REPRESENTATION NUMBER OF A CATERPILLAR

LALEH YAHYAEI[1] AND S. A. KATRE*[2]

Department of Mathematics
S. P. Pune University
Pune-411007, INDIA

Email 1: *laleh_yahyaie@yahoo.com*
Email 2: *sakatre@math.unipune.ac.in, sakatre@gmail.com*

ABSTRACT. For a finite graph G with vertices $\{v_1, \dots, v_r\}$, a representation of G modulo n is a set $\{a_1, \dots, a_r\}$ of distinct, nonnegative integers with $0 \le a_i < n$, satisfying $\gcd(a_i - a_j, n) = 1$ if and only if v_i is adjacent to v_j. The representation number, Rep(G), is the smallest n such that G has a representation modulo n.

Evans et al obtained the representation number of paths. They also obtained the representation number of a cycle except for cycles of length $2^k + 1$, $k \ge 3$. In the present paper we obtain upper and lower bounds for the representation number of a caterpillar, and get its exact value in some cases.

Keywords: Representation number of a graph, Product dimension, Caterpillar, Graph labeling, Path, Cycle.

1. INTRODUCTION

For a finite graph G, with vertices $\{v_1, \dots, v_r\}$, a representation of G modulo n is a set $\{a_1, \dots, a_r\}$ of distinct, nonnegative integers, $0 \le a_i < n$ satisfying $\gcd(a_i - a_j, n) = 1$ if and only if v_i is adjacent to v_j. P. Erdős and A. B. Evans [1] have shown that any finite graph can be represented modulo some positive integer. The representation number, Rep(G), of a graph G, is the smallest n such that G has a representation modulo n.

Modular representations have appeared in several recent publications. Erdős and Evans [1] obtained an upper bound for Rep(G) in terms the number of edges in the complement of G and the order (i.e. the number of vertices) of G.

D. A. Narayan [6] later refined this bound by proving that a graph of order $r > 1$ can be represented modulo a positive integer less than or equal

2012 AMS Subject Classification: 05C78 (primary)
* corresponding author

to the product of the first $r-1$ primes greater than or equal to $r-1$. This bound is the best possible as it is attained by $K_{r-1}+K_1$ (disjoint union). See ([3], Theorem 5.2) and [5], [6].

In this paper we shall obtain close upper and lower bounds for the representation number for some classes of caterpillars. In some cases we are able to get the representation number exactly. Section 2 contains results regarding the representation number of general graphs in the literature. In Section 3 we concentrate on the relation between dimension and representation number for general graphs. In Section 4 we state the results for paths and cycles obtained by A. B. Evans et al [3]. In Sections 5 and 6, we get lower and upper bounds (resp.) for the representation number of a caterpillar. In Section 7, we present results regarding the representation number of certain classes of caterpillar.

2. General Theory

A graph G is *reduced* if no two vertices of G have the same open neighborhood. A *reduction* of G is any reduced graph obtained from G by repeatedly identifying pairs of vertices with common open neighborhoods. Any two reductions of G are isomorphic.

We denote by G_n the graph with representation $\{0,\ldots,n-1\}$ modulo n. Then $\mathrm{Rep}(G_n)=n$. A a graph H is representable modulo n if and only if H is isomorphic to an induced subgraph of G_n. Let $n=p_{i_1}^{k_1}\ldots p_{i_m}^{k_m}$, p_{i_j} distinct primes, $k_j\geq 1$. In G_n, two vertices v_1 and v_2 are adjacent if and only if their representations a_1,a_2 have the property that a_1-a_2 is coprime to $p_{i_1}\ldots p_{i_m}$. Also v_1,v_2 have the same open neighborhood if and only if $a_1\equiv a_2 \pmod{p_{i_1}\ldots p_{i_m}}$. G_n is reduced if and only if n is squarefree.

Evans et al ([2], Lemma 2.2) proved that a graph is representable modulo $p_{i_1}^{k_1}\ldots p_{i_s}^{k_s}$, for some $k_1,\ldots,k_s\geq 1$, where $p_{i_1},\ldots,p_{i_s}$ are distinct primes, if and only if its reduction is representable modulo $p_{i_1}\ldots p_{i_s}$. It follows that the representation number of a reduced graph is squarefree. See also [3] for more information and results about representation number of graphs.

3. Dimensions and Representations

A *product representation* of *length* s of a graph G assigns distinct vectors of nonnegative integers, of length s, to the vertices of G so that vertices u and v are adjacent if and only if their associated vectors differ in every position. The *product dimension* of a graph, denoted $\mathrm{pdim}(G)$, is the minimum length of such a representation of G.

There is a close relation between product representation and modular representation. From a representation of a graph G modulo the product of distinct primes $p_1,\ldots,p_s$, we obtain a product representation of length s as follows. If the vertex v is represented by a modulo $p_1\ldots p_s$, then the associated s-tuple for v is $(v_1, \ldots, v_s)$, where $v_i \equiv a \pmod{p_i}$ and $v_i \in \{0, \ldots, p_{i-1}\}$ for $1 \le i \le s$. If $(u_1, \ldots, u_s)$ and $(v_1, \ldots, v_s)$ are the vectors associated to u and v, then the modular representation implies that u and v are adjacent if and only if $u_i \neq v_i$ for all i, making this assignment a product representation. Thus $\text{pdim}(G) \le s$.

Conversely, given a product representation, a modular representation can be obtained by choosing distinct primes for the coordinates, provided that the prime for each coordinate is larger than the number of values used in that coordinate. The numbers assigned to the vertices can then be obtained using Chinese Remainder Theorem. The resulting modular representation generated from the product representation is called the *coordinate representation.*

Thus we may think of $\text{pdim}(G)$ as the smallest number of prime factors we can have in a representation of G modulo a product of powers of distinct primes. A related representation parameter is given by Silva [7]. The *degree* of a representation modulo n is defined to be the number of prime divisors of n, counting multiplicities. The *representation degree*, $d_r(G)$, of a graph is the smallest degree of any representation of G. Clearly $\text{pdim}(G) \ge d_r(G)$, and if G is reduced then $\text{pdim}(G) = d_r(G)$. Also $\text{pdim}(G) \le$ the number of primes in the factorization of $\text{Rep}(G)$.

The following result by A. B. Evans et al [3] tells us about the possible size of the prime factors of $\text{Rep}(G)$ in terms of the chromatic number $\chi(G)$ of G.

Theorem 3.1. (A. B. Evans et al) ([3], Theorem 2.11) *If G is representable modulo n and p is a prime divisor of n then $p \ge \chi(G)$. Thus, if G is reduced then* $\text{Rep}(G) \ge p_i p_{i+1} \cdots p_{i+m-1}$, *where p_i is the smallest prime satisfying $p_i \ge \chi(G)$ and $m = d_r(G) = \text{pdim}(G)$.*

Proof. See Theorem 1.2 in A. B. Evans et al [2] and Silva [7] and use the observed fact that $\text{pdim}(G) = d_r(G)$ for reduced graphs. □

An elementary but useful result is the following.

Proposition 3.2. *If H is an induced subgraph of G, then*

$$\text{Rep}(H) \le \text{Rep}(G).$$

In this paper, as in [3], we denote by P_n, a path on n vertices. Note that in [5], P_n denotes a path on n edges, i.e. $n+1$ vertices.

Note that the inequality in Proposition 3.2 does not hold for non-induced subgraphs. For example, the path P_2 of length 1 is represented by $0, 1 \pmod 2$ and has $\text{Rep}(P_2) = 2$, whereas its subgraph consisting of just 2 points and no edge (thus not an induced subgraph) is represented by $0, 2 \pmod 4$ and has representation number 4.

4. Paths and Cycles

In this section we review the representation number of paths and cycles. For most of our examples in this section we will have

$$\mathrm{Rep}(G) = p_i p_{i+1} \dots\ p_{i+m-1} \text{ where } m = d_r(G) = \mathrm{pdim}(G),$$

where p_i denotes the i^{th} prime in the increasing sequence of primes $2, 3, 5, \dots$.

Paths form an important class of graphs. They have a role to play in determining lower bounds for representation numbers as, if P_n, the path on n vertices, is an induced subgraph of G, then $\mathrm{Rep}(P_n) \leq \mathrm{Rep}(G)$ by Theorem 3.2. For paths the representation numbers are known. If $n > 1$, then $\{0, 1, \dots, n\}$ is a representation of P_n modulo $n!$. This is of course also a representation modulo the product of all primes less than or equal to n. However the representation number of a path is usually still smaller. The representation numbers for P_n for small n are

$$\mathrm{Rep}(P_1) = 1,\ \mathrm{Rep}(P_2) = 2,\ \mathrm{Rep}(P_3) = 4,$$
$$\mathrm{Rep}(P_4) = 6, \text{ and } \mathrm{Rep}(P_5) = 6.$$

In all these cases, the representation $\{0, 1, \dots, n-1\}$ works for P_n for the given modulus. The following theorem gives the representation numbers for P_n, $n \geq 5$.

In what follows, for x real, $(x)^+$ denotes the smallest integer $\geq x$.

Theorem 4.1. (A. B. Evans et al)([3], Theorem 3.1) *For $n \geq 4$,*

$$\mathrm{Rep}(P_n) = 2 \times 3 \times \cdots \times p_{(\log_2(n-1))^+}.$$

Remark 4.2. *From Theorem 4.1 and Proposition 3.2, we see that for a tree T of diameter d,*

$$\mathrm{Rep}(T) \geq 2 \times 3 \times \cdots \times p_{(\log_2 d)^+}.$$

Cycles form another important class of graphs and the representation numbers are known for most cycles. For small values of n the representation numbers are

$$\mathrm{Rep}(C_3) = 3,\ \mathrm{Rep}(C_4) = 4,$$
$$\mathrm{Rep}(C_5) = 3 \times 5 \times 7 = 105,\ \mathrm{Rep}(C_7) = 3 \times 5 \times 7 \times 11 = 1155.$$
$$\mathrm{Rep}(C_9) \text{ is not known at present.}$$

For even values of n, the representation numbers are given in the following theorem.

Theorem 4.3. (A. B. Evans et al)([3], Theorem 3.2) $\mathrm{Rep}(C_4) = 4$, *and if* $n \geq 3$, $\mathrm{Rep}(C_{2n}) = 2 \times 3 \times \cdots \times p_{(\log_2(n-1))^+ + 1}$.

For the odd cases, $2n + 1$, the representation numbers of cycles have been completely determined when n is odd or an odd power of 2. These results are given in the next theorem.

Theorem 4.4. (A. B. Evans et al)([3], Theorem 3.3) $\text{Rep}(C_5) = 3 \times 5 \times 7$, $\text{Rep}(C_7) = 3 \times 5 \times 7 \times 11$, *and if* $n \geq 4$ *and* n *is not a power of* 2 *then*

$$\text{Rep}(C_{2n+1}) = 3 \times 5 \times \cdots \times p_{(\log_2 n)^+ + 1}.$$

Note that the result of Evans et al [2] shows that when n is not a power of 2, $\text{pdim}(C_{2n+1}) = (\log_2 n)^+ + 1$. This had been left as a question in Remark 6.4 of Lovász et al [5]. Therefore by Theorem 3.1 (this paper) of A. B. Evans et al [3] and by Theorem 3.2 in [2], $\text{Rep}(C_{2n+1}) = 3 \times 5 \times \cdots \times p_{(\log_2 n)^+ + 1}$ when n is not a power of 2. For the remaining cases C_{2n+1}, where n is of the form $n = 2^s$, the representation number is unknown. However, for dimension, in Lovász et al [5], it is shown that for these cases

$$(\log_2 n)^+ + 1 \leq \text{pdim}(C_{2n+1}) \leq (\log_2 n)^+ + 2$$

and upper bound holds for n of the form 2^{2k+1}. Note that except for P_3 and C_4, paths and cycles are reduced graphs and their representation numbers are squarefree. $\text{Rep}(P_3) = \text{Rep}(C_4) = 4$, is not squarefree.

5. A LOWER BOUND FOR THE REPRESENTATION NUMBER OF A CATERPILLAR

A caterpillar is a tree in which there is a path (called a spine) that contains at least one end-point of every edge. Such a path of minimum length in a caterpillar is called an *m-spine*. Its vertex set consists of all non-pendent vertices The m-spine is uniquely determined. There are many maximal spines called *M-spines* which are spines of maximum length. These have all the non-pendent vertices and also two pendent vertices of the caterpillar as extreme vertices of the spine. The m-spine is the intersection of all M-spine (see more in [4]). Note that a caterpillar is a reduced graph if and only if it has exactly 1 M-spine.

In this paper we consider certain families of caterpillars R_n of length $n-1$ and for all these families we get

$$2 \times 3 \times \cdots \times p_{(\log_2(n-1))^+} \leq \text{Rep}(R_n) \leq 2 \times 3 \times \cdots \times p_{(\log_2(n+1))^+ + 1}.$$

For particular families we get better bounds in which the upper and lower bound differ by at most one prime. We get cases when the bounds become equal and in those cases we are able to find $\text{Rep}(R_n)$.

To prove our results we mainly use the ideas of A. B. Evans et al [3] (Theorem 3.1 in this paper). Thus, to get a result for the representation of the caterpillar R_n, the result for the dimension of the caterpillar R_n from [4] is required. Note that:

In [4], R_n *is a caterpillar on* $n+1$ *vertices in M-spine whereas here we use* R_n *for a caterpillar on* n *vertices in M-spine.*

Theorem 5.1. *Let* R_n, $n \geq 5$, *be a caterpillar of length* $n-1$ *with* $x^1, \ldots, x^n$ *as vertices of an M-spine and* $\deg(x^i)$, $2 \leq i \leq n-1$, *be at most* 3. *Let the caterpillar* R_n *contain* t_0 *bunches of gap (non-leg) vertices* x^i *consisting of odd number vertices. Let* $t_1 = 1$ *provided at least one of* *the initial and final bunches of non-leg vertices consists of exactly* 1 *vertex and* $t_1 = 0$ *otherwise. The representation number of such a caterpillar* R_n *satisfies the inequality,*

$$2 \times 3 \times \cdots \times p_{(\log_2(|V|-t_0+t_1))^+} \leq \text{Rep}(R_n).$$

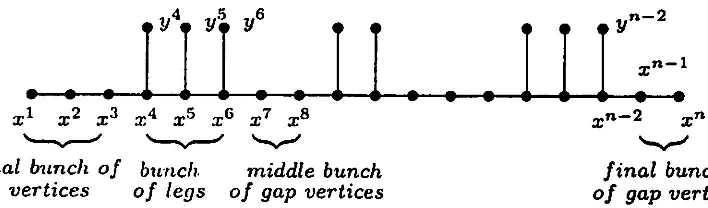

Figure 1

Proof. By Theorem 2.5 of [4], we see that if n is the number of vertices of an M-spine of such a caterpillar, then for $n \geq 5$, $\log_2(|V| - t_0 + t_1))^+ \leq \dim(R_n)$. Hence, by Theorem 3.1 of this paper, as R_n is a reduced graph and $\chi(R_n) = 2$, so $2 \times \cdots \times p_{(\log_2(|V|-t_0+t_1))^+} \leq \text{Rep}(R_n)$. □

6. An upper bound for the representation number of a caterpillar

Theorem 6.1. *Let* R_n, $n \geq 5$, *be a caterpillar of length* $n-1$ *and let* $x^1, \ldots, x^n$ *be the vertices of an M-spine of* R_n. *Let* $\deg(x^i) = 3$ *for* $3 \leq i \leq n-2$ *and* $\deg(x^i) = 2$ *for* $i = 2, n-1$. *For* $2 \leq i \leq n-2$, *it is given that the caterpillar* R_n *has legs starting from* x^i *and these are paths of length* 1 *given by* x^i-y^i. *Such a caterpillar* R_n *can be represented modulo the product of the first* f *distinct primes, where* $f = \dim(R_n) = (\log_2(n-1))^+ + 1$. *Thus, as* $n \geq 5$, *we have* $f \geq 3$, $n \leq 2^{f-1}+1$ *and* $\text{Rep}(R_n)$ *is at most the product of the first* f *primes,*

$$\text{Rep}(R_n) \leq 2 \times 3 \times \cdots \times p_f.$$

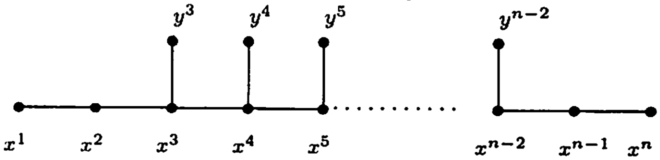

Figure 2

Proof. It has been proved by the authors in [4], that

$$\dim(R_n) = (\log_2(n-1))^+ + 1.$$

Thus if $n = 2^{f-1}+1$, then $\dim(R_n) = f$. We first prove the required upper bound for $\text{Rep}(R_n)$ by induction for all n of the form $2^f + 1$. We assume that there is a representation of R_n modulo the product of first f primes for $n = 2^{f-1}+1$. Note that in any such representation modulo a product of f distinct primes q_i, any associated number m corresponding to a vertex of R_n can be equivalently written as a vector of length f where the i^{th} entry is a number b_i, $0 \leq b_i < q_i$, and $m \equiv b_i \pmod{q_i}$. It is easily checked that $\text{Rep}(R_5) = 2 \times 3 \times 5$, with $V(x) = \{2, 9, 16, 27, 14\}$ and $V(y) = \{23\}$ where $V(x)$ is the sequence of labeling of base-leg vertices and $V(y)$ is the labeling of the pendent vertex y^3 which is adjacent to x^3.

Note that each of the numbers could be represented by a 3-tuple consisting of residues of the numbers modulo the 3 primes 2, 3 and 5. Thus 2 corresponds to the 3-tuple $(0, 2, 2)$ and 23 corresponds to the 3-tuple $(1, 2, 3)$. In the chosen representation of R_n modulo the product of $f = k+1$ distinct primes we represent each vertex x^i by the vector $v_k(i)$, $1 \leq i \leq 2^k + 1$, and the pendent vertex y^i by the vector $v'_k(i)$, $3 \leq i \leq 2^k - 1$. Here $v_k(i)$, $v'_k(i)$ are f-tuples (or $(k+1)$-tuples) or strings of length $f = k + 1$ of integers as above. Thus $V(R_n)$ is represented by

$$(v_k(1), \ldots, v_k(2^k + 1), v'_k(3), \ldots, v'_k(2^k - 1)).$$

Since x^i is adjacent to $x^{i \pm 1}$ and y^i, we can assume that there is no agreement in any position between $v_k(i)$ and $v_k(i \pm 1)$ as well as $v_k(i)$ and $v'_k(i)$ (i.e. the corresponding coordinates are different). For any other pair there is at least one agreement.

Now we give a representation for $R_{n'}$ with $n' = 2^f + 1 = 2^{k+1} + 1$ modulo the product of first $f + 1$ primes by using $(f + 1)$-tuples. We construct a sequence $\{v_{k+1}(i)\}$, $1 \leq i \leq 2^{k+1} + 1$, of distinct $(f + 1)$-tuples (or $(k + 2)$-tuples) as

$$v_k(1)0, v_k(2)1, \ldots, v_k(2^k - 1)0, v_k(2^k)1, v_{k,1}(2^k + 1)2,$$
$$v_k(2^k)0, v_k(2^k - 1)1, \ldots, v_k(2)0, v_k(1)1,$$

(the beginning $v_k(i)$ for odd i are followed by 0 and for even i follows by 1; the opposite in the last part) and a sequence $\{v'_{k+1}(i)\}$, $3 \leq i \leq 2^{k+1} - 1$, of $(f + 1)$-tuples as

$$v'_{k+1}(3) = v'_k(3)1, v'_{k+1}(4) = v'_k(4)0, \ldots,$$
$$v'_{k+1}(2^k - 2) = v'_k(2^k - 1)0,\ v'_{k+1}(2^k - 1) = v'_k(2^k - 1)1,$$
$$v'_{k+1}(2^k) = 020\ldots0\ldots010,$$
$$v'_{k+1}(2^k + 1) = 121\ldots1\ldots101,$$
$$v'_{k+1}(2^k + 2) = 020\ldots0\ldots011,$$
$$v'_{k+1}(2^k + 3) = v'_k(2^k - 1)0,\ v'_{k+1}(2^k + 4) = v'_k(2^k - 2)1,$$
$$\ldots, v'_{k+1}(2^{k+1} - 2) = v'_k(4)1, v'_{k+1}(2^{k+1} - 1) = v'_k(3)0.$$

Now we have a sequence of length $f + 1$ that corresponds to a labeling of R_{2^f+1} modulo a product of $f + 1$ distinct primes. We see that the labeling used in ([4], Theorem 4.1) for upper bound of the dimension of R_n has been utilized here. We can see that this labeling works of $R_{n'}$ as required. This proves by induction that for $n = 2^{f-1} + 1$, $\text{Rep}(R_n) \leq 2 \times 3 \times \cdots \times p_f$.

If $n \leq 2^{f-1} + 1$, then R_n is an induced subgraph of $R_{2^{f-1}+1}$, and so

$\text{Rep}(R_n) \leq$ product of first f primes. Hence,

$$\text{Rep}(R_n) \leq 2 \times 3 \times \cdots \times p_f$$

where $f = (\log_2(n-1))^+ + 1$. □

7. Representation number of a caterpillar

In this section we shall get results about the representation number of certain types of caterpillars using results of Sections 5 and 6.
First we get close bounds for the representation number of a general caterpillar considered in Theorem 5.1. Then we consider special types of caterpillars for which we get representation number for most n.

Theorem 7.1. *For a caterpillar R_n of diameter $n-1$ considered in Theorem 5.1,*

$$2 \times 3 \times \cdots \times p_{(\log_2(n-1))^+} \leq \text{Rep}(R_n) \leq 2 \times 3 \times \cdots \times p_{(\log_2(n+1))^+ + 1}.$$

If one of the initial and final sets of gap vertices has 2 *or more vertices, then* $\text{Rep}(R_n) \leq 2 \times 3 \times \cdots \times p_{(\log_2 n)^+ + 1}$. *If both the initial and final sets of gap vertices have* 2 *or more vertices then* $\text{Rep}(R_n) \leq 2 \times 3 \times \cdots \times p_{(\log_2(n-1))^+ + 1}$.

Proof. In Theorem 4.1 of [4], it is shown that for the caterpillar R_n on n vertices in M-spine, $(\log_2(n-1))^+ \leq \dim(R_n)$. Thus by Theorem 3.1 of this paper, as R_n is a reduced graph and $\chi(R_n) = 2$, we get

$$2 \times \cdots \times p_{(\log_2(n-1))^+} \leq \text{Rep}(R_n).$$

Now R_n is an induced subgraph of the caterpillar considered in Theorem 6.1, but having length $n+1$. Hence $\text{Rep}(R_n) \leq 2 \times 3 \times \cdots \times p_{(\log_2(n+1))^+ + 1}$. If any one or both (resp.) of the initial and final sets of gap vertices has 2 or more vertices, then $n+1$ can be replaced by n or $n-1$ (resp.). □

Theorem 7.2. *For the caterpillar R_n, $n \geq 5$, considered in Theorem 6.1* $\text{Rep}(R_n) = 2 \times 3 \times \cdots \times p_{(\log_2(n-1))^+ + 1}$ *if n is not of the form $2^k + 2$.*
For $n = 2^k + 2$, $2 \times 3 \times \cdots \times p_{k+1} \leq \text{Rep}(R_n) \leq 2 \times 3 \times \cdots \times p_{k+2}$.

Proof. By Theorem 4.2 from [4], $(\log_2(n-2))^+ + 1 \leq \dim(R_n)$. As R_n is a reduced graph, so by Theorem 3.1 of this paper, $2 \times 3 \times \cdots \times p_{(\log_2(n-2))^+ + 1} \leq \text{Rep}(R_n)$. By Theorem 6.1, $\text{Rep}(R_n) \leq 2 \times 3 \times \cdots \times p_{(\log_2(n-1))^+ + 1}$. For $n \neq 2^k + 2$, both the bounds are equal and we get

$$\text{Rep}(R_n) = 2 \times 3 \times \cdots \times p_{(\log_2(n-1))^+ + 1}.$$

For $n = 2^k + 2$, $2 \times 3 \times \cdots \times p_{k+1} \leq \text{Rep}(R_n) \leq 2 \times 3 \times \cdots \times p_{k+2}$. □

Theorem 7.3. *Let R_n, $n \geq 3$, be a caterpillar of length $n-1$. If $x^2, x^3, \ldots, x^{n-1}$ are the vertices of the m-spine of R_n and* $\deg(x^i) = 3$ *for* $2 \leq i \leq n-1$, *then* $\text{Rep}(R_n)$ *satisfies the inequality,*

$$2 \times 3 \times \cdots \times p_{(\log_2(n-1))^+ + 1} \leq \text{Rep}(R_n) \leq 2 \times 3 \times \cdots \times p_{(\log_2(n+1))^+ + 1}.$$

In particular,
$\text{Rep}(R_n) = 2 \times 3 \times \cdots \times p_{(\log_2(n-1))^+ + 1}$ *if n is not of the form 2^k or $2^k + 1$, and* $2 \times 3 \times \cdots \times p_{k+1} \leq \text{Rep}(R_n) \leq 2 \times 3 \times \cdots \times p_{k+2}$ *if $n = 2^k$, $2^k + 1$.*

Proof. In Theorem 4.3 from [4], it is shown that for the caterpillar R_n on n vertices in M-spine, $(\log_2(n-1))^+ + 1 \le \dim(R_n)$. Thus by Theorem 3.1 of this paper, as R_n is a reduced graph and $\chi(R_n) = 2$, we get

$$2 \times 3 \times \cdots \times p_{(\log_2(n-1))^+ + 1} \le \text{Rep}(R_n).$$

Now joining x^1 to a new vertex x^0, and x^n to a new vertex x^{n+1} by an edge, we get a new caterpillar say R'_{n+2} which is of the same type as Theorem 6.1. R_n being an induced subgraph of R'_{n+2}, we get

$$\text{Rep}(R_n) \le \text{Rep}(R'_{n+2}).$$

By Theorem 6.1, $\text{Rep}(R'_{n+2}) \le 2 \times 3 \times \cdots \times p_{(\log_2(n+1))^+ + 1}$. Thus

$$2 \times 3 \times \cdots \times p_{(\log_2(n-1))^+ + 1} \le \text{Rep}(R_n) \le 2 \times 3 \times \cdots \times p_{(\log_2(n+1))^+ + 1}.$$

Hence for n not of the form 2^k and $2^k + 1$,

$$\text{Rep}(R_n) = 2 \times 3 \times \cdots \times p_{(\log_2(n-1))^+ + 1}.$$

For $n = 2^k$ or $2^k + 1$, $2 \times 3 \times \cdots \times p_{k+1} \le \text{Rep}(R_n) \le 2 \times 3 \times \cdots \times p_{k+2}$. □

Now we shall consider a caterpillar with sets of bunches with $p - 1$ leg vertices followed by a gap vertex.

Theorem 7.4. *Let R_n, $n \ge p + 1$, be a caterpillar of length $n - 1$ and let $x^2, x^3, \ldots, x^{n-1}$ be the vertices of the m-spine of R_n. For $2 \le i \le n-1$, let $\deg(x^i) = 3$ or 2 according as $p \nmid (i-1)$ or $p | (i-1)$. Let $n \equiv r \pmod{p}$, $0 \le r \le p-1$, $h = 1$ if $r = 2$ and $h = 0$ if $r = 0, 1, 3, \ldots, p-1$. Under these conditions,* $\text{Rep}(R_n)$ *satisfies the inequality,*

$$2 \times 3 \times \cdots \times p_{(\log_2(n - \lceil \frac{n-1}{p} \rceil + h))^+ + 1} \le \text{Rep}(R_n) \le 2 \times 3 \times \cdots \times p_{(\log_2(n+1))^+ + 1}.$$

For $r = 2$, $\text{Rep}(R_n) \le 2 \times 3 \times \cdots \times p_{(\log_2 n)^+ + 1}$.

In particular, for $2^{k-1} + \frac{2^{k-1}-1}{p-1} + 1 < n \le 2^k - 1$

$$\text{Rep}(R_n) = 2 \times 3 \times \cdots \times p_{k+1}.$$

If $n = 2^k$, where $n \equiv 2 \pmod{p}$, i.e. $r = 2$, $\text{Rep}(R_n) = 2 \times 3 \times \cdots \times p_{k+1}$.

For $n = 2^k + 1$, $2 \times 3 \times \cdots \times p_{k+1} \le \text{Rep}(R_n) \le 2 \times 3 \times \cdots \times p_{k+2}$.

If $2^{k-1} + 2 \le n \le 2^{k-1} + \frac{2^{k-1}-2}{p-1} + 1$,

$$2 \times 3 \times \cdots \times p_k \le \text{Rep}(R_n) \le 2 \times 3 \times \cdots \times p_{k+1}.$$

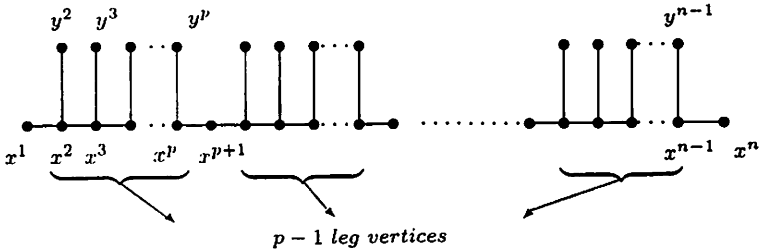

Figure 3

Proof. In Theorem 3.4 from [4], it is shown that for the caterpillar R_n, $(\log_2(n - \lceil \frac{n-1}{p} \rceil + h))^+ + 1 \le \dim(R_n)$. Thus by Theorem 3.1 of this paper, as R_n is a reduced graph and $\chi(R_n) = 2$, we get

$$2 \times 3 \times \cdots \times p_{(\log_2(n-\lceil\frac{n-1}{p}\rceil+h))^{+}+1} \leq \mathrm{Rep}(R_n).$$

Now the caterpillar R_n is an induced subgraph of the caterpillar R'_n which is of the same type as Theorem 7.3. Thus $\mathrm{Rep}(R_n) \leq \mathrm{Rep}(R'_n)$. By Theorem 7.3, $\mathrm{Rep}(R'_n) \leq 2 \times 3 \times \cdots \times p_{(\log_2(n+1))^{+}+1}$. Thus

$$2\times 3\times\cdots\times p_{(\log_2(n-\lceil\frac{n-1}{p}\rceil+h))^{+}+1} \leq \mathrm{Rep}(R_n) \leq 2\times 3\times\cdots\times p_{(\log_2(n+1))^{+}+1}.$$

By Theorem 3.4 from [4], if $2^{k-1} + \frac{2^{k-1}-2}{p-1} < n \leq 2^k - 2$, so $\dim(R_n) = k + 1$. In the notation of Theorem 3.4 from [4], n is the length of M-spine whereas here n is the number of vertices of M-spine. Therefore, if $2^{k-1} + \frac{2^{k-1}-1}{p-1} + 1 < n \leq 2^k - 1$, $\mathrm{Rep}(R_n) = 2\times 3\times\cdots\times p_{k+1}$. When $n \equiv 2 \pmod p$, the final set of non-leg vertices has two vertices, so by Theorem 7.1,

$$2 \times 3 \times \cdots \times p_{(\log_2(n-\lceil\frac{n-1}{p}\rceil+h))^{+}+1} \leq \mathrm{Rep}(R_n) \leq 2 \times 3 \times \cdots \times p_{(\log_2 n)^{+}+1}.$$

If $n = 2^k$ and $r = 2$, i.e. $2^k \equiv 2 \pmod p$, then
$\mathrm{Rep}(R_n) \leq 2 \times 3 \times \cdots \times p_{k+1}$. Thus $\mathrm{Rep}(R_n) = 2 \times 3 \times \cdots \times p_{k+1}$.

Also $2 \times 3 \times \cdots \times p_{k+1} \leq \mathrm{Rep}(R_n) \leq 2 \times 3 \times \cdots \times p_{k+2}$ if $n = 2^k + 1$, and $2 \times 3 \times \cdots \times p_k \leq \mathrm{Rep}(R_n) \leq 2 \times 3 \times \cdots \times p_{k+1}$ if $2^{k-1} + 2 \leq n \leq 2^{k-1} + \frac{2^{k-1}-1}{p-1} + 1$. □

Example 7.5. If $p = 2$ in Theorem 7.4, then for $n = 2^k$ we have

$$\mathrm{Rep}(R_n) = 2 \times 3 \times \cdots \times p_{k+1}.$$

If $p = 3$, then for $n = 2^k$ with *k odd*, $2^k - 1 \equiv 1 \pmod 3$. Hence

$$\mathrm{Rep}(R_n) = 2 \times 3 \times \cdots \times p_{k+1}.$$

In the following theorem we consider a variation of Theorem 7.4 for $p = 2$.

Theorem 7.6. *Let R_n, $n \geq 4$, be a caterpillar of length $n - 1$ and let $x^1, \ldots, x^n$ be the vertices of the M-spine. If for $2 \leq i \leq n - 1$, $\deg(x^i) = 3$ or 2 according as i is odd or even, then $\mathrm{Rep}(R_n)$ satisfies the inequality,*

$$2 \times 3 \times \cdots \times p_{(\log_2(n+1))^{+}} \leq \mathrm{Rep}(R_n) \leq 2 \times 3 \times \cdots \times p_{(\log_2(n-1))^{+}+1}.$$

In particular, if $n = 2^k + 1$ or 2^k,

$$\mathrm{Rep}(R_n) = 2 \times 3 \times \cdots \times p_{k+1}.$$

If $2^{k-1} + 2 \leq n \leq 2^k - 1$, then $2\times 3\times\cdots\times p_k \leq \mathrm{Rep}(R_n) \leq 2\times 3\times\cdots\times p_{k+1}$.

Proof. By Theorem 4.6 from [4], $(\log_2(n + 1))^{+} \leq \dim(R_n)$ for the caterpillar R_n of length $n - 1$. As the caterpillar R_n is a reduced graph, so by Theorem 3.1 we get $2 \times 3 \times \cdots \times p_{(\log_2(n+1))^{+}} \leq \mathrm{Rep}(R_n)$.
Now by Theorem 7.1, since for n even R_n has 2 initial gap vertices and

one initial gap vertex, and for n odd R_n has 2 initial as well as final gap vertices, we have,

$$\begin{aligned} \operatorname{Rep}(R_n) &\leq 2 \times 3 \times \cdots \times p_{(\log_2 n)^+ +1} && \text{if } n \text{ even,} \\ \operatorname{Rep}(R_n) &\leq 2 \times 3 \times \cdots \times p_{(\log_2 (n-1))^+ +1} && \text{if } n \text{ odd.} \end{aligned}$$

But if n is even, $(\log_2 n)^+ = (\log_2(n-1))^+$, so for all $n \geq 4$,

$$2 \times 3 \times \cdots \times p_{(\log_2(n+1))^+} \leq \operatorname{Rep}(R_n) \leq 2 \times 3 \times \cdots \times p_{(\log_2(n-1))^+ +1}.$$

Hence for $2^{k-1} + 2 \leq n \leq 2^k + 1$, $\operatorname{Rep}(R_n) = 2 \times 3 \times \cdots \times p_{k+1}$, if $n = 2^k$ or $2^k + 1$, otherwise $2 \times 3 \times \cdots \times p_k \leq \operatorname{Rep}(R_n) \leq 2 \times 3 \times \cdots \times p_{k+1}$. □

ACKNOWLEDGEMENT

The second author thanks Research and Development Grant of S. P. Pune University. The authors also thank the referee for carefully going through the manuscript and suggesting modifications related to literature survey.

REFERENCES

[1] P. Erdös, A. B. Evans, *Representations of graphs and orthogonal latin square graphs*, J. Graph Theory **13** (1989), 593-595.

[2] A. B. Evans, G. H. Fricke, C. C. Maneri, T. A. McKee, M. Perkel, *Representation of graphs modulo n*, J. Graph Theory **18** (1994), 801-815.

[3] A. B. Evans, G. Isaak and D. A. Narayan, *Representation of graphs modulo n*, Discrete Math. **223** (2000) 109-123.

[4] S. A. Katre, L. Yahyaei, *Dimension of a caterpillar* (to appear in JCMCC, 2016).

[5] L. Lovász, J. Nesetril, and A. Pultr, *On a product dimension of graphs*, J. Combinatorial Theory, Series B, **29** (1980), 47-67.

[6] D. A. Narayan, *An upper bound for representation number of graphs with fixed order*, Integers **3** (2003), A12, 1-4.

[7] F. C. Silva, *Integral representation of graphs*, Portugaliae Mathematics **53** (1996), 137-142.