FRIENDLY INDICES AND FULLY CORDIAL GRAPHS

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ABSTRACT. For a graph $G = (V, E)$ with a coloring $f : V(G) \to \mathbb{Z}_2$, let $v_f(i) = |f^{-1}(i)|$. We say f is friendly if $|v_f(1) - v_f(0)| \leq 1$. The coloring f induces an edge labeling $f_+ : E \to \mathbb{Z}_2$ defined by $f_+(uv) = f(u) + f(v)$ mod 2, for each $uv \in E$. Let $e_f = |f_+^{-1}(i)|$. The friendly index set of the graph G, denoted by $FI(G)$, is defined by $\{|e_f(1) - e_f(0)| : f$ is a friendly coloring of $G\}$. We say G is fully cordial if $FI(G) = \{|E|, |E| - 2, |E| - 4, ..., |E| - 2[\frac{|E|}{2}]\}$. In this paper, we develop a new technique to calculate friendly index sets without labeling vertices, and we develop a technique to create fully cordial graphs from smaller fully cordial graphs. In particular, we show the first examples of fully cordial graphs that are not trees, as well as show new infinite classes of fully cordial graphs.

INTRODUCTION

Throughout this paper, all graphs are assumed to be finite, simple and connected. An outstanding conjecture in graph theory is the Ringel-Kotzig conjecture, named after Gerhard Ringel and Anton Kotzig, which states that all trees are graceful graphs (see [Gol72], [Ros67] for details on graceful graphs and graceful labelings). Cahit introduced the concept of cordial labelings as a weakened version of graceful labelings [Cah87]. There are several papers on cordial labelings [CE00], [You09]. Hovey generalized this concept to A-cordial labeling, where A is an abelian group [Hov91]. In this paper, we focus on the group $A = \mathbb{Z}_2$, which leads us to the concepts of friendly index numbers and fully cordial graphs.

For a graph $G = (V, E)$ with vertex set $V = V(G)$, edge set $E = E(G)$, and a binary coloring (vertex labeling)

$$f : V \to \mathbb{Z}_2,$$

we let $v_f(i)$ be the number of vertices labeled i under the map f. This is equivalent to saying that

$$v_f(i) = |f^{-1}(i)|.$$

Definition 0.1. We say that f is *friendly* if and only if $|v_f(1) - v_f(0)| \leq 1$.

The coloring f induces an edge labeling $f_+ : E \to \mathbb{Z}_2$ defined by $f_+(xy) = f(x) + f(y) \bmod 2$, for all $xy \in E$. Let $e_f(i)$ be the number of edges labeled i under the map f_+.

Definition 0.2. We say $N(f) = |e_f(1) - e_f(0)|$ is the friendly index number of f. We define the friendly index set of G, denoted $FI(G)$, by the set $\{N(f) : f \text{ is a friendly labeling of } G\}$.

See figure 1 below for an example of a friendly vertex label. There have been many papers on friendly indices and friendly labels [LN08], [Sal10], [SL06]. It has been shown that

$$FI(G) \subset \{e, e-2, e-4, ..., e-2[\frac{e}{2}]\}.$$

If equality holds then G is said to be fully cordial. A similar definition holds for fully product-cordial sets [Sal10]. In section 1, we show several characteristics of fully cordial graph G. In section 2, we develop a new tool for calculating friendly index numbers using degrees and adjacency of the vertices. In section 3, we develop a method for combining two fully cordial graphs into larger fully cordial graphs. Finally, in section 4, we find new infinite classes of fully cordial graphs.

FIGURE 1. A graph with a friendly vertex label.

1. Fully Cordial Graphs

Throughout this section, we use the notation $n = |V(G)|$ and $e = |E(G)|$. We also use the notation $G = (n, e)$ to denote a graph with n vertices and e edges. For any $G = (n, e)$, let us consider a friendly vertex label $f : V \to \mathbb{Z}_2$. Since G is connected, there must be at least one edge connecting a vertex labeled 1 and a vertex labeled 0. Hence, under any friendly vertex label f, we must have $e_f(1) \geq 1$. Therefore, the only way a graph can have $N(f) = e$ is if all of the edges are labeled 1. Recall that if G is fully cordial, then $FI(G) = \{e, e-2, e-4, ..., e-2[\frac{e}{2}]\}$. Hence, if G is fully cordial then it must have e as a friendly index number.

Lemma 1.1. *A graph G has e as a friendly index number if and only if G is isomorphic to a spanning subgraph of $K_{m,m}$ or $K_{m,m+1}$.*

Proof. Suppose e is a friendly index number of G. Then for some vertex label f, we must have $e_f(0) = 0$ and $e_f(1) = e$, since $e_f(1) \geq 1$. In this labeling, since $e_f(0) = 0$, none of the vertices labeled 0 connects with another vertex labeled 0, and the same thing holds for vertices labeled 1. Thus, G must be a bipartite graph for e to be a friendly index number. Specifically, if $n = 2m$ then G is isomorphic to a spanning subgraph of $K_{m,m}$ to guarantee f is friendly, and if $n = 2m + 1$ then G is isomorphic to a spanning subgraph of $K_{m,m+1}$ to guarantee f is friendly.

Now suppose G is isomorphic to a spanning subgraph of $K_{m,m}$. Then there exists two separate groups of m vertices, say V_0 and V_1, such that none of the vertices in V_i connect to each other, for each i. We let g be the friendly label defined as follows:

$$g(v) = \begin{cases} 0, & v \in V_0 \\ 1, & v \in V_1. \end{cases}$$

Then $|v_g(1) - v_g(0)| = 0 \leq 1$ and $e_g(1) = e, e_g(0) = 0$. Thus, g is friendly and $N(g) = e$, as desired. The case when G is isomorphic to a spanning subgraph of $K_{m,m+1}$ is proved in a similar manner. □

There are examples of complete bipartite graphs with e edges that do not contain e as a friendly index number, such as $K_{2,5}$. If G is a graph that is not isomorphic to a spanning subgraph of a complete bipartite graph of the form $K_{m,m}$ or $K_{m,m+1}$ then we know that G is not fully cordial. Note that since a fully cordial graph must be isomorphic to a subgraph of one of these complete bipartite graphs by lemma 1.1, it is easy to see that $e \leq [\frac{n^2}{4}]$. Many examples of trees being fully cordial have been shown, and it is known that complete graphs have only one friendly index number [LN08], [SL06]. This may suggest that the more edges a graph has, the less friendly index numbers it should have. As far as I know, there has not been a paper published on the existence of fully cordial graphs that are not trees. We show later that there are infinitely many graphs that are not trees which are fully cordial. In particular, we prove the following theorem in section 4:

Theorem 4.1. *For any k, there exists an n and a graph $G = (n, n + k)$ such that G is fully cordial.*

This result shows that there cannot be an upper bound on e in the form of $n + k$, where k is fixed. However, we can improve upon the upper bound on the number of edges of a fully cordial graph. First, we need to establish the following lemma.

Lemma 1.2. *Let G be the graph obtained by removing exactly x edges from $K_{m,m}$. Then $FI(G) \subset (\{m^2-x\} \cup \{(m-2i)^2 \pm j : 1 \leq i \leq \frac{m}{2}, 0 \leq j \leq x\})$. Similarly, if G is the graph obtained from removing exactly x edges from $K_{m,m+1}$ then $FI(G) \subset (\{m^2 \pm m - x\} \cup \{(m-2i)^2 \pm (m-2i) \pm j : 1 \leq i \leq \frac{m}{2}, 0 \leq j \leq x\})$.*

Proof. First, let G be the graph obtained from removing exactly x edges from $K_{m,m}$. It has been shown that $FI(K_{m,m}) = \{m^2, (m-2)^2, (m-4)^2, ..., (m-2[\frac{m}{2}])^2\}$, but more specifically, for each $(m-2i)^2 \in FI(K_{m,m})$, we have that $e_0 = 2i(m-i)$ and $e_1 = (m-2i)^2 + 2i(m-i)$ [LN08]. When x edges are removed, it is possible that x_0 of those edges were labeled 0 (pending on the vertex label) and x_1 labeled 1 where $0 \leq x_0 \leq x$, with $x_0 + x_1 = x$. We cannot guarantee that every combination can be achieved. However, we know that at best, the friendly index number $(m-2i)^2 \in FI(K_{m,m})$ becomes at least one of the following friendly index number of $FI(G)$: $(m-2i)^2 - x, (m-2i)^2 - (x-2), (m-2i)^2 - (x-4), ..., (m-2i)^2 + x$, except that when $i = 0$ only $m^2 - x$ is possible. Thus, the result follows in this case.

Now, we let G be the graph obtained from removing exactly x edges from $K_{m,m+1}$. It has been shown that $FI(K_{m,m+1}) = \{m^2 \pm m, (m-2)^2 \pm (m-2), (m-4)^2 \pm (m-4), ..., (m-2[\frac{m}{2}])^2 \pm (m-2[\frac{m}{2}])\}$, but more specifically, for each $(m-2i)^2 \pm (m-2i) \in FI(K_{m,m+1})$, we have that $e_0 = (m-i)(2i+1)$ and $e_1 = (m-i)^2 + i(i+1)$, or $e_0 = i(2m-2i+1)$ and $e_1 = i^2 + (m-i)(m+1-i)$ [LN08]. When x edges are removed, it is possible that x_0 of those edges were labeled 0 (pending on the vertex label) and x_1 labeled 1 where $0 \leq x_0 \leq x$, with $x_0 + x_1 = x$. We cannot guarantee that every combination can be achieved. However, we know that at best, the friendly index number $(m-2i)^2 \pm (m-2i) \in FI(K_{m,m+1})$ becomes at least one of the following friendly index number of $FI(G)$: $(m-2i)^2 \pm (m-2i) - x, (m-2i)^2 \pm (m-2i) - (x-2), (m-2i)^2 \pm (m-2i) - (x-4), ..., (m-2i)^2 \pm (m-2i) + x$, except that when $i = 0$ only $m^2 \pm m - x$ is possible. Thus, the result follows in this case. □

Corollary 1.3. *For any fully cordial graph G, if $n = 2m$ then $e \leq (m-1)^2 + 2$. If $n = 2m+1$ then $e \leq m^2 + 1$.*

Proof. First, let $n = 2m$. Then G must be obtained by removing x edges from $K_{m,m}$. Hence, by lemma 1.2, $e = m^2 - x$. When the expression $(m-2)^2 + x$ is less than $m^2 - x$, then it is the second largest possible friendly index number. If the expression $(m-2)^2 + x$ is greater than or equal to $e = m^2 - x$, then the second friendly index number (which is $e-2$) must be less than $(m-2)^2 + x$ since it cannot exceed the total number of edges of the graph. Either case, we must have $e - 2 \leq (m-2)^2 + x$.

Solving $m^2 - x - 2 \leq (m-2)^2 + x$ for x gives $x \geq 2m - 3$. Hence, $e, e-2 \in FI(G) \Rightarrow x \geq 2m-3$. Thus, $e \leq (m-1)^2 + 2$.

Next, let $n = 2m + 1$. Then G must be obtained by removing x edges from $K_{m,m+1}$. Hence, by lemma 1.2, $e = m^2 + m - x$. Since $m^2 - m + x$ is the second largest possible friendly index number or exceeds the total number of edges of the graph, in order for $e - 2 \in FI(G)$ we must have that $e - 2 \leq m^2 - m + x$. Solving $m^2 + m - x - 2 \leq m^2 - m + x$ for x gives $x \geq m - 1$. Hence, $e, e-2 \in FI(G) \Rightarrow x \geq m - 1$. Thus, $e \leq m^2 + 1$. □

The bounds obtained in corollary 1.3 cannot be improved upon since there exists fully cordial graphs $G = (5,5)$ and $G = (6,6)$, and $5 = 2^2 + 1, 6 = (3-1)^2 + 2$ (these two graphs are shown in figures 4 and 5, toward the end of section 2).

2. Degrees and Adjacency

Next, we establish a tool to further investigate fully cordial graphs. Pick any graph G such that $e \in FI(G)$ and let f be the vertex label such that $N(f) = e$. Before continuing, we need to show that there is only two such vertex labelings f where $N(f) = e$.

Lemma 2.1. *Pick any graph G such that $e \in FI(G)$. Then there exists only one vertex label f (and its inverse label $g = 1 - f$) such that $N(f) = N(g) = e$.*

Proof. Suppose by way of contradiction that there exists a third vertex label h such that $N(h) = e$. Under f, we divide $V(G)$ into two distinct groups A and B, where $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$ (hence, G is isomorphic to subgraph of $K_{|A|,|B|}$). Since we must have $e_f(0) = 0$, no two vertices are adjacent in A and no two vertices are adjacent in B. Under h, we divide A into two distinct groups A_0 and A_1 and we divide B into two distinct groups B_0 and B_1, such that $h(x) = 0$ for all $x \in A_0 \cup B_0$ and $h(x) = 1$ for all $x \in A_1 \cup B_1$. Since $e_h(0)$ must be 0, no two vertices are adjacent in A_0 and B_0 and no two vertices are adjacent in A_1 and B_1. However, this makes $A_0 \cup B_1$ disjoint from $A_1 \cup B_0$, which makes G disconnected, a contradiction. Thus, no such h exists. □

Since f and $g = 1 - f$ basically behave the same (by which we mean it results in the same isomorphic subgraph of $K_{m,m}$ or $K_{m,m+1}$), we can let f be *the* vertex label such that $N(f) = e$. For a new vertex label h, we label half the vertices $u_1, u_2, ..., u_m$ and the other half $v_1, v_2, ..., v_m$ (if n is odd then we also add either u_{m+1} or v_{m+1}) such that $f(u_i) = 0$ and $f(v_i) = 1$, for all i, and $h(x) = f(x)$ for all $x \in V(G)$ except that $h(u_1) = 1$ and $h(v_1) = 0$. If u_1 and v_1 are not adjacent then $N(h) = |e - 2(\deg(u_1)+\deg(v_1))|$ since this results in $e_g(1) = e-(\deg(u_1)+\deg(v_1))$ and $e_g(0) = \deg(u_1)+\deg(v_1)$. Similarly, if u_1 and v_1 are adjacent then

$N(h) = |e - 2(\deg(u_1)+\deg(v_1) - 2)|$. Another way to say this is $N(h) = |e - 2(\deg(u_1)+\deg(v_1) - 2A(u_1, v_1))|$, where

$$A(u,v) = \begin{cases} 0, & u \text{ not adjacent to } v, \\ 1, & u \text{ adjacent to } v. \end{cases}$$

More generally, if h is a vertex label such that $h(x) = f(x)$ for all $x \in V(G)$ except that there exists $u_1, ..., u_r$ and $v_1, ..., v_s$ such that $h(u_i) = 1, h(v_j) = 0, f(u_i) = 0, f(v_j) = 1$ for $1 \le i \le r, 1 \le j \le s$ then $N(h) = |e - 2K(r,s)|$, where

$$K(r,s) = \sum_{i=1}^{r} \deg(u_i) + \sum_{j=1}^{s} \deg(v_s) - 2 \sum_{1 \le i,j \le s} A(u_i, v_j).$$

Thus, $e - 2k \in FI(G)$ if and only if $K(r,s) = k$ or $K(r,s) = e - k$ for some $0 \le r, s \le m+1$ with corresponding vertices $u_1, ..., u_r, v_1, ..., v_s$.

Remark 1. Note that for each fixed value of r and $s, K(r,s)$ is a multivalued function since the different choices of the vertices $u_1, ...u_r, v_1, ..., v_s$ can possibly lead to different values of $K(r,s)$. Also, $K(r,s)$ is only defined for graphs that have e as a friendly index number, since it can only be applied to graphs that are isomorphic to a spanning subgraph of $K_{m,m}$ or $K_{m,m+1}$. Note that $K(r,s)$ is the same as $e_h(0)$.

Remark 2. In order for h to be friendly, $|r - s| \le 1$, and $|r - s| = 1$ only in the case when n is odd. When n is even, we calculate $K(r,r)$ only, and when n is odd, we can calculate $K(r,r)$ or $K(r,r+1)$. Also, if $n = 2m$ and we pick $r > \frac{m}{2}$ vertices originally labeled 0 under f (where $N(f) = e$) and we pick $r > \frac{m}{2}$ vertices originally labeled 1 under f resulting in $K(r,r) = k$, then picking the other $m - r$ vertices labeled 0 under f and the other $m - r$ vertices labeled 1 under f results in $K(m-r, m-r) = k$ (this is essentially the same as how $e_f(0) = e_{1-f}(0)$). A similar result holds for $K(r,s)$ when $r, s > \frac{m}{2}$ in the $n = 2m+1$ case. Hence, to find every friendly index number, we only need to choose $0 \le r, s \le \frac{m}{2}$.

This allows us a way to find the complete friendly index set of a graph without needing to label the vertices. We calculate the friendly index set of a graph by observing the degree and adjacency of each vertex. The following theorem is an application of this.

Theorem 2.2. *Let $G = (n,e)$ with n and e both even. If the degree of any two vertices differs by an even number, then G is not fully cordial. More specifically, if $e \in FI(G)$ then $FI(G) \subset \{e, e-4, e-8, ..., e-4[\frac{e}{4}]\}$.*

Proof. First, if e is not an element of $FI(G)$ then G is clearly not fully cordial. Assume $e \in FI(G)$. Thus, G is isomorphic to a spanning subgraph of $K_{m,m}$. Let us label the vertices $u_1, ..., u_m$ and $v_1, ..., v_m$ such that $f(u_i) =$

$0, f(v_i) = 1$, and $N(f) = e$. Let the maximum degree of the vertices be d. Then for any $0 \le r \le \frac{m}{2}$, since every vertex must have degree d minus an even number, $\sum_{i=0}^{r} \deg(u_i) = dr -$ (even number), and $\sum_{i=0}^{r} \deg(v_i) = dr -$ (even number). Thus, $K(r,r) = 2dr -$ (even number), which can never be odd. Thus, $FI(G) \subset \{e, e-4, e-8, ...\}$. Since e was even, there does not exist a j so that $|e - 4j| = e - 2(2k+1)$ (this is why we need e to be even). □

Recall that hypercubes, denoted Q_n, are defined by $Q_1 = K_2$ and $Q_n = Q_{n-1} \times K_2$, for all $n \ge 2$ (see figure 2 for an example). An immediate consequence of theorem 2.2 is that all hypercubes are not fully cordial, since each Q_n is a graph with $n2^{n-1}$ edges and 2^n vertices, all of which have the same degree.

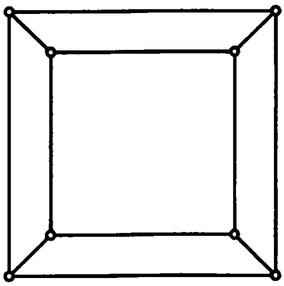


FIGURE 2. The graph of Q_3.

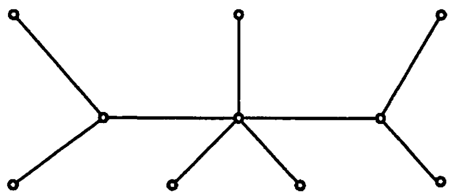
Here is another application, proving a result already known.

Lemma 2.3. *For any* $m, FI(K_{m,m}) = \{m^2, (m-2)^2, ...(m-2[\frac{m}{2}])^2\}$, *and* $FI(K_{m,m+1}) = \{m^2 \pm m, (m-2)^2 \pm (m-2), ..., (m-2[\frac{m}{2}])^2 \pm (m-2[\frac{m}{2}])\}$.

Proof. First, we consider $K_{m,m}$. Pick any $0 \le r \le \frac{m}{2}$. Notice that the degree of each vertex is m. For any group of $2r$ vertices chosen to calculate $K(r,r)$, all r^2 pairs of vertices are adjacent. Hence, regardless of our choice of $r, K(r,r) = 2mr - 2r^2$. Then $e - 2K(r,r) = m^2 - 2(2mr - 2r^2) = (m-2r)^2$. Thus, $FI(K_{m,m}) = \{(m-2r)^2 : 0 \le r \le \frac{m}{2}\}$, as desired.

Next, we consider $K_{m,m+1}$. Let A be the group of m vertices not adjacent to each other and let B be the group of $m+1$ vertices not adjacent to each other. Notice that the degree of each vertex in A is $m+1$ and the degree of each vertex in B is m. First, pick any $0 \le r \le \frac{m}{2}$. Then all r^2 pairs of vertices are adjacent and $K(r,r) = mr + (m+1)r - 2r^2$. Next, pick any $0 \le r \le \frac{m}{2} - 1$. Then all $r(r+1)$ pairs of vertices are adjacent and $K(r,r+1) = (m+1)r + m(r+1) - 2r(r+1)$. Hence, since $e = m^2 + m$, we get $e - 2K(r,r) = (m-2r)^2 + (m-2r)$ and $e - 2K(r,r+1) = (m-2r)^2 - (m-2r)$, for each $0 \leq r \leq \frac{m}{2}$. Hence, $FI(K_{m,m+1}) = \{m^2 \pm m, (m-2)^2 \pm (m-2), ..., (m-2[\frac{m}{2}])^2 \pm (m-2[\frac{m}{2}])\}$, as desired. □

The next application involves caterpillar graphs. Recall that a caterpillar graph of diameter $d+1$ can be represented as $C(a_1, a_2, ..., a_d)$ where a_i is the degree of the i^{th} inner vertex along the "spine" (see figure 3 for an example). Caterpillars of diameter 3 and 4 have been fully classified by Daniel Corral and Ebrahim Salehi [CS]. Next, we reproduce the classification of caterpillars of diameter 3 using the degree and adjacency trick. Note that $C(a, b)$ has $a+b$ vertices and $a+b-1$ edges.



C(3, 5, 3)

FIGURE 3. The graph of $C(3,5,3)$.

Lemma 2.4. *Let $C(a, b)$ be a caterpillar of diameter 3 with $a \leq b$. Then $C(a, b)$ is fully cordial if and only if $|a-b| \leq 1$ with $a, b \geq 2$.*

Proof. Notice that $e = a+b-1$ is a friendly index number if and only if $|b-a| \leq 1$. Also, if the degree of a or b is 1, then the diameter is at most 2, which is why we need a and b to have degree of 2 or more. Let u and v be the vertices of degree a and b, respectively. First, let $a+b$ be even (hence, $a = b$). We know e is a friendly index number if and only if there exists a label f such that without loss of generality, $f(x) = 0$ for u and the pendant vertices adjacent to v, and $f(x) = 1$ for v and the pendant vertices adjacent to u. For a new label h, without loss of generality, we assume that u_i vertices are chosen from u and the $b-1$ degree 1 vertices adjacent to v, and the v_i vertices are chosen from the rest. Pick any $0 \leq r \leq [\frac{a+b}{4}]$. If u and v are not chosen then since none of the pendant vertices are adjacent, $K(r, r) = 2r$ for $0 \leq r \leq [\frac{a+b}{4}]$. Hence,

$$\{a+b-1, a+b-5, a+b-9, ..., a+b-1-4[\frac{a+b}{4}]\} \subset FI(C(a, b)).$$

If we choose $u_1 = u$ and all the rest of the u_i's and v_i's are pendant, then since every pendant vertex chosen for the v_i's are adjacent to u, we get $K(r, r) = a + 2r - 1 - 2r = a - 1$, for each $1 \leq r \leq [\frac{a+b+c-1}{4}]$. Similarly,

$K(r,r) = b+2r-1-2r = b-1$, if $v_1 = v$ and the rest are pendant vertices. Hence, $e - 2K(r,r) = |b-a+1|, |a-b+1| = 1 \in FI(C(a,b))$. Also, when $u_1 = u, v_1 = v$, and the rest are pendant, $K(r,r) = a+b+2r-2-2(2r-1) = a+b-2r$, for each $1 \leq r \leq [\frac{a+b}{4}]$, since we have $2r-1$ pair of adjacent vertices. Hence, $|e - 2K(r,r)| = a+b+1-4r$, and

$$\{a+b-3, a+b-7, ..., a+b+1-4[\frac{a+b}{4}]\} \subset FI(C(a,b)).$$

Thus, $C(a,b)$ is fully cordial. The $a+b$ is odd case is proved in a similar manner. □

The next theorem shows the existence of fully cordial graphs that are not trees.

Theorem 2.5. *Let G be a graph isomorphic to a spanning subgraph of $K_{m,m+1}$ with A representing the m vertices not connected to each other and B representing the $m+1$ vertices not connected to each other. For $m = 2$, if the degrees of the vertices of B are 1, 2, 2 then G is fully cordial. For $3 \leq m$, if the degrees of the vertices in B are 1, 1, 2, ..., m, the degrees of the vertices in A are $m+1, m-1, m-2, ..., 1$, and the vertex of degree j from A is only adjacent to the vertices of degree m to $m-j+1$ from B, then G is fully cordial.*

Proof. Let G be a graph satisfying these conditions. Then we get $K(0,1) = j$, for each $1 \leq j \leq m$, by choosing v_1 to be the vertex with degree j. For $m = 2$, this gives $K(0,1) = 1, 2$. Since $K(0,0) = 0$, $FI(G) = \{5,3,1\}$, and G is fully cordial.

Now let $m \geq 3$. We know that $e - 2j \in FI(G)$ for each $1 \leq j \leq m$. Since $G = (2m+1, \frac{m^2+m+2}{2})$, we also need to prove $e - 2j \in FI(G)$ for $m+1 \leq j \leq [\frac{m^2+m+2}{4}]$. Pick any $1 \leq r \leq [\frac{m}{2}]$. Then choose $u_1, ..., u_r$ from A as the vertices of degree 1, ..., r, and choose $v_1, ..., v_r$ from B as the vertices of degree $m, m-1, ..., m-r+1$, and choose v_{r+1} as any vertex of degree j for $1 \leq j \leq m-r$. Then there are $\frac{r(r+1)}{2}$ pairs of adjacent vertices and

$$K(r, r+1) = (\frac{r(r+1)}{2}) + (\frac{2mr - r^2 + r}{2} + j) - 2(\frac{r(r+1)}{2}) = r(m-r) + j,$$

where $1 \leq r \leq [\frac{m}{2}]$ and $1 \leq j \leq m-r$. Hence, $K(r, r+1)$ achieves every number from 1 to $[\frac{m^2+2m}{4}]$, which guarantees G is fully cordial. □

Figure 4 below shows examples of fully cordial graphs that are not trees, all of which come from theorem 2.5. The next theorem proves that there exists a graph $G = (6,6)$ that is fully cordial.

Theorem 2.6. *Let $G = (6,6)$ be the graph isomorphic to a spanning subgraph of $K_{3,3}$ with A representing the 3 vertices not connected to each other*

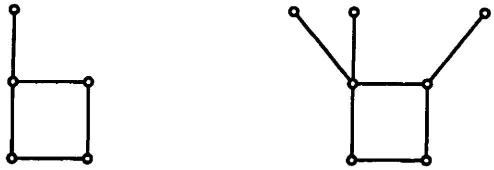


FIGURE 4. Examples of fully cordial graphs that are not trees.

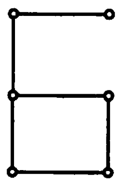
and B representing the remaining 3 vertices not connected to each other. If the degrees of the vertices in A are all 2, and the degree of the vertices in B are 1, 2, 3, then G is fully cordial.

Proof. Clearly, $K(0,0) = 0$. Pick u_1 to be the vertex of degree j from B, and let v_1 be the vertex of degree 2 adjacent to u_1. Then, for each $1 \leq j \leq 3$,

$$K(1,1) = (j) + (2) - 2(1) = j.$$

Thus, $K(1,1) = j$, for each $1 \leq j \leq 3$, which proves that $FI(G) = \{6,4,2,0\}$, as desired. □

Figure 5 gives the graph from theorem 2.6, which we denote by H_6 (H_6 is used again in section 4).



H_6

FIGURE 5. Graph of H_6.

3. Joining Graphs

In this section, we show how to construct larger fully cordial graphs from two smaller fully cordial graphs. Pick any two graphs (not necessarily fully cordial) $G_1 = (n_1, e_1)$ and $G_2 = (n_2, e_2)$. Pick any vertex $v_1 \in V(G_1)$ and any vertex $v_2 \in V(G_2)$. We denote the joining of two graphs by making $v_1 = v_2$ by $G_1 * G_2$ (see figure 6 for an example). We denote the joining of graphs G_1 and G_2 with the edge v_1v_2 by $G_1 ** G_2$ (see figure 7 for an example). Note that for $G = G_1 * G_2, G = (n_1 + n_2 - 1, e_1 + e_2)$, and for $G = G_1 ** G_2, G = (n_1 + n_2, e_1 + e_2 + 1)$.

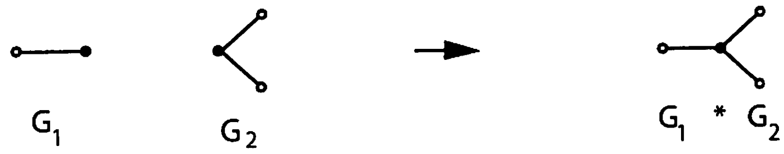


FIGURE 6. The joining operation *.

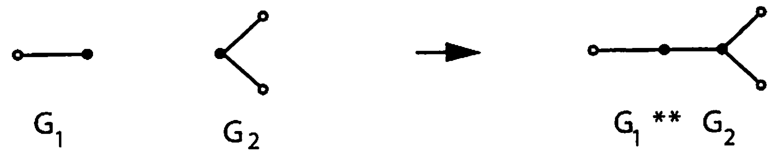


FIGURE 7. The joining operation **.

Before continuing, it is important to emphasize that the joining of two fully cordial graphs does not always result in a fully cordial graph. For example, in figure 6, if G_1 is the fully cordial path P_3, G_2 is the fully cordial path P_2, and we choose v_1 to be the inner vertex of G_1 then $G_1 * G_2$ is $K_{1,3}$, which is not fully cordial. Also, if $e - 2k \in FI(G)$ then either $e_f(1) = e - k, e_f(0) = k$, or $e_f(1) = k, e_f(0) = e - k$. If we want to join two graphs, we need to be careful about how the vertex labels achieve the friendly index number $e - 2k$. This leads us to the following definitions.

Definition 3.1. Let $G = (n, e)$ be any graph. Then we say $e - 2k$, with $0 \leq k \leq \frac{e}{2}$, is a *positive friendly index number of* G if and only if there exists a friendly label f such that $e_f(1) = e - k$ and $e_f(0) = k$. We define $FI_+(G) = \{N(f) : N(f) \in FI(G)$ and $N(f)$ is positive $\}$. We say G is *positive* if and only if $FI(G) = FI_+(G)$. For n odd, we say that vertex $v \in V(G)$ is *join-friendly* if and only if for every $e - 2k \in FI_+(G)$, there exists a label f such that $e_f(1) = e - k, e_f(0) = k$, and if $v_f(1) = v_f(0) + 1$ (or $v_f(0) = v_f(1) + 1$) then $f(v) = 1$ (or 0).

Remark 3. Another way to say G is positive is that for every friendly index number there exists a friendly label f such that $e_f(1) \geq e_f(0)$. Notice that not every graph is positive. For example, $C(3,3)$ is not positive because the only way to achieve the friendly index number 3 is with an f such that $e_f(1) = 1$ and $e_f(0) = 4$. Also, if we examine the proof of lemma 2.4 closely, we see that for each $a \geq 2$, $\{2a-1, 2a-5, 2a-9, ..., 2a-1-4[\frac{2a}{4}]\} \subset FI_+(C(a,a))$. We use this fact in section 4.

For an example of a join-friendly vertex, consider the path P_3, where we label the vertices as u, v, and w with v being the inner vertex. Then

u (and similarly w) is a join-friendly vertex because for the friendly index number 2, we use the label f so that $f(u) = f(w) = 1$ and $f(v) = 0$, and for the friendly index number 0, we use the label f so that $f(u) = f(v) = 1$ and $f(w) = 0$. In both cases, $v_f(1) = v_f(0) + 1$ and $f(u) = 1$. v is not join-friendly since the only way to obtain 2 as a friendly index number is with f such that $f(u) = f(w) = 1$ (or 0) and $f(v) = 0$ (or 1), and either way, if $v_f(i) = v_f(1-i) + 1$ then $f(v) = 1 - i$, instead of $f(v) = i$.

Recall that if f is a friendly label on G then so is $g = 1 - f$, and $N(f) = N(g)$. More importantly, if $e_f(1) = e - k$ and $e_f(0) = k$ then $e_g(1) = e - k$ and $e_g(0) = k$. This fact is used in lemma 3.2.

Let us define $A +_1 B = \{a + b : a \in A, b \in B\}$, and define $A +_2 B = (\{a+b+1 : a \in A, b \in B\} \cup \{a+b-1 : a \in A, b \in B\})$, for any sets A and B. For example, if $A = \{1,3,7\}$ and $B = \{2,4\}$ then $A +_1 B = \{3,5,7,9,11\}$ and $A +_2 B = \{2,4,6,8,10,12\}$. We are now ready to prove the following important technical result, which immediately proves the main result in this paper, theorem 3.3.

Lemma 3.2. *Let $G_1 = (n_1, e_1)$ and $G_2 = (n_2, e_2)$ be any two graphs, and let $G = G_1 * G_2$ and $GG = G_1 * *G_2$, where the joins use some $v_1 \in V(G_1)$ and some $v_2 \in V(G_2)$. Then if n_1 and n_2 are both even, $FI_+(G_1) +_1 FI_+(G_2) \subset FI_+(G)$, and if n_1 is even or n_2 is even then $FI_+(G_1) +_2 FI_+(G_2) \subset FI_+(GG)$. If n_1 is odd (or n_2 is odd or both) and v_1 is join-friendly (or v_2 is join-friendly), then $FI_+(G_1) +_1 FI_+(G_2) \subset FI_+(G)$.*

Proof. First, let us consider $G = G_1 * G_2$, where the join occurs at vertices $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, and we call the resulting vertex v_3. Pick any $e_1 - 2k_1 \in FI_+(G_1)$ and any $e_2 - 2k_2 \in FI_+(G_2)$. Then there exists two vertex labels f and g such that $e_f(1) = e_1 - k_1, e_f(0) = k_1$, and $e_g(1) = e_2 - k_2, e_g(0) = k_2$.

Case 1: $f(v_1) = g(v_2)$. Then we define a vertex label h on G as follows:

$$h(v) = \begin{cases} f(v), & \text{if } v \in V(G_1) \setminus \{v_1\} \\ g(v), & \text{if } v \in V(G_2) \setminus \{v_2\} \\ f(v_1)(= g(v_2)), & \text{if } v = v_3(= v_1 = v_2). \end{cases}$$

Then $N(h) = e_1 + e_2 - 2(k_1 + k_2) \in FI_+(G)$, as long as h is friendly, since $e_f(1) \geq e_f(0)$ and $e_g(1) \geq e_g(0)$ implies that $e_h(1) \geq e_h(0)$. If n_1 and n_2 are both even then clearly h is friendly with $v_h(1) - v_h(0) = \pm 1$, where the plus or minus depends on whether $h(v_3) = 1$ or $h(v_3) = 0$. If not, say n_1 is odd, then we must have v_1 be join-friendly. This is enough to guarantee h is friendly. Note that if n_1 and n_2 are both odd, we only need one of v_1 or v_2 to be join-friendly, but we can have both be join-friendly as well.

Case 2: $f(v_1) \neq g(v_2)$. Then use $\bar{f} = 1 - f$. Then $N(\bar{f}) = N(f), e_{\bar{f}}(i) = e_f(i)$ for $i = 0, 1$, and we are back in case 1 with $\bar{f}$ and g. Notice that in the case when n_1 is odd, v_1 stays join-friendly even when we change f to $\bar{f}$.

In all cases, $e_1 - 2k_1 \in FI_+(G_1)$ and $e_2 - 2k_2 \in FI_+(G_2)$ implies that $e_1 + e_2 - 2(k_1 + k_2) \in FI_+(G)$. Thus, $FI_+(G_1) +_1 FI_+(G_2) \subset FI_+(G)$.

Next, we consider $GG = G_1 * *G_2$, where the join occurs at vertices $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, using edge v_1v_2. Assume that n_1 is even or n_2 is even. Pick any $e_1 - 2k_1 \in FI_+(G_1)$ and any $e_2 - 2k_2 \in FI_+(G_2)$. Then there exists two vertex labels f and g such that $e_f(1) = e_1 - k_1, e_f(0) = k_1$, and $e_g(1) = e_2 - k_2, e_g(0) = k_2$.

Define vertex label h as follows:

$$h(v) = \begin{cases} f(v), & \text{if } v \in V(G_1) \\ g(v), & \text{if } v \in V(G_2). \end{cases}$$

Then if we let $\bar{f} = 1 - f$, define vertex label $\bar{h}$ as follows:

$$\bar{h}(v) = \begin{cases} \bar{f}(v), & \text{if } v \in V(G_1) \\ g(v), & \text{if } v \in V(G_2). \end{cases}$$

The condition that n_1 is even or n_2 is even guarantees that both h and $\bar{h}$ are friendly labels. Let h_+ and $\bar{h}_+$ be the corresponding edge labels to h and $\bar{h}$. Note that $N(f) = N(\bar{f})$, but $f(v_1) \neq \bar{f}(v_1)$. Thus, edge v_1v_2 will be labeled differently under h_+ and $\bar{h}_+$. Recall that $e_f(1) = e_1 - k_1, e_f(0) = k_1$ and $e_g(1) = e_2 - k_2, e_g(0) = k_2$. Either $h_+(v_1v_2) = 0, \bar{h}_+(v_1v_2) = 1$, or $h_+(v_1v_2) = 1, \bar{h}_+(v_1v_2) = 0$. Without loss of generality, assume $h_+(v_1v_2) = 0, \bar{h}_+(v_1v_2) = 1$. Then $N(h) = e_1 + e_2 - 2(k_1 + k_2) - 1$ and $N(\bar{h}) = e_1 + e_2 - 2(k_1 + k_2) + 1$. Thus, if $e_1 - 2k_1 \in FI_+(G_1)$ and $e_2 - 2k_2 \in FI_+(G_2)$ then $e_1 + e_2 \pm 1 - 2(k_1 + k_2) \in FI_+(GG)$. Therefore, $FI_+(G_1) +_2 FI_+(G_2) \subset FI_+(GG)$. □

Theorem 3.3. *For any two graphs G_1 and G_2, let $G = G_1 * G_2$ and $GG = G_1 * *G_2$. If G_1 and G_2 are both fully cordial and positive with n_1 and n_2 both even then so is G; if G_1 and G_2 are both fully cordial and positive with at least n_1 odd and v_1 join-friendly then so is G; and if G_1 and G_2 are both fully cordial and positive with n_1 even or n_2 even then so is GG. Also, if $\{e_i, e_i - 4, e_i - 8, \ldots, e_i - 4[\frac{e_i}{4}]\} \subset FI_+(G_i)$, for each $1 \leq i \leq 2$, and n_1 is even or n_2 is even, then GG is fully cordial and positive.*

Proof. If G_1 and G_2 are both positive, then $FI(G_i) = FI_+(G_i)$ for each i. Thus, by lemma 3.2, if G_1 and G_2 are both fully cordial and positive with n_1 and n_2 both even then so is G; if G_1 and G_2 are both fully cordial and positive with at least n_1 odd and v_1 join-friendly then so is G; and if G_1 and G_2 are both fully cordial and positive with n_1 even or n_2 even then so is GG. If $\{e_i, e_i - 4, e_i - 8, \ldots, e_i - 4[\frac{e_i}{4}]\} \subset FI_+(G_i)$, for each $1 \leq i \leq 2$, then $FI_+(G_1) +_2 FI_+(G_2) =$

$$\{e_1 + e_2 + 1, e_1 + e_2 - 1, ..., e_1 + e_2 + 1 - 2[\frac{e_1 + e_2 + 1}{2}]\} = FI(GG),$$

which proves the final statement in the theorem. □

Remark 4. Going back to the example of G_1 being the path P_3 and G_2 being the path P_2, when we join (by using the * operation) at the inner vertex v_1 of G_1 and any v_2 of G_2, we have a label f and a label g so that $N(f) = 2, N(g) = 1, f(v_1) = i, g(v_2) = i$, and then for any other vertex, $f(x) = 1 - i$ and $g(x) = 1 - i$. Then $G = G_1 * G_2$ has a resulting label h such that $N(h) = 3$ but $v_h(i) = 1, v_h(1 - i) = 3$, so h is not friendly and only $1 \in FI(K_{1,3})$. This is why we need to join an odd graph using the * operation with a join-friendly vertex.

Remark 5. The last part of theorem 3.3 is important, because it allows us to join two graphs that do not have all of its friendly index numbers being positive, and still obtain a fully cordial and positive graph. We demonstrate this in the next section.

4. Applications

Recall that in section 2, we found a graph H_6 of 6 vertices and 6 edges that is fully cordial. Note that H_6 is positive (we did not prove this but it is easy to verify). We can join H_6 with itself using the ** operation and create a new fully cordial graph, say G_2, with 12 vertices and 13 edges. If we continue this process over and over again, we can create a fully cordial graph $G_n = G_{n-1} * H_6 = (6n, 7n - 1)$ (see figure 8 for an example). Thus, we have the following result:

Theorem 4.1. *For any k, there exists an n and a graph $G = (n, n + k)$ such that G is fully cordial.*

Hence, even though fully cordial graphs have to satisfy the upper bound in corollary 1.3, the value of $e - n$ can be arbitrarily large for fully cordial graphs. Recall that we showed previously that $\{2a-1, 2a-5, 2a-9, ..., 2a-1-4[\frac{2a}{4}]\} \subset FI_+(C(a, a))$, for each $a \geq 2$. The ** join of $C(a, a)$ with itself, if we choose the correct vertices, results in $C(a, a + 1, a + 1, a)$. Hence, $C(a, a + 1, a + 1, a)$ is fully cordial and positive. We can arbitrarily join any number of caterpillars of the form $C(a_i, a_i)$ and get a fully cordial and positive graph. We summarize this in the next corollary.

Corollary 4.2. *For any positive integers $a_i \geq 2$, and any $i \geq 2$, caterpillars of the form $C(a_1, a_1 + 1, a_2 + 1, a_2 + 1, a_3 + 1, a_3 + 1, ..., a_i + 1, a_i)$ is fully cordial and positive.*

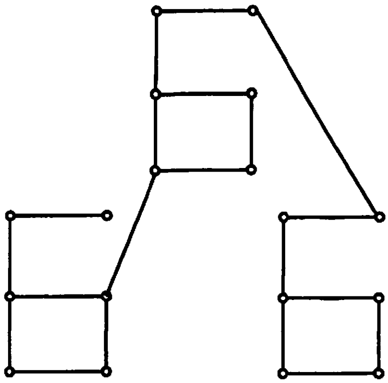


FIGURE 8. The graph of G_3.

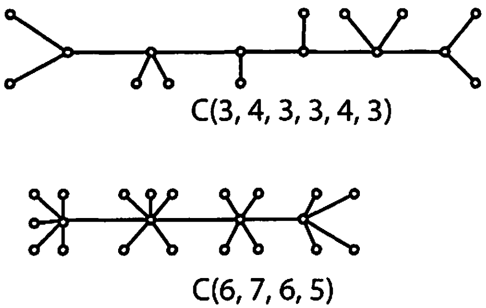


FIGURE 9. Graph of $C(3,4,3,3,4,3)$ and $C(6,7,6,5)$.

Figure 9 shows some examples of these fully cordial caterpillars.

We can create more complicated graphs by ** joining caterpillars with other caterpillars (see figure 10 for some example). We conclude by mentioning that we now have the technique in place to create many infinite classes of fully cordial graphs by simply joining fully cordial graphs together over and over again (see figure 11 for an example).

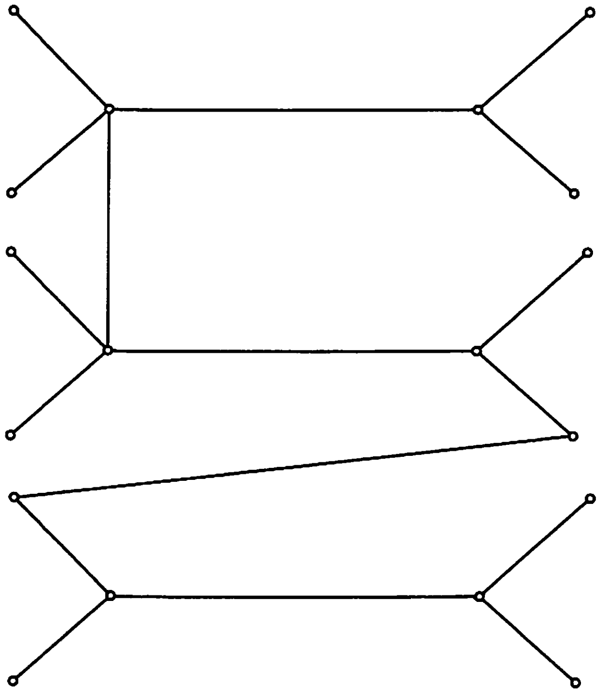


FIGURE 10. A fully cordial graph from joining multiple caterpillars.

Acknowledgements: I would like to thank Daniel Corral for introducing me to the concept of friendly indices and fully cordial graphs. I would also like to thank Dr. Salehi for all of his help and support.

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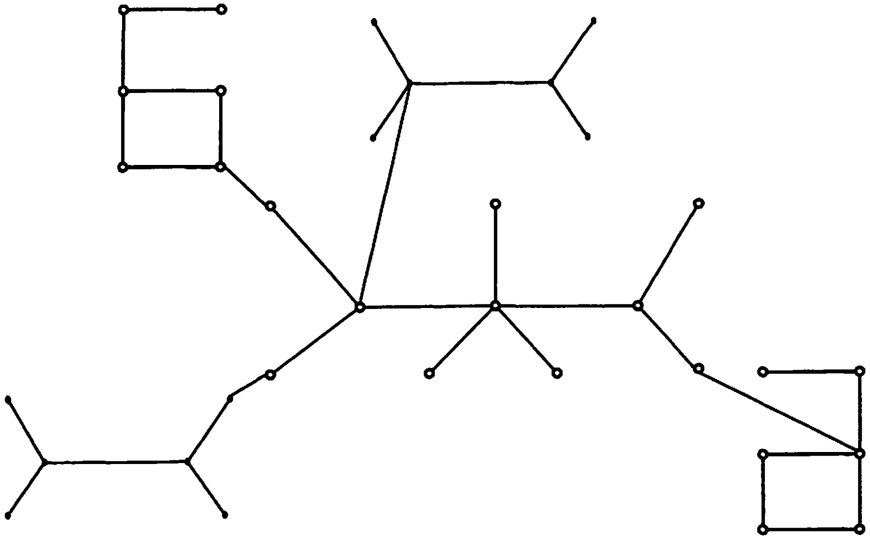

FIGURE 11. Another fully cordial graph that is not a tree.

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