Generalized Stanton–type Graphs

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ABSTRACT. Stanton–type graphs were introduced recently. In this paper, we define generalized Stanton–type graphs. We also identify LO and OE graphs, find the minimum λ for decomposition of λK_n into these graphs, and show that for all viable values of λ, the necessary conditions are sufficient for LO– and OE–decompositions using cyclic decompositions from base graphs.

1. Introduction

A complete multigraph λK_n (for $\lambda \geq 1$) is a graph on n points with λ edges between every pair of distinct points. A complete bipartite multigraph $\lambda K_{m,n}$ (for $\lambda \geq 1$) has λ copies of each edge in a complete bipartite graph $K_{m,n}$ (also denoted $\lambda K_{S,T}$ when S and T are partite sets of $K_{m,n}$, or simply as $\lambda K_{\{s_1,\ldots,s_m\},\{t_1,\ldots,t_n\}}$).

Decompositions of graphs into subgraphs is a well–known classical problem; for an excellent survey on graph decompositions, see [**1**]. Recently, several people including Chan [**4**], El–Zanati, Lapchinda, Tangsupphathawat and Wannasit [**5**], Hein [**6, 7, 8**], Sarvate [**9**], Winter [**11, 12**] and Zhang [**13**] have worked on decomposing λK_n into multigraphs. In fact, similar decompositions have been attempted earlier in various papers; see Priesler and Tarsi [**10**]. Ternary designs also provide such decompositions; see Billington [**2, 3**].

To this end, these authors have defined *Stanton graphs* (see [**4**]) and *Stanton–type graphs* (see [**8**]):

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DEFINITION 1. *A* Stanton graph S_n *on* $n \geq 2$ *vertices has exactly one edge of frequency* i *for every* $1 \leq i \leq \binom{n}{2}$.

DEFINITION 2. *Let* $n \geq 2$ *and* m *be fixed integers such that* $n-1 \leq m \leq \binom{n}{2}$. *A* Stanton-type graph $S(n,m)$ *on* n *vertices is a connected graph that has exactly one edge of frequency* i *for every* $1 \leq i \leq m$.

2. Preliminaries

We now introduce the concept of *generalized Stanton-type graphs*:

DEFINITION 3. *Let* $n \geq 2$, m *and* ℓ *be fixed integers such that* $n-1 \leq m \leq \binom{n}{2}$ *and* $1 \leq \ell \leq m+2-n$. *A* generalized Stanton-type graph $S(n,\ell,m)$ *on* n *vertices is a connected graph that has exactly one edge of frequency* i *for every* $\ell \leq i \leq m$.

EXAMPLE 1. $S(2,1,1)$:
This is the same as an $S(2,1)$ *or an* S_2, *which is of course a* K_2. ▲

EXAMPLE 2. $S(3,1,2)$: 
This is the same as an $S(3,2)$. ▲

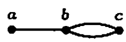
EXAMPLE 3. $S(3,1,3)$: 
This is the same as an S_3 *or an* $S(3,3)$. ▲

EXAMPLE 4. $S(3,2,3)$:  ▲

EXAMPLE 5. $S(4,1,3)$ *is the same as an* $S(4,3)$, *and* $S(4,1,6)$ *is the same as an* $S(4,6)$ *or an* S_4. ▲

For simplicity of notation, we adopt the "alphabetic labeling" used in [**6, 7, 8, 11, 12, 13**]:

DEFINITION 4. *An* LO graph $\lfloor a,b,c \rfloor$ *on* $V = \{a,b,c\}$ *is a graph with 3 edges where the frequencies of edges* $\{a,b\}$ *and* $\{b,c\}$ *are 1 and 2 (respectively).*



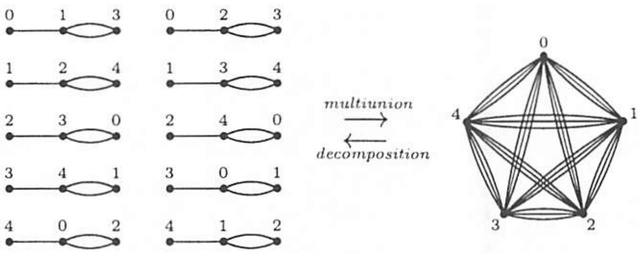
DEFINITION 5. *An* OE graph $|a,b,c|$ *on* $V = \{a,b,c\}$ *is a graph with 5 edges where the frequencies of edges* $\{a,b\}$ *and* $\{b,c\}$ *are 2 and 3 (respectively).*

NOTE 1. *An LO graph was called an* H_1 *graph in* [**9**] *and an* $S(3,1,2)$ *in Example 2 above. An OE graph was called an* $S(3,2,3)$ *in Example 4 above.*

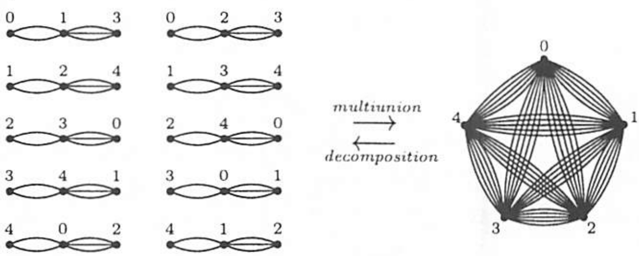
DEFINITION 6. *For positive integers $n \geq 3$ and $\lambda \geq 2$, an* LO–decomposition *of λK_n (denoted $LO(n, \lambda)$) is a collection of LO graphs such that the multiunion of their edge sets contains λ copies of all edges in a K_n. Similarly, for $\lambda \geq 3$, we have* OE–decompositions *of λK_n (denoted $OE(n, \lambda)$).*

One of the powerful techniques to construct combinatorial designs is based on *difference sets* and *difference families*; see Stinson [**14**] for details. This technique is modified to achieve our decompositions of λK_n — in general, we exhibit the *base graphs*, which can be developed to obtain the decomposition.

EXAMPLE 6. *Considering the set of points to be $V = \mathbb{Z}_5$, the LO base graphs $\lfloor 0, 1, 3 \rfloor$ and $\lfloor 0, 2, 3 \rfloor$ (when developed modulo 5) constitute an $LO(5, 3)$.*

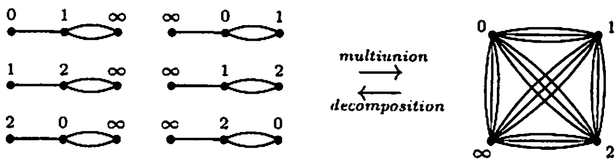


EXAMPLE 7. *Considering the set of points to be $V = \mathbb{Z}_5$, the OE base graphs $|0, 1, 3|$ and $|0, 2, 3|$ (when developed modulo 5) constitute an $OE(5, 5)$.*

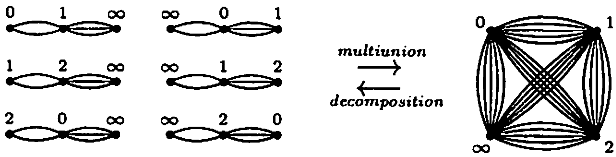


We note that special attention is needed with the base graphs containing the "dummy element" ∞; the non–∞ elements are developed, while ∞ is simply rewritten each time.

EXAMPLE 8. *Considering the set of points to be $V = \mathbb{Z}_3 \cup \{\infty\}$, the LO base graphs $\lfloor 0, 1, \infty \rfloor$ and $\lfloor \infty, 0, 1 \rfloor$ (when developed modulo 3) constitute an $LO(4, 3)$.*



EXAMPLE 9. *Considering the set of points to be* $V = \mathbb{Z}_3 \cup \{\infty\}$, *the OE base graphs* $|0, 1, \infty|$ *and* $|\infty, 0, 1|$ *(when developed modulo 3) constitute an* $OE(4, 5)$.



3. LO–Decompositions

We are now in a position to prove main results of the paper. We first remark that an LO graph has 3 vertices; that is, we consider $n \geq 3$. Also, necessarily $\lambda \geq 2$. We note that we use difference sets to achieve our decompositions of λK_n. In general, we exhibit the base graphs, which can be developed (modulo either n or $n-1$) to obtain the decomposition. We also note that the frequency of the edges is fixed by position, as per the LO graph.

We first address the minimum values of λ in an LO(n, λ).

THEOREM 3.1. *Let* $n \geq 3$. *The minimum values of* λ *for which an* $LO(n, \lambda)$ *exists are* $\lambda = 2$ *when* $n \equiv 0, 1 \pmod 3$ *and* $\lambda = 3$ *when* $n \equiv 2 \pmod 3$.

PROOF. Since there are $\frac{\lambda n(n-1)}{2}$ edges in a λK_n, and 3 edges in an LO graph, we must have that $\lambda n(n-1) \equiv 0 \pmod 6$ (where $\lambda \geq 2$ and $n \geq 3$) for LO–decompositions. By considering cases on $n \pmod 6$, we have that the minimum values of λ for which an LO(n, λ) exists are $\lambda \geq 2$ when $n \equiv 0, 1 \pmod 3$ and $\lambda \geq 3$ when $n \equiv 2 \pmod 3$.

We now show that these bounds are achieved. Let $n \geq 3$. We proceed by cases on $n \pmod 6$.

If $n = 6t$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{6t-1} \cup \{\infty\}$. The number of graphs required for LO$(6t, 2)$ is $\frac{2(6t)(6t-1)}{6} = 2t(6t-1)$. Thus, we need $2t$ base graphs (modulo $6t-1$). Then, the differences we must achieve (modulo $6t-1$) are $1, 2, \ldots, 3t-1$. For the first two base graphs, use $\lfloor \infty, 0, 3t-1 \rfloor$ and $\lfloor \infty, 0, 3t-2 \rfloor$. We also use the $2t-2$ base graphs $\lfloor 0,1,3t-2 \rfloor$, $\lfloor 0,1,3t-3 \rfloor$, $\lfloor 0,2,3t-3 \rfloor$, $\lfloor 0,2,3t-4 \rfloor, \ldots, \lfloor 0,t-2,2t+1 \rfloor$, $\lfloor 0,t-2,2t \rfloor$, $\lfloor 0,t-1,2t \rfloor$ and $\lfloor 0,t-1,2t-1 \rfloor$ if necessary. Hence, in this case, LO$(6t, 2)$ exists.

If $n = 6t+1$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{6t+1}$. The number of graphs required for LO$(6t+1, 2)$ is $\frac{2(6t+1)(6t)}{6} = 2t(6t+1)$. Thus, we need $2t$ base graphs (modulo $6t+1$). Then, the differences we must achieve (modulo $6t+1$) are $1, 2, \ldots, 3t$. We use the base graphs $\lfloor 0,1,3t+1 \rfloor$, $\lfloor 0,1,3t \rfloor$, $\lfloor 0,2,3t \rfloor$, $\lfloor 0,2,3t-1 \rfloor, \ldots, \lfloor 0,t-1,2t+3 \rfloor$, $\lfloor 0,t-1,2t+2 \rfloor$, $\lfloor 0,t,2t+2 \rfloor$ and $\lfloor 0,t,2t+1 \rfloor$. Hence, in this case, LO$(6t+1, 2)$ exists.

If $n = 6t+2$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{6t+1} \cup \{\infty\}$. The number of graphs required for LO$(6t+2, 3)$ is $\frac{3(6t+2)(6t+1)}{6} = (3t+1)(6t+1)$. Thus, we need $3t+1$ base graphs (modulo $6t+1$). Then, the differences we must achieve (modulo $6t+1$) are $1, 2, \ldots, 3t$. For the first base graph, use $\lfloor 0, \infty, 1 \rfloor$. For the last $3t$ base graphs, use $\lfloor 0,1,3t+1 \rfloor$, $\lfloor 0,2,3t+1 \rfloor, \ldots, \lfloor 0,3t-2,3t+1 \rfloor$, $\lfloor 0,3t-1,3t+1 \rfloor$ and $\lfloor 0,3t,3t+1 \rfloor$. Hence, in this case, LO$(6t+2, 3)$ exists.

If $n = 6t+3$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{6t+2} \cup \{\infty\}$. The number of graphs required for LO$(6t+3, 2)$ is $\frac{2(6t+3)(6t+2)}{6} = (2t+1)(6t+2)$. Thus, we need $2t+1$ base graphs (modulo $6t+2$). Then, the differences we must achieve (modulo $6t+2$) are $1, 2, \ldots, 3t+1$. For the first base graph, use $\lfloor 3t+1, 0, \infty \rfloor$. We also use the $2t$ base graphs $\lfloor 0,1,3t+1 \rfloor$, $\lfloor 0,1,3t \rfloor$, $\lfloor 0,2,3t \rfloor$, $\lfloor 0,2,3t-1 \rfloor, \ldots, \lfloor 0,t,2t+2 \rfloor$ and $\lfloor 0,t,2t+1 \rfloor$ if necessary. Hence, in this case, LO$(6t+3, 2)$ exists.

If $n = 6t+4$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{6t+4}$. The number of graphs required for LO$(6t+4, 2)$ is $\frac{2(6t+4)(6t+3)}{6} = (2t+1)(6t+4)$. Thus, we need $2t+1$ base graphs (modulo $6t+4$). Then, the differences we must achieve (modulo $6t+4$) are $1, 2, \ldots, 3t+2$. For the first base graph, use $\lfloor 0, 3t+2, 6t+3 \rfloor$. We also use the $2t$ base graphs $\lfloor 0,1,3t+1 \rfloor$, $\lfloor 0,1,3t \rfloor$, $\lfloor 0,2,3t \rfloor$, $\lfloor 0,2,3t-1 \rfloor, \ldots, \lfloor 0,t,2t+2 \rfloor$ and $\lfloor 0,t,2t+1 \rfloor$ if necessary. Hence, in this case, LO$(6t+4, 2)$ exists.

If $n = 6t+5$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{6t+5}$. The number of graphs required for LO$(6t+5, 3)$ is $\frac{3(6t+5)(6t+4)}{6} = (3t+2)(6t+5)$. Thus, we need $3t+2$ base graphs (modulo $6t+5$). Then, the differences we must achieve (modulo $6t+5$) are $1, 2, \ldots, 3t+2$. For the first two base graphs, use $\lfloor 0, 3t+1, 6t+3 \rfloor$ and $\lfloor 0, 3t+2, 3t+3 \rfloor$. For the last $3t$ base graphs, use $\lfloor 0,1,3 \rfloor$, $\lfloor 0,2,5 \rfloor$, $\lfloor 0,3,7 \rfloor, \ldots, \lfloor 0,3t,6t+1 \rfloor$ if necessary. Hence, in this case, LO$(6t+5, 3)$ exists. ■

We now address the sufficiency of existence of LO(n, λ).

THEOREM 3.2. *Let $n \geq 3$ and $\lambda \geq 2$. For LO(n, λ), the necessary condition for n is that $n \equiv 0, 1, 3, 4 \pmod 6$ when $\lambda \equiv 1, 2 \pmod 3$. There is no condition for n when $\lambda \equiv 0 \pmod 3$.*

PROOF. Since there are $\frac{\lambda n(n-1)}{2}$ edges in a λK_n, and 3 edges in an LO graph, we must have that $\lambda n(n-1) \equiv 0 \pmod 6$ (where $\lambda \geq 2$ and $n \geq 3$) for LO–decompositions. The result follows by considering cases on $\lambda \pmod 6$. ∎

LEMMA 3.1. *There exists an LO$(n, 2)$ for the necessary $n \geq 3$.*

PROOF. From Theorem 3.2, the necessary condition is $n \equiv 0, 1, 3, 4 \pmod 6$. In these cases, LO$(n, 2)$ exists from Theorem 3.1. ∎

LEMMA 3.2. *There exists an LO$(n, 3)$ for any $n \geq 3$.*

PROOF. From Theorem 3.2, there is no condition for n. We consider cases when $n \geq 3$ is odd or even.

If $n = 2t + 1$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{2t+1}$. The number of graphs required for LO$(2t + 1, 3)$ is $\frac{3(2t+1)(2t)}{6} = t(2t + 1)$. Thus, we need t base graphs (modulo $2t + 1$). The differences we must achieve (modulo $2t + 1$) are $1, 2, \ldots, t$. We use the base graphs $\lfloor 0, 1, t+1 \rfloor$, $\lfloor 0, 2, t+1 \rfloor, \ldots, \lfloor 0, t-1, t+1 \rfloor$, $\lfloor 0, t, t+1 \rfloor$. Hence, in this case, LO$(2t + 1, 3)$ exists.

If $n = 2t$ (for $t \geq 2$), we consider the set V as $\mathbb{Z}_{2t-1} \cup \{\infty\}$. The number of graphs required for LO$(2t, 3)$ is $\frac{3(2t)(2t-1)}{6} = t(2t-1)$. Thus, we need t base graphs (modulo $2t - 1$). The differences we must achieve (modulo $2t-1$) are $1, 2, \ldots, t-1$. For the first two base graphs, use $\lfloor t-1, 0, \infty \rfloor$ and $\lfloor \infty, 0, t-1 \rfloor$. For the last $t - 2$ base graphs, use $\lfloor 0, 1, t-1 \rfloor$, $\lfloor 0, 2, t-1 \rfloor, \ldots, \lfloor 0, t-3, t-1 \rfloor$, $\lfloor 0, t-2, t-1 \rfloor$ if necessary. Hence, in this case, LO$(2t, 3)$ exists. ∎

THEOREM 3.3. *An LO(n, λ) exists for all $\lambda \geq 2$ and necessary $n \geq 3$.*

PROOF. We proceed by cases on $\lambda \pmod 3$.

For $\lambda \equiv 0 \pmod 3$ (so that $\lambda = 3t$ for $t \geq 1$), by taking t copies of an LO$(n, 3)$ (given in Lemma 3.2), we have an LO$(n, 3t)$.

For $\lambda \equiv 1 \pmod 3$ (so that $\lambda = 3t + 1 = 3(t-1) + 4$ for $t \geq 1$), we first take two copies of an LO$(n, 2)$ (given in Lemma 3.1). (This gives us $\lambda = 4$ thus far.) We then adjoin this to $t - 1$ copies of an LO$(n,3)$ (given in Lemma 3.2) if necessary. Hence, we have an LO$(n,3t+1)$.

For $\lambda \equiv 2 \pmod 3$ (so that $\lambda = 3t+2$ for $t \geq 0$), we first take an LO$(n,2)$ (given in Lemma 3.1). (This gives us $\lambda = 2$ thus far.) We then adjoin this to t copies of an LO$(n,3)$ (given in Lemma 3.2) if necessary. Hence, we have an LO$(n,3t+2)$. ■

4. OE–Decompositions

We are again in a position to prove main results of the paper. We remark that an OE graph has 3 vertices; that is, we consider $n \geq 3$. Also, necessarily $\lambda \geq 3$. We again use difference sets to achieve our decompositions of λK_n. We also note that the frequency of the edges is fixed by position, as per the OE graph.

We first address the minimum value of λ in an OE(n,λ).

Evidently, there exists an LO$(n,3)$ if and only if there exists an OE$(n,5)$. (Compare Example 6 with Example 7, and Example 8 with Example 9.) This observation is the basis of the following result:

THEOREM 4.1. *Let $n \geq 3$. The minimum value of λ for which an OE(n,λ) exists is 5.*

PROOF. Suppose that G is an OE graph in some OE$(n,3)$. Let edge e have frequency 2 in G. Then, e yet needs a frequency of 1, which cannot occur with OE graphs. Hence, the minimum λ cannot be 3. Now, suppose that G is an OE graph in some OE$(n,4)$. Let edge e have frequency 3 in G. Then, e yet needs a frequency of 1, which cannot occur with OE graphs. Hence, the minimum λ cannot be 4. Thus, the minimum λ in all cases of n must be at least 5.

We now show that this bound is achieved. Let $n \geq 3$.

Take an LO$(n,3)$ (as given in Lemma 3.2). We replace each LO graph $\lfloor a,b,c \rfloor$ by the corresponding OE graph $|a,b,c|$. Hence, we have an EO$(n,5)$. ■

We now address the sufficiency of existence of OE(n,λ).

THEOREM 4.2. *Let $n \geq 3$ and $\lambda \geq 5$. For OE(n,λ), the necessary condition for n is that $n \equiv 0,1,5,6 \pmod{10}$ when $\lambda \not\equiv 0 \pmod 5$. There is no condition for n when $\lambda \equiv 0 \pmod 5$.*

PROOF. Since there are $\frac{\lambda n(n-1)}{2}$ edges in a λK_n, and 5 edges in an OE graph, we must have that $\lambda n(n-1) \equiv 0 \pmod{10}$ (where

$\lambda \geq 5$ and $n \geq 3$) for OE–decompositions. The result follows by considering cases on λ (mod 10). ■

LEMMA 4.1. *There exists an OE$(n, 5)$ for any $n \geq 3$.*

PROOF. From Theorem 4.2, there is no condition for n. Thus, OE$(n, 5)$ exists from Theorem 4.1. ■

LEMMA 4.2. *There exists an OE$(n, 6)$ for necessary $n \geq 3$.*

PROOF. From Theorem 4.2, the necessary condition is $n \equiv 0, 1, 5, 6 \pmod{10}$.

If $n = 10t$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{10t-1} \cup \{\infty\}$. The number of graphs required for OE$(10t, 6)$ is $\frac{6(10t)(10t-1)}{10} = 6t(10t-1)$. Thus, we need $6t$ base graphs (modulo $10t-1$). The differences we must achieve (modulo $10t-1$) are $1, 2, \ldots, 5t-1$. For the first six base graphs, use $|\infty, 0, 5t-1|$ twice, $|\infty, 0, 5t-2|$, $|0, 1, 5t-1|$ and $|0, 1, 5t-2|$ twice. For the last $6t-6$ base graphs, use $|0, 2, 5t-2|$ twice, $|0, 2, 5t-3|$, $|0, 3, 5t-2|$, $|0, 3, 5t-3|$ twice, $\ldots$, $|0, 2t-2, 4t|$ twice, $|0, 2t-2, 4t-1|$, $|0, 2t-1, 4t|$ and $|0, 2t-1, 4t-1|$ twice if necessary. Hence, in this case, OE$(10t, 6)$ exists.

If $n = 10t+1$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{10t+1}$. The number of graphs required for OE$(10t+1, 6)$ is $\frac{6(10t+1)(10t)}{10} = 6t(10t+1)$. Thus, we need $6t$ base graphs (modulo $10t+1$). The differences we must achieve (modulo $10t+1$) are $1, 2, \ldots, 5t$. We use the base graphs $|0, 1, 5t+1|$ twice, $|0, 1, 5t|$, $|0, 2, 5t+1|$, $|0, 2, 5t|$ twice, $\ldots$, $|0, 2t-1, 4t+2|$ twice, $|0, 2t-1, 4t+1|$, $|0, 2t, 4t+2|$ and $|0, 2t, 4t+1|$ twice. Hence, in this case, OE$(10t+1, 6)$ exists.

If $n = 10t+5$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{10t+4} \cup \{\infty\}$. The number of graphs required for OE$(10t+5, 6)$ is $\frac{6(10t+5)(10t+4)}{10} = (6t+3)(10t+4)$. Thus, we need $6t+3$ base graphs (modulo $10t+4$). The differences we must achieve (modulo $10t+4$) are $1, 2, \ldots, 5t+2$. For the first three base graphs, use $|\infty, 0, 5t+2|$ and $|\infty, 0, 5t+1|$ twice. For the last $6t$ base graphs, use $|0, 1, 5t+1|$ twice, $|0, 1, 5t|$, $|0, 2, 5t+1|$, $|0, 2, 5t|$ twice, $\ldots$, $|0, 2t-1, 4t+2|$ twice, $|0, 2t-1, 4t+1|$, $|0, 2t, 4t+2|$ and $|0, 2t, 4t+1|$ twice if necessary. Hence, in this case, OE$(10t+5, 6)$ exists.

If $n = 10t+6$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{10t+6}$. The number of graphs required for OE$(10t+6, 6)$ is $\frac{6(10t+6)(10t+5)}{10} = (6t+3)(10t+6)$. Thus, we need $6t+3$ base graphs (modulo $10t+6$). The differences we must achieve (modulo $10t+6$) are $1, 2, \ldots, 5t+3$.

For the first three base graphs, use $|0, 1, 5t + 4|$ and $|0, 1, 5t + 3|$ twice. For the last $6t$ base graphs, use $|0, 2, 5t+3|$ twice, $|0, 2, 5t+2|$, $|0, 3, 5t + 3|$, $|0, 3, 5t + 2|$ twice, ..., $|0, 2t, 4t + 4|$ twice, $|0, 2t, 4t + 3|$, $|0, 2t + 1, 4t + 4|$ and $|0, 2t + 1, 4t + 3|$ twice if necessary. Hence, in this case, OE$(10t + 6, 6)$ exists. ■

LEMMA 4.3. *There does not exist an OE$(n, 7)$.*

PROOF. The only edge frequencies in an OE graph are 2 and 3. The only way to write $\lambda = 7$ as a sum of 2s and 3s is as $7 = 3+2+2$. In an OE$(n, 7)$, the number of times each edge needs to occur triply is half the number of times it needs to occur doubly. However, as there are equal numbers of double edges and triple edges in an OE graph, such a decomposition is not possible. ■

LEMMA 4.4. *There exists an OE$(n, 8)$ for necessary $n \geq 3$.*

PROOF. From Theorem 4.2, the necessary condition is $n \equiv 0, 1, 5, 6 \pmod{10}$.

If $n = 10t$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{10t-1} \cup \{\infty\}$. The number of graphs required for OE$(10t, 8)$ is $\frac{8(10t)(10t-1)}{10} = 8t(10t-1)$. Thus, we need $8t$ base graphs (modulo $10t - 1$). The differences we must achieve (modulo $10t - 1$) are $1, 2, \ldots, 5t - 1$. For the first eight base graphs, use $|0, \infty, 1|$, $|4t, 0, \infty|$, $|0, 4t - 1, 8t - 2|$, $|0, 4t, 8t - 1|$, $|0, 4t-2, 8t-4|$, $|0, 4t, 8t-2|$, $|0, 4t-3, 8t-6|$ and $|0, 4t, 8t-3|$. For the last $8t - 8$ base graphs, use $|0, 1, 2|$, $|0, 5t - 1, 5t|$, $|0, 2, 4|$, $|0, 5t - 1, 5t+1|$, $|0, 3, 6|$, $|0, 5t-1, 5t+2|$, $|0, 4, 8|$, $|0, 5t-1, 5t+3|$, $|0, 5, 10|$, $|0, 5t-2, 5t+3|$, $|0, 6, 12|$, $|0, 5t-2, 5t+4|$, $|0, 7, 14|$, $|0, 5t-2, 5t+5|$, $|0, 8, 16|$, $|0, 5t - 2, 5t + 6|, \ldots, |0, 4t - 7, 8t - 14|$, $|0, 4t + 1, 8t - 6|$, $|0, 4t-6, 8t-12|$, $|0, 4t+1, 8t-5|$, $|0, 4t-5, 8t-10|$, $|0, 4t+1, 8t-4|$, $|0, 4t-4, 8t-8|$ and $|0, 4t+1, 8t-3|$ if necessary. Hence, in this case, OE$(10t, 8)$ exists.

If $n = 10t + 1$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{10t+1}$. The number of graphs required for OE$(10t + 1, 8)$ is $\frac{8(10t+1)(10t)}{10} = 8t(10t + 1)$. Thus, we need $8t$ base graphs (modulo $10t + 1$). The differences we must achieve (modulo $10t+1$) are $1, 2, \ldots, 5t$. We use the base graphs $|0, 1, 2|$, $|0, 5t, 5t + 1|$, $|0, 2, 4|$, $|0, 5t, 5t + 2|$, $|0, 3, 6|$, $|0, 5t, 5t+3|$, $|0, 4, 8|$, $|0, 5t, 5t+4|$, $|0, 5, 10|$, $|0, 5t-1, 5t+4|$, $|0, 6, 12|$, $|0, 5t - 1, 5t + 5|$, $|0, 7, 14|$, $|0, 5t - 1, 5t + 6|$, $|0, 8, 16|$, $|0, 5t - 1, 5t + 7|, \ldots, |0, 4t - 3, 8t - 6|$, $|0, 4t + 1, 8t - 2|$, $|0, 4t - 2, 8t - 4|$, $|0, 4t + 1, 8t-1|$, $|0, 4t-1, 8t-2|$, $|0, 4t+1, 8t|$, $|0, 4t, 8t|$ and $|0, 4t+1, 8t+1|$. Hence, in this case, OE$(10t + 1, 8)$ exists.

If $n = 10t+5$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{10t+4} \cup \{\infty\}$. The number of graphs required for OE$(10t+5, 8)$ is $\frac{8(10t+5)(10t+4)}{10} = (8t+4)(10t+4)$. Thus, we need $8t+4$ base graphs (modulo $10t+4$). The differences we must achieve (modulo $10t+4$) are $1, 2, \ldots, 5t+2$. For the first four base graphs, use $|0, 4t+1, 8t+2|$, $|0, 5t+2, 9t+3|$, $|0, \infty, 1|$ and $|5t+2, 0, \infty|$. For the last $8t$ base graphs, use $|0, 1, 2|$, $|0, 5t+1, 5t+2|$, $|0, 2, 4|$, $|0, 5t+1, 5t+3|$, $|0, 3, 6|$, $|0, 5t+1, 5t+4|$, $|0, 4, 8|$, $|0, 5t+1, 5t+5|$, $|0, 5, 10|$, $|0, 5t, 5t+5|$, $|0, 6, 12|$, $|0, 5t, 5t+6|$, $|0, 7, 14|$, $|0, 5t, 5t+7|$, $|0, 8, 16|$, $|0, 5t, 5t+8|, \ldots, |0, 4t-3, 8t-6|$, $|0, 4t+2, 8t-1|$, $|0, 4t-2, 8t-4|$, $|0, 4t+2, 8t|$, $|0, 4t-1, 8t-2|$, $|0, 4t+2, 8t+1|$, $|0, 4t, 8t|$ and $|0, 4t+2, 8t+2|$ if necessary. Hence, in this case, OE$(10t+5, 8)$ exists.

If $n = 10t+6$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{10t+6}$. The number of graphs required for OE$(10t+6, 8)$ is $\frac{8(10t+6)(10t+5)}{10} = (8t+4)(10t+6)$. Thus, we need $8t+4$ base graphs (modulo $10t+6$). The differences we must achieve (modulo $10t+6$) are $1, 2, \ldots, 5t+3$. For the first four base graphs, use $|0, 4t+1, 8t+2|$, $|0, 5t+3, 9t+4|$, $|0, 4t+2, 8t+4|$ and $|0, 5t+3, 9t+5|$. For the last $8t$ base graphs, use $|0, 1, 2|$, $|0, 5t+2, 5t+3|$, $|0, 2, 4|$, $|0, 5t+2, 5t+4|$, $|0, 3, 6|$, $|0, 5t+2, 5t+5|$, $|0, 4, 8|$, $|0, 5t+2, 5t+6|$, $|0, 5, 10|$, $|0, 5t+1, 5t+6|$, $|0, 6, 12|$, $|0, 5t+1, 5t+7|$, $|0, 7, 14|$, $|0, 5t+1, 5t+8|$, $|0, 8, 16|$, $|0, 5t+1, 5t+9|, \ldots, |0, 4t-3, 8t-6|$, $|0, 4t+3, 8t|$, $|0, 4t-2, 8t-4|$, $|0, 4t+3, 8t+1|$, $|0, 4t-1, 8t-2|$, $|0, 4t+3, 8t+2|$, $|0, 4t, 8t|$ and $|0, 4t+3, 8t+3|$ if necessary. Hence, in this case, OE$(10t+6, 8)$ exists. ■

EXAMPLE 10. *The OE graphs* $|b_1, b_2, b_3|$, $|b_1, b_2, b_4|$, $|b_1, b_2, b_5|$, $|b_1, b_3, b_2|$, $|b_1, b_3, b_4|$, $|b_1, b_3, b_5|$, $|b_1, b_4, b_2|$, $|b_1, b_4, b_3|$, $|b_1, b_4, b_5|$, $|b_2, b_5, b_3|$, $|b_4, b_5, b_1|$, $|b_4, b_5, b_3|$, $|b_5, b_1, b_2|$, $|b_5, b_1, b_3|$, $|b_5, b_1, b_4|$, $|b_5, b_2, b_3|$, $|b_5, b_2, b_4|$ *and* $|b_5, b_4, b_3|$ *constitute an OE*$(5, 9)$ *with point set* $V = \{b_1, \ldots, b_5\}$. ▲

EXAMPLE 11. *The OE graphs* $|b_1, b_2, b_3|$, $|b_1, b_2, b_4|$, $|b_1, b_2, b_5|$, $|b_1, b_6, b_2|$, $|b_1, b_6, b_3|$, $|b_1, b_6, b_4|$, $|b_2, b_6, b_1|$, $|b_2, b_6, b_3|$, $|b_2, b_6, b_5|$, $|b_3, b_5, b_1|$, $|b_3, b_5, b_4|$, $|b_4, b_3, b_1|$, $|b_4, b_3, b_2|$, $|b_4, b_3, b_5|$, $|b_4, b_6, b_3|$, $|b_4, b_6, b_5|$, $|b_4, b_6, b_5|$, $|b_5, b_1, b_3|$, $|b_5, b_1, b_4|$, $|b_5, b_1, b_4|$, $|b_5, b_2, b_1|$, $|b_5, b_2, b_3|$, $|b_5, b_2, b_4|$, $|b_5, b_3, b_1|$, $|b_5, b_4, b_1|$, $|b_5, b_4, b_2|$ *and* $|b_5, b_4, b_3|$ *constitute an OE*$(6, 9)$ *with point set* $V = \{b_1, \ldots, b_6\}$. ▲

EXAMPLE 12. *We see that the OE graphs* $|a_1, b, a_2|$, $|a_1, b, a_3|$, $|a_1, b, a_4|$, $|a_2, b, a_3|$, $|a_2, b, a_4|$, $|a_2, b, a_5|$, $|a_5, b, a_1|$, $|a_5, b, a_3|$ *and* $|a_5, b, a_4|$ *constitute an OE–decomposition of* $9K_{\{a_1,a_2,a_3,a_4,a_5\},\{b\}}$. ▲

LEMMA 4.5. *There exists an* $OE(n, 9)$ *for necessary* $n \geq 3$.

PROOF. From Theorem 4.2, the necessary condition is $n \equiv 0, 1, 5, 6 \pmod{10}$.

If $n = 10t$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{10t-1} \cup \{\infty\}$. The number of graphs required for OE$(10t, 9)$ is $\frac{9(10t)(10t-1)}{10} = 9t(10t-1)$. Thus, we need $9t$ base graphs (modulo $10t-1$). The differences we must achieve (modulo $10t-1$) are $1, 2, \ldots, 5t-1$. For the first nine base graphs, use $|0, 3t-2, 6t-4|$, $|0, 3t-2, 6t-1|$ twice, $|0, 3t-1, 6t-2|$, $|0, 3t-1, 6t|$, $|3t-1, 0, \infty|$, $|0, 3t, 6t|$ and $|3t, 0, \infty|$ twice. For the last $9t-9$ base graphs, use $|0, 1, 2|$, $|0, 1, 5t|$ twice, $|0, 2, 4|$, $|0, 2, 5t+1|$, $|0, 2, 5t|$, $|0, 3, 6|$, $|0, 3, 5t+1|$ twice, $|0, 4, 8|$, $|0, 4, 5t+1|$ twice, $|0, 5, 10|$, $|0, 5, 5t+2|$, $|0, 5, 5t+1|$, $|0, 6, 12|$, $|0, 6, 5t+2|$ twice, $\ldots$, $|0, 3t-5, 6t-10|$, $|0, 3t-5, 6t-2|$ twice, $|0, 3t-4, 6t-8|$, $|0, 3t-4, 6t-1|$, $|0, 3t-4, 6t-2|$, $|0, 3t-3, 6t-6|$ and $|0, 3t-3, 6t-1|$ twice if necessary. Hence, in this case, OE$(10t, 9)$ exists.

If $n = 10t+1$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{10t+1}$. The number of graphs required for OE$(10t+1, 9)$ is $\frac{9(10t+1)(10t)}{10} = 9t(10t+1)$. Thus, we need $9t$ base graphs (modulo $10t+1$). The differences we must achieve (modulo $10t+1$) are $1, 2, \ldots, 5t$. We use the base graphs $|0, 1, 2|$, $|0, 1, 5t+1|$ twice, $|0, 2, 4|$, $|0, 2, 5t+2|$, $|0, 2, 5t+1|$, $|0, 3, 6|$, $|0, 3, 5t+2|$ twice, $|0, 4, 8|$, $|0, 4, 5t+2|$ twice, $|0, 5, 10|$, $|0, 5, 5t+3|$, $|0, 5, 5t+2|$, $|0, 6, 12|$, $|0, 6, 5t+3|$ twice, $\ldots$, $|0, 3t-2, 6t-4|$, $|0, 3t-2, 6t|$ twice, $|0, 3t-1, 6t-2|$, $|0, 3t-1, 6t+1|$, $|0, 3t-1, 6t|$, $|0, 3t, 6t|$ and $|0, 3t, 6t+1|$ twice. Hence, in this case, OE$(10t+1, 9)$ exists.

Recall that an OE$(5, 9)$ on $\{b_1, \ldots, b_5\}$ is given in Example 10. If $n = 10t+5$ (for $t \geq 1$), we consider the set V as $\{a_1, \ldots, a_{10t}, b_1, \ldots, b_5\}$. To obtain an OE$(10t+5, 9)$, use an OE$(10t, 9)$ on $\{a_1, \ldots, a_{10t}\}$ (given two cases above), an OE$(5, 9)$ on $\{b_1, \ldots, b_5\}$ (given in Example 10), and an OE–decomposition of $9K_{\{a_{5i-4}, \ldots, a_{5i}\}, \{b_j\}}$ for all $i = 1, \ldots, 2t$ and for all $j = 1, \ldots, 5$ (given in Example 12). Hence, in this case, OE$(10t+5, 9)$ exists.

Recall that an OE$(6, 9)$ on $\{b_1, \ldots, b_6\}$ is given in Example 11. If $n = 10t+6$ (for $t \geq 1$), we consider the set V as $\{a_1, \ldots, a_{10t}, b_1, \ldots, b_6\}$. To obtain an OE$(10t+6, 9)$, use an OE$(10t, 9)$ on $\{a_1, \ldots, a_{10t}\}$ (given three cases above), an OE$(6, 9)$ on $\{b_1, \ldots, b_6\}$ (given in Example 11), and an OE–decomposition of $9K_{\{a_{5i-4}, \ldots, a_{5i}\}, \{b_j\}}$ for all $i = 1, \ldots, 2t$ and for all $j = 1, \ldots, 6$ (given in Example 12). Hence, in this case, OE$(10t+6, 9)$ exists. ∎

THEOREM 4.3. *An OE(n,λ) exists for all $\lambda \geq 5$ (except $\lambda = 7$, according to Lemma 4.3) and necessary $n \geq 3$.*

PROOF. We proceed by cases on λ (mod 5).

For $\lambda \equiv 0 \pmod 5$ (so that $\lambda = 5t$ for $t \geq 1$), by taking t copies of an OE$(n, 5)$ (given in Lemma 4.1), we have an OE$(n, 5t)$.

For $\lambda \equiv 1 \pmod 5$ (so that $\lambda = 5t + 1 = 5(t-1) + 6$ for $t \geq 1$), we first take an OE$(n, 6)$ (given in Lemma 4.2). (This gives us $\lambda = 6$ thus far.) We then adjoin this to $t - 1$ copies of an OE$(n, 5)$ (given in Lemma 4.1) if necessary. Hence, we have an OE$(n, 5t + 1)$.

For $\lambda \equiv 2 \pmod 5$ (so that $\lambda = 5t+2 = 5(t-2)+12$ for $t \geq 2$), we first take two copies of an OE$(n, 6)$ (given in Lemma 4.2). (This gives us $\lambda = 12$ thus far.) We then adjoin this to $t - 2$ copies of an OE$(n, 5)$ (given in Lemma 4.1) if necessary. Hence, we have an OE$(n, 5t + 2)$.

For $\lambda \equiv 3 \pmod 5$ (so that $\lambda = 5t + 3 = 5(t-1) + 8$ for $t \geq 1$), we first take an OE$(n, 8)$ (given in Lemma 4.4). (This gives us $\lambda = 8$ thus far.) We then adjoin this to $t - 1$ copies of an OE$(n, 5)$ (given in Lemma 4.1) if necessary. Hence, we have an OE$(n, 5t + 3)$.

For $\lambda \equiv 4 \pmod 5$ (so that $\lambda = 5t + 4 = 5(t-1) + 9$ for $t \geq 1$), we first take an OE$(n, 9)$ (given in Lemma 4.5). (This gives us $\lambda = 9$ thus far.) We then adjoin this to $t - 1$ copies of an OE$(n, 5)$ (given in Lemma 4.1) if necessary. Hence, we have an OE$(n, 5t + 4)$. ■

5. Conclusion

We have introduced generalized Stanton–type graphs $S(n, \ell, m)$, identified LO and OE graphs, found the minimum λ for decomposition of λK_n into these graphs, and showed that for all viable values of λ, the necessary conditions are sufficient for LO- and OE-decompositions. We note that we mostly used cyclic decompositions from base graphs; however, OE–decompositions of bipartite graphs were needed in two cases of Lemma 4.5.

Since decompositions of λK_n using near–triangle graphs H_1, H_2 and H_3 (see [**9**]) as well as generalized Stanton–type graphs $S(3,1,2)$, $S(3,1,3)$ (see [**4**]), $S(3,2,3)$ and $S(4,1,3)$ (see [**6, 7, 8**]) have been done, we leave it as an open problem to construct decompositions of λK_n using the generalized Stanton–type graphs $S(4,1,4)$, $S(4,1,5)$ and $S(4,1,6)$.

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