} + y_{5,3}) + 288(y_{4,2} + y_{5,2}) + 144(y_{4,1} + y_{5,1}) \geq 54432$. Then, $3(y_{4,3} + y_{5,3}) + 2(y_{4,2} + y_{5,2}) + (y_{4,1} + y_{5,1}) \geq 378$. But $y_{4,3} + y_{5,3} \leq 6$ and $y_{4,2} + y_{5,2} \leq 150$, so $y_{4,3} + y_{4,2} + y_{4,1} + y_{5,3} + y_{5,2} + y_{5,1} \geq 216$. Hence, $y_4 + y_5 \geq y_{4,3} + y_{4,2} + y_{4,1} + y_{5,3} + y_{5,2} + y_{5,1} \geq 216$. Since also $y_3 + y_4 \geq 165$ and $y_3 + y_5 \geq 83$ by Proposition 4.2 , $2(y_3 + y_4 + y_5) \geq 464$ which implies that $y_3 + y_4 + y_5 \geq 232$. □

Proposition 4.7 *If C is a minimal covering of A_{11}, then $y_1 = 11$ and $y_2 = 55$. Consequently $\sigma(A_{11}) = 2751$.*

Proof. Note that the union of classes $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, and $\mathcal{M}_6$ is a cover of A_{11} by 2751 maximal subgroups, so if C is a minimal cover, $|C| \leq 2751$. By Proposition 4.1, we must have $y_1 + y_2 + y_3 + y_4 + y_5 \leq 231$. By Proposition 4.3, we must have $y_1 = 11$, and by Propositions 4.4 and 4.6 $y_2 = 55$. Proposition 4.2 says that $y_3 + y_4 \geq 165$, proving that $y_1 + y_2 + y_3 + y_4 \geq 231$, and so by Proposition 4.1, $|C| \geq 2751$. □

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