Unitary Cayley Graphs over Ring of Dual Numbers

Mohammad Abudayah *
Department of Mathematics
German Jordanian University
mohammad.abudayah@gju.edu.jo

Hasan Al-Ezeh
Department of Mathematics
University of Jordan
alezehh@ju.edu.jo

November 18, 2015

Abstract

This paper studies the unitary Cayley graph associated with ring of dual numbers, $Z_n[\alpha]$. It determines the exact diameter, vertex chromatic number and edge chromatic number. In addition, it classifies all perfect graphs within this class.

Keywords: Ring of Dual Numbers, Unitary Cayley Graph, Chromatic, Clique, Perfect Graph.
AMS Subject Classification: 05C25, 05C12, 05C17.

1 Introduction

Given an integer $n > 1$, the unitary Cayley graph, denoted by $X_n = Cay(Z_n, Z_n^*)$, is defined as the graph whose vertex set is Z_n, the set of integers modulo n, and two distinct vertices x, y are adjacent, if $x - y$ is a unit in the ring Z_n, that is, $x - y \in Z_n^*$.

*Corresponding author

This graph was extensively studied in the literature, see [1] and [5]. The elegant relationship between the graph theoretic properties and Number theory was studied by Klotz and Sander in [3] and Akhtar et al in [2]. They generalized this graph to study the properties of the Cayley graph, $Cay(R, U(R))$, for any commutative ring R with unity 1 and its group of units, U(R).

Recall that, for any commutative ring R with unity, an important ring extension of R, called the ring of dual numbers over R, denoted by $R[\alpha]$, is defined as the quotient ring of the polynomial ring $R[X]/(X^2)$. Equivalently, this can be represented as $a + b\alpha$ where $a, b \in R$ and $\alpha^2 = 0$ with addition and multiplication defined as:

$$(a + b\alpha) + (c + d\alpha) = (a + c) + (b + d)\alpha,$$
$$(a + b\alpha)(c + d\alpha) = (ac) + (ad + bc)\alpha.$$

Clearly this is a commutative ring extension of R with unity 1. This ring was extensively studied in the literature, see [9] and [10].

Motivated by Klotz's work, we want to make a good attempt to study the graph $X_n(\alpha)$, with vertices as the elements of the ring $Z_n(\alpha)$, and two distinct vertices are adjacent, if their difference is a unit in the ring $Z_n(\alpha)$. The precise relationship between this graph theoretic properties and Number theory will be emphasized.

The relationship between the graph $X_n(\alpha)$ and X_n will be studied. Also, it should be noticed that X_n is an induced subgraph of the graph $X_n(\alpha)$.

A graph is said to be representable modulo n, if it is isomorphic to an induced subgraph of $Cay(Z_n, Z_n^*)$. In this paper we study the other direction, that is, we construct an extension of X_n.

One can easily show that the units of the ring $Z_n(\alpha)$ is the set

$$U_n = \{a + b\alpha : gcd(a, n) = 1\}.$$

$X_n(\alpha)$ has a vertex set $V(X_n(\alpha)) = Z_n(\alpha)$, and an edge set

$$E(X_n(\alpha)) = \{\{a + b\alpha, c + d\alpha\} : gcd(c - d, n) = 1\}.$$

It is obvious that X_n is an induced subgraph of $X_n(\alpha)$. We will show that $X_n(\alpha)$ is not only an extension of X_n, but it also has many properties of X_n.

The graph $X_n(\alpha)$ is regular of degree $| U_n |= n\varphi(n)$, where $\varphi(n)$ denotes the Euler ϕ-function. Walter [3] has shown that if $n = p$, p is a prime number, then, $X_n = K_p$, the complete graph on n vertex. However, $X_n(\alpha)$ can not be the complete graph even if $n = p$.

In section two, we investigate some basic graph theoretic invariants of $X_n(\alpha)$. We show that Chromatic and Clique numbers of $X_n(\alpha)$ are equal. Moreover, we characterize the Chromatic and Clique numbers for complementary graphs.

We conclude section 2 with an interesting result, that is, the graph $X_n(\alpha)$ is never self complementary. In section 3, we discuss the perfectness of $X_n(\alpha)$.

2 Basic Invariant

In this section, we start with the Chromatic, $\chi(X_n(\alpha))$, and Clique, $\omega(X_n(\alpha))$, numbers. For the remaining of this paper, n will be an integer greater than 2, and $[i]_p = \{a + k\alpha : a \equiv i \bmod p\}$.

Theorem 1 *If p is the smallest prime divisor of n then,*

a. $\chi(X_n(\alpha)) = \omega(X_n(\alpha))$, *and*

b. $\chi(\overline{X}_n(\alpha)) = \omega(\overline{X}_n(\alpha))$, *where $\overline{X}_n(\alpha)$ is the complementary graph of the graph $X_n(\alpha)$.*

Proof. Consider the vertices $\{0, 1, 2, ..., p-1\}$.
If $i, j \epsilon \{0, 1, 2, ..., p-1\}$, then, since p is the smallest prime divisor of n, we have, $gcd(i-j, n) = 1$, which means that i and j are adjacent. Hence, the vertices $\{0, 1, 2, ..., p-1\}$ induce a clique in $X_n(\alpha)$. Therefore,

$$\chi(X_n(\alpha)) \geq \omega(X_n(\alpha)) \geq p.$$

On the other hand, consider the sets $[0]_p, [1]_p, [2]_p, ..., [p-1]_p$. These sets form a partition of $Z_n(\alpha)$. Moreover, there are no two adjacent vertices in the same set. Therefore, these sets constitute a proper colouring. Thus,

$$\chi(X_n(\alpha)) \leq p, \text{ and hence}$$
$$\chi(X_n(\alpha)) = \omega(X_n(\alpha)) = p.$$

For the complementary graph $\overline{X}_n(\alpha)$, each set of $[0]_p, [1]_p, [2]_p, ..., [p-1]_p$ induces a clique. Since the order of $X_n(\alpha)$ is n^2, and the sets $[0]_p, [1]_p, [2]_p, ..., [p-1]_p$ form a partition of $X_n(\alpha)$, then, the number of vertices in each set is $\frac{n^2}{p}$. Therefore,

$$\chi(\overline{X}_n(\alpha)) \geq \omega(\overline{X}_n(\alpha)) \geq \tfrac{n^2}{p}.$$

On the other hand, for $0 \leq k \leq \frac{n}{p} - 1$, consider the sets

$$M_i = \{kp + j + i\alpha : 0 \leq j \leq p - 1\}.$$

Since $kp+j+i\alpha-(kp+m+i\alpha) = j-m < p$, we have, $gcd(j-m, n) = 1$. Therefore, the vertices of M_i are independent in $\overline{X}_n(\alpha)$. These vertices constitute a proper colouring of $\overline{X}_n(\alpha)$. Hence, $\chi(\overline{X}_n(\alpha)) \leq \frac{n^2}{p}$. This completes the proof.

□

Corollary 1 *The graph $X_n(\alpha)$ is bipartite if and only if n is even.*

Corollary 2 *There is no self complementary unitary Cayley graph over $Z_n[\alpha]$.*

Proof. Suppose that $X_n(\alpha)$ is self complementary graph. Then,

$$\chi(X_n(\alpha)) = \chi(\overline{X}_n(\alpha)),$$

and hence $n = p$. Since $\chi(X_n(\alpha)) \cup \chi(\overline{X}_n(\alpha))$ is the complete graph, we have

$$2p\varphi(p) = p^2 - 1.$$

This is a contradiction.

□

Since X_n is an induced subgraph of $X_n(\alpha)$, $X_n(\alpha)$ can not be a complete multipartite graph if $n \neq p^k$, see [3]. For the converse, consider the following theorem:

Theorem 2 *If $n = p^k$, then, $X_n(\alpha)$ is a complete multipartite graph with partite sets $\{[i]\}_{i=0}^{p-1}$.*

Proof. Assume that $n = p^k$, for $x + y\alpha\epsilon[i]$ and $a + b\alpha\epsilon[j]$, with $i \neq j$, then,

$$x - a \equiv i - j \bmod p, \text{ and } gcd(x - a, p^k) = 1,$$

So, $x + y\alpha$ and $a + b\alpha$ are adjacent, and therefore, $X_n(\alpha)$ is a complete multipartite graph.

□

If $X_n(\alpha)$ is a complete multipartite graph, then, for any two vertices $x + y\alpha$ and $a + b\alpha$, there are two cases:

Case 1 *If $x + y\alpha$ and $a + b\alpha$ are in the same partite set, then, the number of common neighbors is $\frac{n^2}{p}(p - 1)$.*

Case 2 *If $x + y\alpha$ and $a + b\alpha$ are in different partite sets, then, the number of common neighbors is $\frac{n^2}{p}(p - 2)$.*

The following theorem gives a general result for common neighbors of two elements:

Theorem 3 *Suppose that $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, for $a + b\alpha$ and $c + d\alpha$, the number of common neighbors is given by:*

$$n^2 \prod_{a \equiv c \bmod p_i} (1 - \tfrac{1}{p_i}) \prod_{a \not\equiv c \bmod p_i} (1 - \tfrac{2}{p_i}).$$

Proof. Since $Z_n(\alpha)$ is a finite ring, we have

$$Z_n(\alpha) = \prod_{i=1}^{k} Z_{p_i^{2\alpha_i}}.$$

For the vertices $a+b\alpha$ and $c+d\alpha$ in $X_n(\alpha)$, assume that $a = (a_1, a_2, a_3, ..., a_k)$ and $c = (c_1, c_2, c_3, ..., c_k)$. For $x = (x_1, x_2, x_3, ..., x_k)$ common neighbor of $a + b\alpha$ and $c + d\alpha$, we have two choices:

1. if $a \equiv c \bmod p_i$, then, $a_i \equiv c_i \bmod p_i$. Therefore, there are $\frac{p_i^{2\alpha_i}}{p_i}(p_i - 1)$ choices for x_i.

2. if $a \not\equiv c \bmod p_i$, then, $a_i \not\equiv c_i \bmod p_i$. Therefore, there are $\frac{p_i^{2\alpha_i}}{p_i}(p_i - 2)$ choices for x_i. Hence, there are

$$\prod_{i=1}^{k} p_i^{2\alpha_i} \prod_{a \equiv c \bmod p_i} (1 - \tfrac{1}{p_i}) \prod_{a \not\equiv c \bmod p_i} (1 - \tfrac{2}{p_i})$$

common neighbors for $a + b\alpha$ and $c + d\alpha$.

□

The distance between two vertices u, v in a graph is the length of the shortest path between the vertices u and v. This distance is denoted by $d(u, v)$. The diameter of the graph G, denoted by $diam(G)$, is the maximum distance between any two vertices in the graph G. Since $X_n(\alpha)$ can not be a complete graph, then, $diam(X_n(\alpha)) > 1$. The following theorem gives the exact diameter of $X_n(\alpha)$:

Theorem 4

$$diam(X_n(\alpha)) = \begin{cases} 2 & \text{if } n = 2^k \text{ or } n \text{ is odd,} \\ 3 & \text{otherwise.} \end{cases}$$

Proof.

1. If $n = 2^k$, then, $X_n(\alpha)$ is a complete bipartite graph and $diam(X_n(\alpha)) = 2$. If n is odd, then, the number of neighbors can not be zero. Therefore, $diam(X_n(\alpha)) \leq 2$. On the other hand, $X_n(\alpha)$ can not be a complete graph, so, $diam(X_n(\alpha)) > 1$, and therefore, $diam(X_n(\alpha)) = 2$.

2. Suppose that n is not an odd number and n has an odd divisor p, then, the number of neighbors is given by:

$$\prod_{i=1}^{k} p_i^{2\alpha_i} \prod_{a \equiv c \bmod p_i} (1 - \tfrac{1}{p_i}) \prod_{a \not\equiv c \bmod p_i} (1 - \tfrac{2}{p_i}).$$

   Since $p \not\equiv 0 \bmod 2$, then, the number of neighbors is 0. Therefore, $diam(X_n(\alpha)) \geq 3$.

On the other hand, assume that $a + b\alpha$ and $c + d\alpha$ are two arbitrary non adjacent vertices of $X_n(\alpha)$ with no common neighbor, then, $a \not\equiv c \bmod 2$. Assume that a is even, then, since n is even, all neighbors of a are odd. Let x be any one of them, since x and c are odd, then, there is a common neighbor to them, say y, then, $d(a + b\alpha, c + d\alpha) \leq 3$. This completes the proof.

□

3 Perfectness

According to Walter Klotz [3], for n even or odd, X_n is perfect if it has at most two different prime divisors. Since X_n is an induced subgraph of $X_n(\alpha)$, then, $X_n(\alpha)$ can not be perfect whenever n is odd and has more than two prime divisors. It remains to study the case when n is even or n is odd with at most two prime divisors. If n is even, then $X_n(\alpha)$ is perfect since $X_n(\alpha)$ is a bipartite as mentioned earlier. To complete our discussion, it remains to study the case when $n = p_1^{k_1} p_2^{k_2}$, $p_2 > p_1 \geq 3$.

Theorem 5 *If n is odd and has exactly two odd prime divisors, then $X_n(\alpha)$ has no odd hole, C_{2k+1}, $k \geq 2$.*

Proof. Suppose that $X_n(\alpha)$ has an odd hole, C_{2k+1}, say,

$$\{a_i + b_i\alpha\}_{i=0}^{i=2k}.$$

Claim: $a_i \neq a_j$, for all $i \neq j$.

Indeed, assume that there is i and j with $i \neq j$ and $a_i = a_j$, then, $a_{i+1} + b_{i+1}\alpha$ is adjacent vertex to both $a_i + b_i\alpha$ and $a_j + b_j\alpha$. Similarly for $a_{i-1} + b_{i-1}\alpha$. So, $C_{2k+1} = C_4$, which is a contradiction. □

Now assume that $X_n(\alpha)$ contains an induced odd hole $C_{2k+1} = \{a_i + b_i\alpha\}_{i=0}^{i=2k}$, since $a_i \neq a_j$ for all $i \neq j$, then, $\{a_i\}_{i=0}^{i=2k}$ is an induced odd hole in X_n. This contradicts the perfectness of X_n.

Theorem 6 *If n is odd and has exactly two odd prime divisors, then $\overline{X}_n(\alpha)$ has no odd hole, C_{2k+1}, $k \geq 2$.*

Proof. Assume that $\overline{X}_n(\alpha)$ contains the induced odd hole $C_{2k+1} = \{a_i + b_i\alpha\}_{i=0}^{i=2k}$, then, the prime divisors of n, say p_1 and p_2, will alternate in dividing $a_i - a_{i+1}$. Since it is impossible for two primes to alternate in dividing odd numbers. Therefore, $\overline{X}_n(\alpha)$ has no odd hole, C_{2k+1}, $k \geq 2$. □

Now we are ready to give the following final theorem:

Theorem 7 *The unitary Cayley graph $X_n(\alpha)$ is perfect if and only if n is even or n is odd of at most two different prime divisors.*

4 Acknowledgment

The authors would like to thank the referee for the valuable and constructive comments which helped to improve the quality of the manuscript.

References

[1] Omar A. AbuGhneim, Emad E. AbdAlJawad and Hasan Al-Ezeh, *The clique number of $(Z_{p^n}(\alpha))$*. Rocky Mountain J. Math. 42 no. 1 (2012), 1-13.

[2] Reza Akhtar, Megan Boggess, Isidora Jimenez, Rachel Karpman, Amanda Kinzel and Dan Pritikin, *On the unitary Cayley graph of a finite ring*, The Electronic Journal of Combinatorics, Vol. 16 (2009), R117.

[3] Walter Klotz and Torsten Sander, *Some properties of unitary Cayley graphs*, The Electronic Journal of Combinatorics, Vol. 14 (2007), R45.

[4] Yan-Quan Feng. *Automorphism groups of cayley graphs on symmetric groups with generating transposition sets.* J. Comb. Theory, Vol. 96: 6772, January, 2006.

[5] Emad. E. AbdAlJawad and Hasan Al-Ezeh, *Some properties of the zero-divisor graph of the ring of dual numbers of a commuatitive ring.* Doctoral Dissertation, University of Jordan, Jordan (2007).

[6] Gene Cooperman and Larry Finkelstein. *New methods for using cayley graphs in interconnection networks* Discrete Applied Mathematics, Vol. 37-38:95118, 1992.

[7] Gene Cooperman et al. *Applications of cayley graphs.* Applied Algebra, Vol. 508: 367378, 1990.

[8] Cai Heng Li and Zai Ping Lu. *Tetravalent edge-transitive cayley graphs with odd number of vertices.* J. Comb. Theory, Vol. 96:164181, January, 2006.

[9] Gu, Y.L. and Luh, J.Y.S., *Dual number transformations and its applications to robotics*, IEEE Journal and Robotics and Automation, vol. RA-3, December 1987.

[10] E. Pennestri and R. Stefanelli *Linear Algebra and Numerical Algorithms Using Dual Numbers.* Multibody System Dynamics Vol. 18(3):32349, 2007.