Reciprocal Sums of Quintuple Product of Generalized Binary Sequences with Indices

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Abstract

By applying the method of generating function, the purpose of this paper is to give several summation of reciprocals related to quintuple product of general second order recurrence $\{W_{rn}\}$ for arbitrary positive integer r. As applications, some identities involving Fibonacci, Lucas numbers are obtained.

Key Words: Second order recurrence; Fibonacci number; Lucas number; Quintuple product; Reciprocal
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1 Introduction

In the notation of Horadam [4], write

$$W_n = W_n(a,\ b;\ P,\ Q),$$

so that

$$W_n = PW_{n-1} - QW_{n-2}, \quad (W_0 = a,\ W_1 = b,\ n \geq 2), \tag{1}$$

where a, b, P and Q are integers, with $PQ \neq 0$. In the sequel we shall suppose that $\Delta = P^2 - 4Q > 0$. Then it is easily to obtain the Binet formula [4]:

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{2}$$

where $\alpha = \frac{P+\sqrt{P^2-4Q}}{2}$, $\beta = \frac{P-\sqrt{P^2-4Q}}{2}$, $A = b - \beta a$ and $B = b - \alpha a$. In particular, we write $U_n = W_n(0,\ 1;\ P,\ Q), V_n = W_n(2,\ P;\ P,\ Q)$, then $\alpha^n - \beta^n = (\alpha - \beta)U_n, \alpha^n + \beta^n = V_n$.

In [1], R. Andre-Jeannin obtained the following series identities:

$$\sum_{n=1}^{m} \frac{Q^n}{W_n W_{n+k}} = U_m \sum_{n=1}^{k} \frac{Q^n}{W_n W_{n+m}}, \tag{3}$$

$$\sum_{n=1}^{\infty} \frac{Q^n}{W_n W_{n+k}} = \frac{1}{ABU_k} \left(\sum_{n=1}^{k} \frac{W_{n+1}}{W_n} - k\alpha \right), \tag{4}$$

where $P > 0$ and k and m are nonnegative integers. (3) and (4) are given by Good [3] in the case $Q = -1$. Brousseau [2] proved (4) for $W_n = F_n$. In [6](see also [7]), T. Mansour used the generating function techniques to get several summation of reciprocals related to generalized Fibonacci numbers.

Regarding taking l-th powers of terms in the sums, the author [9] generalized the results of (3) and (4). For example, he derived the following infinitive reciprocal sums:

$$\begin{aligned} &\sum_{n=k}^{\infty} \frac{Q^n}{W_n^l W_{n+m}^l} \sum_{i=0}^{l-1} \Big[(W_{k+1} - W_k\beta) Q^{n-k} \alpha^m - (W_{k+1} - W_k\alpha) \\ &\quad \times \beta^{2(n-k)+m} \Big]^{l-1-i} \Big[(W_{k+1} - W_k\beta) Q^{n-k} \beta^m \\ &\quad - (W_{k+1} - W_k\alpha) \beta^{2(n-k)+m} \Big]^i \\ &= \frac{(P^2 - 4Q)^{l/2} Q^k}{(\alpha^m - \beta^m)(W_{k+1} - W_k\beta)} \sum_{i=0}^{m-1} \frac{\beta^{li}}{W_{k+i}^l}. \end{aligned} \tag{5}$$

In [5], the author gave the generalize the result of (5) regarding reciprocal sums of l-th powers of the terms with indices.

In [8], the author gave the summation of reciprocals related to quintuple product of generalized Fibonacci sequences. By applying the method of generating function, the purpose of this paper is to give generalize the result of [8] regarding reciprocal sums of triple product of the terms with indices.

Throughout the paper, r, k, l, m and t are six positive integers with $t \geq k$.

2 Main Result

In this section we consider finite both infinite reciprocal sums of quintuple product of r-consecutive terms of sequence $\{W_n\}$. First of all, we give the following general second order recurrence $\{W_{rn}\}$ of [5] which are to be used later.

Let W_n be the nth term of sequence $\{W_n\}$. Then for $n, r > 0$,

$$W_{rn} = Y_r W_{r(n-1)} - Z_r W_{r(n-2)}, \tag{6}$$

where $Y_r = \alpha^r + \beta^r$ and $Z_r = (\alpha\beta)^r$.

We define $D_{rn,s}$ to be $\prod_{j=0}^{s} W_{r(n+j)}$, so the product

$$W_{rn} W_{r(n+1)} W_{r(n+2)} W_{r(n+3)} W_{r(n+4)}$$

becomes $D_{rn,4}$.

Theorem 2.1 *Let $P > 0$. Then*

$$\begin{aligned}
&\sum_{n=k}^{t} \frac{Q^{rn}}{D_{rn,4}(\alpha^r - \beta^r)} \Big\{ \mu_{rk}^3 \alpha^{3r(n+2-k)} - \nu_{rk}^3 \beta^{3r(n+2-k)} \\
&\quad + \mu_{rk}\nu_{rk} Q^{r(n+1-k)} (V_r^2 - Q^r) \left[\mu_{rk} \alpha^{r(n+2-k)} - \nu_{rk} \beta^{r(n+2-k)} \right] \Big\} \\
&= \frac{(\alpha^r - \beta^r) Q^{rk}}{\mu_{rk} V_r} \left[\frac{1}{W_{rk}} - \frac{\beta^r}{W_{r(k+1)}} - \frac{\beta^{2r}}{W_{r(k+2)}} + \frac{\beta^{3r}}{W_{r(k+3)}} - \frac{\beta^{r(t+1-k)}}{W_{r(t+1)}} \right. \\
&\quad \left. + \frac{\beta^{r(t+2-k)}}{W_{r(t+2)}} + \frac{\beta^{r(t+3-k)}}{W_{r(t+3)}} - \frac{\beta^{r(t+4-k)}}{W_{r(t+4)}} \right],
\end{aligned} \tag{7}$$

where $\mu_{rk} = W_{r(k+1)} - W_{rk}\beta^r$, $\nu_{rk} = W_{r(k+1)} - W_{rk}\alpha^r$.

Proof Let $f(x) = \sum_{n=k}^{\infty} W_{rn} x^n$, $\mu_{rk} = W_{r(k+1)} - W_{rk}\beta^r$, $\nu_{rk} = W_{r(k+1)} - W_{rk}\alpha^r$. From (6) we have

$$f(x) - W_{rk} x^k - W_{r(k+1)} x^{k+1} = Y_r x [f(x) - W_{rk} x^k] - Z_r x^2 f(x).$$

Hence the following generating function is obtained:

$$f(x) = x^k \frac{W_{rk} + x[W_{r(k+1)} - Y_r W_{rk}]}{1 - Y_r x + Z_r x^2}.$$

Since $1 - Y_r x + Z_r x^2 = (1 - \alpha^r x)(1 - \beta^r x)$, we can decompose $f(x)$ into partial fractions:

$$f(x) = \frac{x^k}{\alpha^r - \beta^r} \left(\frac{\mu_{rk}}{1 - \alpha^r x} - \frac{\nu_{rk}}{1 - \beta^r x} \right).$$

Comparing the coefficients of x^n in both sides of above equation, we obtain that

$$W_{rn} = \frac{\mu_{rk}}{\alpha^r - \beta^r} \alpha^{r(n-k)} - \frac{\nu_{rk}}{\alpha^r - \beta^r} \beta^{r(n-k)}. \tag{8}$$

Let

$$T_s = \frac{\beta^{r(n+s-k)}}{W_{r(n+s)}} = \frac{\beta^{r(n+s-k)}}{\frac{\mu_{rk}}{\alpha^r-\beta^r}\alpha^{r(n+s-k)} - \frac{\nu_{rk}}{\alpha^r-\beta^r}\beta^{r(n+s-k)}} (s \in N). \tag{9}$$

Then

$$T_0 - T_1 = \frac{\beta^{r(n-k)}}{W_{rn}} - \frac{\beta^{r(n+1-k)}}{W_{r(n+1)}} = \frac{\mu_{rk}Q^{r(n-k)}}{D_{rn,1}}.$$

$$T_1 - T_2 = \frac{\mu_{rk}Q^{r(n+1-k)}}{W_{r(n+1)}W_{r(n+2)}}.$$

Hence we have

$$T_0 - T_1 - T_1 + T_2 = \frac{\mu_{rk}Q^{r(n-k)}}{D_{rn,2}} \left[\mu_{rk}\alpha^{r(n+1-k)} + \nu_{rk}\beta^{r(n+1-k)}\right].$$

$$T_2 - T_3 - T_3 + T_4 = \frac{\mu_{rk}Q^{r(n+2-k)}}{W_{r(n+2)}W_{r(n+3)}W_{r(n+4)}} \times \left[\mu_{rk}\alpha^{r(n+3-k)} + \nu_{rk}\beta^{r(n+3-k)}\right].$$

$$\begin{aligned}
&(T_0 - T_1 - T_1 + T_2) - (T_2 - T_3 - T_3 + T_4) \\
&= \frac{\mu_{rk}Q^{r(n-k)}}{D_{rn,2}} \left[\mu_{rk}\alpha^{r(n+1-k)} + \nu_{rk}\beta^{r(n+1-k)}\right] \\
&\quad - \frac{\mu_{rk}Q^{r(n+2-k)}}{W_{r(n+2)}W_{r(n+3)}W_{r(n+4)}} \left[\mu_{rk}\alpha^{r(n+3-k)} + \nu_{rk}\beta^{r(n+3-k)}\right] \\
&= \frac{\mu_{rk}Q^{r(n-k)}V_r}{D_{rn,4}(\alpha^r - \beta^r)} \Big\{\mu_{rk}^3\alpha^{3r(n+2-k)} - \nu_{rk}^3\beta^{3r(n+2-k)} \\
&\quad + \mu_{rk}\nu_{rk}Q^{r(n+1-k)}(V_r^2 - Q^r)\left[\mu_{rk}\alpha^{r(n+2-k)} - \nu_{rk}\beta^{r(n+2-k)}\right]\Big\}.
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=k}^{t} \frac{Q^{rn}}{D_{rn,4}(\alpha^r - \beta^r)} \Big\{\mu_{rk}^3\alpha^{3r(n+2-k)} - \nu_{rk}^3\beta^{3r(n+2-k)} \\
&\quad + \mu_{rk}\nu_{rk}Q^{r(n+1-k)}(V_r^2 - Q^r)\left[\mu_{rk}\alpha^{r(n+2-k)} - \nu_{rk}\beta^{r(n+2-k)}\right]\Big\} \\
&= \frac{(\alpha^r - \beta^r)Q^{rk}}{\mu_{rk}V_r} \left[\frac{1}{W_{rk}} - \frac{\beta^r}{W_{r(k+1)}} - \frac{\beta^{2r}}{W_{r(k+2)}} + \frac{\beta^{3r}}{W_{r(k+3)}} - \frac{\beta^{r(t+1-k)}}{W_{r(t+1)}}\right. \\
&\quad \left. + \frac{\beta^{r(t+2-k)}}{W_{r(t+2)}} + \frac{\beta^{r(t+3-k)}}{W_{r(t+3)}} - \frac{\beta^{r(t+4-k)}}{W_{r(t+4)}}\right].
\end{aligned}$$

The proof of the theorem is completed. □

Theorem 2.2 *Let $P > 0$. Then*

$$\sum_{n=k}^{\infty} \frac{Q^{rn}}{D_{rn,4}(\alpha^r - \beta^r)} \Big\{ \mu_{rk}^3 \alpha^{3r(n+2-k)} - \nu_{rk}^3 \beta^{3r(n+2-k)} + \mu_{rk}\nu_{rk} Q^{r(n+1-k)} (V_r^2 - Q^r) \left[\mu_{rk}\alpha^{r(n+2-k)} - \nu_{rk}\beta^{r(n+2-k)} \right] \Big\} = \frac{(\alpha^r - \beta^r)Q^{rk}}{\mu_{rk}V_r} \left[\frac{1}{W_{rk}} - \frac{\beta^r}{W_{r(k+1)}} - \frac{\beta^{2r}}{W_{r(k+2)}} + \frac{\beta^{3r}}{W_{r(k+3)}} \right], \quad (10)$$

where $\mu_{rk} = W_{r(k+1)} - W_{rk}\beta^r$, $\nu_{rk} = W_{r(k+1)} - W_{rk}\alpha^r$.

Proof By (7), we have

$$\sum_{n=k}^{\infty} \frac{Q^{rn}}{D_{rn,4}(\alpha^r - \beta^r)} \Big\{ \mu_{rk}^3 \alpha^{3r(n+2-k)} - \nu_{rk}^3 \beta^{3r(n+2-k)} + \mu_{rk}\nu_{rk} Q^{r(n+1-k)} (V_r^2 - Q^r) \left[\mu_{rk}\alpha^{r(n+2-k)} - \nu_{rk}\beta^{r(n+2-k)} \right] \Big\}$$

$$= \frac{(\alpha^r - \beta^r)Q^{rk}}{\mu_{rk}V_r} \lim_{t\to\infty} \left[\frac{1}{W_{rk}} - \frac{\beta^r}{W_{r(k+1)}} - \frac{\beta^{2r}}{W_{r(k+2)}} + \frac{\beta^{3r}}{W_{r(k+3)}} - \frac{\beta^{r(t+1-k)}}{W_{r(t+1)}} + \frac{\beta^{r(t+2-k)}}{W_{r(t+2)}} + \frac{\beta^{r(t+3-k)}}{W_{r(t+3)}} - \frac{\beta^{r(t+4-k)}}{W_{r(t+4)}} \right].$$

where the limiting process has been justified by

$$\left| \frac{\alpha}{\beta} \right| = \left| \frac{P + \sqrt{P^2 - 4Q}}{P - \sqrt{P^2 - 4Q}} \right| > 1, \quad \text{for} \quad P > 0,$$

and

$$\lim_{n\to\infty} T_s = \lim_{n\to\infty} \frac{\beta^{r(n+s-k)}}{W_{r(n+s)}} = \lim_{n\to\infty} \frac{1}{\frac{W_{r(k+1)} - W_{rk}\beta^r}{\alpha^r - \beta^r} (\frac{\alpha}{\beta})^{r(n+s-k)} - \frac{W_{r(k+1)} - W_{rk}\alpha^r}{\alpha^r - \beta^r}} = 0. \quad (11)$$

The proof of the theorem is completed. □

We define $E_{rn,m,l,s}$ to be $W_{rn}\prod_{j=0}^{s} W_{r(n+m+jl)}W_{r(n+2m+(j+1)l)}$, so the product

$$W_{rn}W_{r(n+m)}W_{r(n+m+l)}W_{r(n+2m+l)}W_{r(n+2m+2l)}$$

becomes $E_{rn,m,l,1}$.

Theorem 2.3 *Let $P > 0$. Then*

$$\sum_{n=k}^{t} \frac{Q^{rn}}{E_{rn,m,l,1}} \Big\{ x\mu_{rk}^3 \alpha^{r(3n+4m+2l-3k)} - y\nu_{rk}^3 \beta^{r(3n+4m+2l-3k)} + \mu_{rk}\nu_{rk} Q^{r(n+m-k)} \big[(x\alpha^{rm}\beta^{rl} + yV_{r(l+m)}) \times \mu_{rk}\alpha^{r(n+m+l-k)} - (y\alpha^{rl}\beta^{rm} + xV_{r(l+m)})\nu_{rk}\beta^{r(n+m+l-k)} \big] \Big\}$$

$$= \frac{(\alpha^r - \beta^r)^4 Q^{rk}}{\mu_{rk}[\alpha^{r(m+l)} - \beta^{r(m+l)})]} \Bigg[\sum_{i=0}^{m-1} \left(\frac{\beta^{ri}}{W_{r(k+i)}} - \frac{\beta^{r(m+l+i)}}{W_{r(k+m+l+i)}} - \frac{\beta^{r(t+i+1-k)}}{W_{r(t+i+1)}} + \frac{\beta^{r(t+m+l+i+1-k)}}{W_{r(t+m+l+i+1)}} \right) - \sum_{i=0}^{l-1} \left(\frac{\beta^{r(m+i)}}{W_{r(k+m+i)}} - \frac{\beta^{r(2m+l+i)}}{W_{r(k+2m+l+i)}} - \frac{\beta^{r(t+m+i+1-k)}}{W_{r(t+m+i+1)}} + \frac{\beta^{r(t+2m+l+i+1-k)}}{W_{r(t+2m+l+i+1)}} \right) \Bigg], \tag{12}$$

where $\mu_{rk} = W_{r(k+1)} - W_{rk}\beta^r$, $\nu_{rk} = W_{r(k+1)} - W_{rk}\alpha^r$, $x = \alpha^{r(m+l)} + \beta^{r(m+l)} - 2\alpha^{rl}\beta^{rm}$, $y = \alpha^{r(m+l)} + \beta^{r(m+l)} - 2\alpha^{rm}\beta^{rl}$.

Proof From (9) and (8), we have

$$T_0 - T_m = \frac{(\alpha^{rm} - \beta^{rm})\mu_{rk}Q^{r(n-k)}}{(\alpha^r - \beta^r)W_{rn}W_{r(n+m)}},$$

$$T_m - T_{m+l} = \frac{(\alpha^{rl} - \beta^{rl})\mu_{rk}Q^{r(n+m-k)}}{(\alpha^r - \beta^r)W_{r(n+m)}W_{r(n+m+l)}}.$$

Hence

$$T_0 - T_m - T_m + T_{m+l} = \frac{(\alpha^{rm} - \beta^{rm})\mu_{rk}Q^{r(n-k)}}{(\alpha^r - \beta^r)W_{rn}W_{r(n+m)}} - \frac{(\alpha^{rl} - \beta^{rl})\mu_{rk}Q^{r(n+m-k)}}{(\alpha^r - \beta^r)W_{r(n+m)}W_{r(n+m+l)}}$$

$$= \frac{\mu_{rk}Q^{r(n-k)}}{(\alpha^r - \beta^r)^2 W_{rn}W_{r(n+m)}W_{r(n+m+l)}} \left[x\mu_{rk}\alpha^{r(n+m-k)} + y\nu_{rk}\beta^{r(n+m-k)} \right],$$

$$T_{m+l} - T_{2m+l} - T_{2m+l} + T_{2m+2l} = \frac{\mu_{rk}Q^{r(n+m+l-k)}}{(\alpha^r - \beta^r)^2 W_{r(n+m+l)}W_{r((n+2m+l)}W_{r(n+2m+2l)}} \times \left[x\mu_{rk}\alpha^{r(n+2m+l-k)} + y\nu_{rk}\beta^{r(n+2m+l-k)} \right],$$

$$
\begin{aligned}
&(T_0 - T_m - T_m + T_{m+l}) - (T_{m+l} - T_{2m+l} - T_{2m+l)} + T_{2m+2l}) \\
&\quad = \frac{\mu_{rk} Q^{r(n-k)}[\alpha^{r(m+l)} - \beta^{r(m+l)})]}{E_{rn,m,l,1}(\alpha^r - \beta^r)^4} \Big\{ x\mu_{rk}^3 \alpha^{r(3n+4m+2l-3k)} \\
&\quad - y\nu_{rk}^3 \beta^{r(3n+4m+2l-3k)} + \mu_{rk}\nu_{rk} Q^{r(n+m-k)} \big[(x\alpha^{rm}\beta^{rl} + yV_{r(l+m)})\mu_{rk} \\
&\quad \times \alpha^{r(n+m+l-k)} - (y\alpha^{rl}\beta^{rm} + xV_{r(l+m)})\nu_{rk}\beta^{r(n+m+l-k)}\big]\Big\}.
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{n=k}^{t} \frac{Q^{rn}}{E_{rn,m,l,1}} \Big\{ x\mu_{rk}^3 \alpha^{r(3n+4m+2l-3k)} \\
&\quad - y\nu_{rk}^3 \beta^{r(3n+4m+2l-3k)} + \mu_{rk}\nu_{rk} Q^{r(n+m-k)} \big[(x\alpha^{rm}\beta^{rl} + yV_{r(l+m)}) \\
&\quad \times \mu_{rk}\alpha^{r(n+m+l-k)} - (y\alpha^{rl}\beta^{rm} + xV_{r(l+m)})\nu_{rk}\beta^{r(n+m+l-k)}\big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(\alpha^r - \beta^r)^4 Q^{rk}}{\mu_{rk}[\alpha^{r(m+l)} - \beta^{r(m+l)})]} \Bigg[\sum_{i=0}^{m-1} \left(\frac{\beta^{ri}}{W_{r(k+i)}} - \frac{\beta^{r(m+l+i)}}{W_{r(k+m+l+i)}} - \frac{\beta^{r(t+i+1-k)}}{W_{r(t+i+1)}} \right. \\
&\left. + \frac{\beta^{r(t+m+l+i+1-k)}}{W_{r(t+m+l+i+1)}} \right) - \sum_{i=0}^{l-1} \left(\frac{\beta^{r(m+i)}}{W_{r(k+m+i)}} - \frac{\beta^{r(2m+l+i)}}{W_{r(k+2m+l+i)}} - \frac{\beta^{r(t+m+i+1-k)}}{W_{r(t+m+i+1)}} \right. \\
&\left. + \frac{\beta^{r(t+2m+l+i+1-k)}}{W_{r(t+2m+l+i+1)}} \right) \Bigg].
\end{aligned}
$$

The proof of the theorem is completed. □

Theorem 2.4 *Let* $P > 0$. *Then*

$$
\begin{aligned}
&\sum_{n=k}^{\infty} \frac{Q^{rn}}{E_{rn,m,l,1}} \Big\{ x\mu_{rk}^3 \alpha^{r(3n+4m+2l-3k)} \\
&\quad - y\nu_{rk}^3 \beta^{r(3n+4m+2l-3k)} + \mu_{rk}\nu_{rk} Q^{r(n+m-k)} \big[(x\alpha^{rm}\beta^{rl} + yV_{r(l+m)}) \\
&\quad \times \mu_{rk}\alpha^{r(n+m+l-k)} - (y\alpha^{rl}\beta^{rm} + xV_{r(l+m)})\nu_{rk}\beta^{r(n+m+l-k)}\big]\Big\} \\
&\quad = \frac{(\alpha^r - \beta^r)^4 Q^{rk}}{\mu_{rk}[\alpha^{r(m+l)} - \beta^{r(m+l)})]} \Bigg[\sum_{i=0}^{m-1} \left(\frac{\beta^{ri}}{W_{r(k+i)}} - \frac{\beta^{r(m+l+i)}}{W_{r(k+m+l+i)}} \right) \\
&\quad - \sum_{i=0}^{l-1} \left(\frac{\beta^{r(m+i)}}{W_{r(k+m+i)}} - \frac{\beta^{r(2m+l+i)}}{W_{r(k+2m+l+i)}} \right) \Bigg], \qquad (13)
\end{aligned}
$$

where $\mu_{rk} = W_{r(k+1)} - W_{rk}\beta^r$, $\nu_{rk} = W_{r(k+1)} - W_{rk}\alpha^r$, $x = \alpha^{r(m+l)} + \beta^{r(m+l)} - 2\alpha^{rl}\beta^{rm}$, $y = \alpha^{r(m+l)} + \beta^{r(m+l)} - 2\alpha^{rm}\beta^{rl}$.

Proof By (12) and (11), we have

$$\sum_{n=k}^{\infty} \frac{Q^{rn}}{E_{rn,m,l,1}} \Big\{ x\mu_{rk}^3 \alpha^{r(3n+4m+2l-3k)} - y\nu_{rk}^3 \beta^{r(3n+4m+2l-3k)} + \mu_{rk}\nu_{rk} Q^{r(n+m-k)} \big[(x\alpha^{rm}\beta^{rl} + yV_{r(l+m)}) \times \mu_{rk}\alpha^{r(n+m+l-k)} - (y\alpha^{rl}\beta^{rm} + xV_{r(l+m)})\nu_{rk}\beta^{r(n+m+l-k)} \big] \Big\}$$

$$= \frac{(\alpha^r - \beta^r)^4 Q^{rk}}{\mu_{rk}[\alpha^{r(m+l)} - \beta^{r(m+l)}]} \lim_{t\to\infty} \Bigg[\sum_{i=0}^{m-1} \left(\frac{\beta^{ri}}{W_{r(k+i)}} - \frac{\beta^{r(m+l+i)}}{W_{r(k+m+l+i)}} - \frac{\beta^{r(t+i+1-k)}}{W_{r(t+i+1)}} \right) + \frac{\beta^{r(t+m+l+i+1-k)}}{W_{r(t+m+l+i+1)}} - \sum_{i=0}^{l-1} \left(\frac{\beta^{r(m+i)}}{W_{r(k+m+i)}} - \frac{\beta^{r(2m+l+i)}}{W_{r(k+2m+l+i)}} - \frac{\beta^{r(t+m+i+1-k)}}{W_{r(t+m+i+1)}} + \frac{\beta^{r(t+2m+l+i+1-k)}}{W_{r(t+2m+l+i+1)}} \right) \Bigg].$$

The theorem is proved. □

Corollary 2.5 *Let $P > 0$. Then*

$$\sum_{n=1}^{\infty} \frac{Q^{rn}}{E_{rn,m,l,1}} \Big\{ x\mu_r^3 \alpha^{r(3n+4m+2l-3)} - y\nu_r^3 \beta^{r(3n+4m+2l-3)} + \mu_r\nu_r Q^{r(n+m-1)} \big[(x\alpha^{rm}\beta^{rl} + yV_{r(l+m)}) \times \mu_r\alpha^{r(n+m+l-1)} - (y\alpha^{rl}\beta^{rm} + xV_{r(l+m)})\nu_r\beta^{r(n+m+l-1)} \big] \Big\}$$

$$= \frac{(\alpha^r - \beta^r)^4 Q^r}{\mu_r[\alpha^{r(m+l)} - \beta^{r(m+l)}]} \Bigg[\sum_{i=0}^{m-1} \left(\frac{\beta^{ri}}{W_{r(1+i)}} - \frac{\beta^{r(m+l+i)}}{W_{r(1+m+l+i)}} \right) - \sum_{i=0}^{l-1} \left(\frac{\beta^{r(m+i)}}{W_{r(1+m+i)}} - \frac{\beta^{r(2m+l+i)}}{W_{r(1+2m+l+i)}} \right) \Bigg],$$

where $\mu_r = W_{2r} - W_r\beta^r$, $\nu_r = W_{2r} - W_r\alpha^r . x = \alpha^{r(m+l)} + \beta^{r(m+l)} - 2\alpha^{rl}\beta^{rm}$, $y = \alpha^{r(m+l)} + \beta^{r(m+l)} - 2\alpha^{rm}\beta^{rl}$.

Proof Take $k = 1$ in the identity (13), respectively. □

3 Some Applications

In this section we can obtain some interesting identities involving Fibonacci, Lucas numbers by taking special values for a, b, P and Q.

3.1 Fibonacci Numbers

In this case, $W_n(0,\ 1;\ 1,\ -1) = F_n$, the Fibonacci number. Then according to above theorems, corollaries and the Binet formula (2) we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^{rn}}{F_{rn}F_{r(n+m)}F_{r(n+m+l)}F_{r(n+2m+l)}F_{r(n+2m+2l)}} \Big\{x\alpha^{r(3n+4m+2l)} - y\beta^{r(3n+4m+2l)} + (-1)^{r(n+m-1)} \Big[(x\alpha^{rm}\beta^{rl} + yV_{r(l+m)})\alpha^{r(n+m+l)} - (y\alpha^{rl}\beta^{rm} + xV_{r(l+m)})\beta^{r(n+m+l)}\Big]\Big\}$$

$$= \frac{5\sqrt{5}(-1)^r}{\alpha^r U_{r(m+l)}} \left[\sum_{i=0}^{m-1} \left(\frac{\beta^{ri}}{F_{r(1+i)}} - \frac{\beta^{r(m+l+i)}}{F_{r(1+m+l+i)}}\right) - \sum_{i=0}^{l-1} \left(\frac{\beta^{r(m+i)}}{F_{r(1+m+i)}} - \frac{\beta^{r(2m+l+i)}}{F_{r(1+2m+l+i)}}\right)\right],$$

In particular,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{F_nF_{n+1}F_{n+3}F_{n+4}F_{n+6}} \Big\{2\sqrt{5}(\alpha^{3n+9} + \beta^{3n+9}) + 2(-1)^{n+1} \big[(10 - \sqrt{5})\alpha^{n+3} - (10 + \sqrt{5})\beta^{n+3}\big]\Big\} = \frac{-113\sqrt{5}}{48}.$$

3.2 Lucas Numbers

In this case, $W_n(2,\ 1;\ 1,\ -1) = L_n$, the Lucas number. Then according to above theorems, corollaries and the Binet formula (2) we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^{rn}}{L_{rn}L_{r(n+m)}L_{r(n+m+l)}L_{r(n+2m+l)}L_{r(n+2m+2l)}} \Big\{x\alpha^{r(3n+4m+2l)} + y\beta^{r(3n+4m+2l)} - (-1)^{r(n+m-1)} \Big[(x\alpha^{rm}\beta^{rl} + yV_{r(l+m)})\alpha^{r(n+m+l)} + (y\alpha^{rl}\beta^{rm} + xV_{r(l+m)})\beta^{r(n+m+l)}\Big]\Big\}$$

$$= \frac{(-1)^r}{\sqrt{5}\alpha^r U_{r(m+l)}} \left[\sum_{i=0}^{m-1} \left(\frac{\beta^{ri}}{L_{r(1+i)}} - \frac{\beta^{r(m+l+i)}}{L_{r(1+m+l+i)}}\right) - \sum_{i=0}^{l-1} \left(\frac{\beta^{r(m+i)}}{L_{r(1+m+i)}} - \frac{\beta^{r(2m+l+i)}}{L_{r(1+2m+l+i)}}\right)\right],$$

In particular,

$$\sum_{n=1}^{\infty} \frac{(-1)^n \left[\alpha^{3n+12} + \beta^{3n+12} - 8(-1)^n \left(\alpha^{n+4} + \beta^{n+4}\right)\right]}{L_nL_{n+2}L_{n+4}L_{n+6}L_{n+8}} = -\frac{755351}{32384880}.$$

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